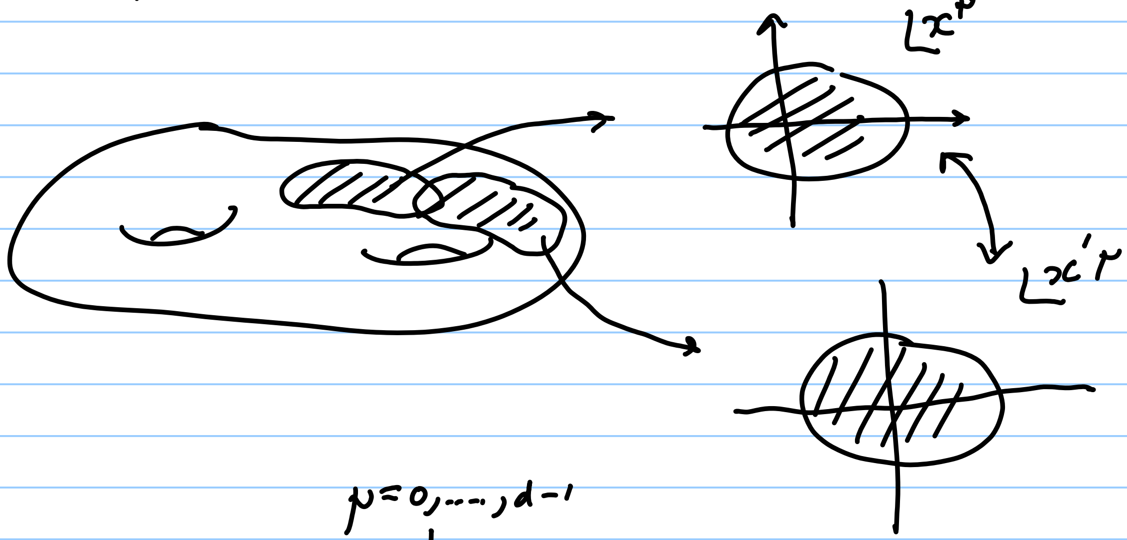
Note Title 20/10/2009

Spacetime : Lecture 1

Vectors, tensors and derivatives



$\mu = 0, \ldots, d-1$

A vector V^μ is a quantity that transforms

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

summation convention

Often it is convenient to define an abstract vector

$$V = V^\mu \overbrace{\frac{\partial}{\partial x^\mu}}^{\equiv\, e_\mu}$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$$

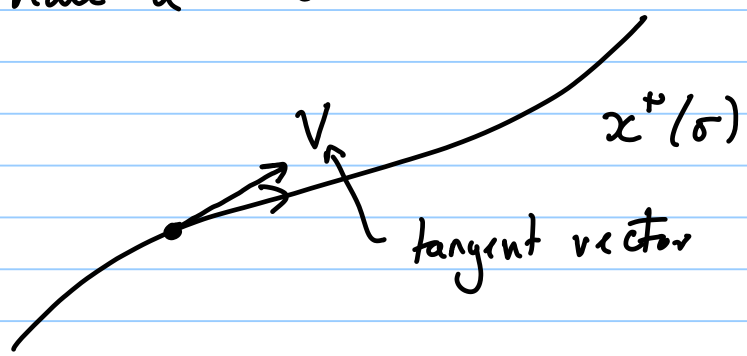
A 1-forms transform as

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

or abstractly as

$$A = A_\mu \, dx^\mu \qquad dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

If we have a line



$$V = \underbrace{\frac{dx^\mu(\sigma)}{d\sigma}}_{V^\mu} \frac{\partial}{\partial x^\mu} = \frac{d}{d\sigma}$$

Lie derivative of a vector Y w.r.t to X

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

$$\underbrace{\mathcal{L}_X Y}_{\text{this is a vector}} = [X, Y]$$

$$= [X^\mu \partial_\mu, Y^\nu \partial_\nu]$$

$$= X^\mu (\partial_\mu Y^\nu) \partial_\nu - Y^\nu (\partial_\nu X^\mu) \partial_\mu$$

$$(\mathcal{L}_X Y)^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu$$

check it is a vector.

(p, q) - tensor $\qquad T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$

$$T'^{\mu_1'\cdots\mu_p'}{}_{\nu_1'\cdots\nu_q'} = \underset{\uparrow}{\frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}}} \cdots \underset{\uparrow}{\frac{\partial x^{\nu_1}}{\partial x'^{\nu_1'}}} \cdots T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$$

Contractions – consistent with definitions of tensors

$$A^\mu B_\mu = \text{scalar}$$

$$A^{\cdots\mu\cdots}{}_{\cdots\cdots} \quad B^{\cdots}{}_{\cdots\mu\cdots}$$

$$A(B,C) = A^{\mu_1\mu_2} B_{\mu_1} C_{\mu_2}$$

The covariant derivative is a rule for defining the derivative of tensors, e.g.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

a connection (not a tensor)

the connection compensates for the fact that $\partial_\mu V^\nu$ doesn't transform like a $(1,1)$ tensor:

$$\partial_\mu' V'^\nu = \frac{\partial x^\sigma}{\partial x'^\mu} \partial_\sigma \left(\frac{\partial x'^\nu}{\partial x^\lambda} V^\lambda \right)$$

$$= \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\lambda} \partial_\sigma V^\lambda + \frac{\partial x^\sigma}{\partial x'^\mu} \underbrace{\frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\lambda}}_{\text{"bad"}} V^\lambda$$

need

$$\Gamma'^{\nu}{}_{\mu\lambda} = \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \Big(\overbrace{\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\lambda}}}^{\text{bad}} + \Gamma^{\sigma}{}_{\xi\eta} \frac{\partial x^{\xi}}{\partial x'^{\mu}} \frac{\partial x^{\eta}}{\partial x'^{\lambda}} \Big)$$

Can also write this as $\nabla_X Y$ which means

$$(X^{\mu} \nabla_{\mu} Y^{\nu}) \frac{\partial}{\partial x^{\nu}}$$

note $\nabla_X Y$ is a vector.

Extend to tensor by linearity

$$\nabla_X (\underbrace{A^{\mu} B_{\mu}}_{\text{scalar}}) = (\nabla_X A)^{\mu} B_{\mu} + A^{\mu} (\nabla_X B)_{\mu}$$

$$= X^{\mu} \partial_{\mu} (A^{\nu} B_{\nu})$$

From a connection we can define 2 related tensorial objects:

TORSION $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

vectors

vector

(1,2) tensor

$$T^{\mu}{}_{\nu\lambda} X^{\nu} Y^{\lambda} = \text{----- fill in ---}$$

$$= (\Gamma^{\mu}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\lambda\nu}) X^{\nu} Y^{\lambda}$$

RIEMANN TENSOR

$$R(Z,X,Y) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

Vectors

Vector

(1,3) tensor

$$R^\mu{}_{\nu\lambda\sigma} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$$

RICCI TENSOR

$$R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}$$

(0,2)

Spacetime has a special additional structure

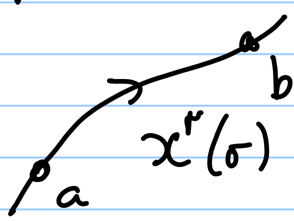
(0,2) symmetric tensor — Metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

"line element"

$$g_{\mu\nu} = g_{\nu\mu}$$

Defines a notion of "distance" between 2 points on a curve



$$L = \int ds = \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\, d\sigma$$

$x^\mu(\sigma)$

Metric $\longrightarrow$ Geometry

Theorem : g uniquely determines a connection Γ if

(1) Torsion vanishes, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$

(2) $\nabla_X g = 0$, $\nabla_\mu g_{\nu\lambda} = 0$

this defines the metric- or Levi-Civita connection

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda})$$

This means that $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ can be used to raise and lower indices.

$$A^\mu = g^{\mu\sigma} A_\sigma$$

General Relativity

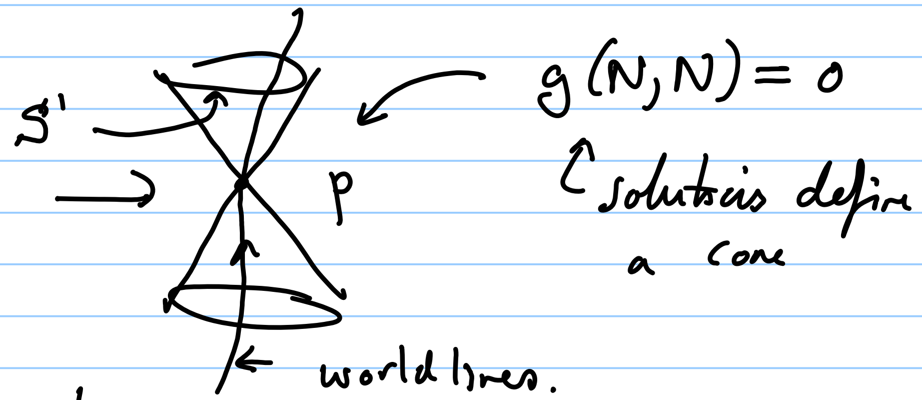
In GR spacetime is a manifold with a pseudo-Riemannian metric. At each point $p \in M$ there exists a basis of vectors E_a such that

$$g(E_a, E_b) = \mathrm{diag}(-1, 1, \ldots, 1)$$

( )

$$\searrow \quad g_{\mu\nu} E^\mu{}_a E^\nu{}_b = \eta_{ab}$$

The metric determines the connection and curvature and also global properties, e.g. the null cone at each point



e.g. Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$g(N,N) = -(N^0)^2 + (N^1)^2 + (N^2)^2 + (N^3)^2 = 0$$

$$N = (N^0, \vec{N}) \qquad \vec{N} = \pm N^0 \vec{\Omega}$$

future/past pointing

arbitrary unit vector on S^2

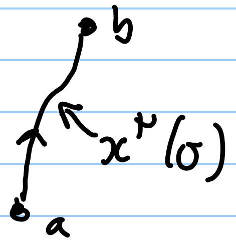
Classify vectors

$$g(v,v) = \begin{cases} <0 & \text{time-like} \\ =0 & \text{null} \\ >0 & \text{space-like} \end{cases}$$

World-lines of particles $x^\mu(\sigma)$ must have tangent vectors $V = d/d\sigma$ which are (i) time-like for massive particles (ii) null for particles.

In case (i), the proper time (as measured by a particle's internal clock) is

$$\tau = \int_a^b |ds|$$



$$= \int_{\sigma_a}^{\sigma_b} \sqrt{g_{\mu\nu}(x(\sigma))\,\dot{x}^\mu(\sigma)\,\dot{x}^\nu(\sigma)}\; d\sigma$$

World-line of free particles maximize the proper time

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu$$

Lagrangian $\qquad = \left(\frac{ds}{d\sigma}\right)^2 = \left(\frac{d\tau}{d\sigma}\right)^2$

Lagrange's equations $\longrightarrow$ GEODESIC EQUATION

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

$$\frac{d}{d\sigma}\left(2\, g_{\mu\nu}(x)\,\dot{x}^\nu\right) = \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\,\dot{x}^\lambda \dot{x}^\nu$$

If $\sigma = \tau$, the proper-time then we also have

$$\mathcal{H}(x^\mu, \dot{x}^\mu) = -1$$

check -ve sign

For the massless case the geodesic equation applies, along with

$$\mathcal{H}(x^\mu, \dot{x}^\mu) = 0 \qquad \sigma \neq \text{proper time}$$

e.g.

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

$$\mathcal{H} = e^{2a\xi}(-\dot{\eta}^2 + \dot{\xi}^2)$$

Time-like geodesics $\sigma = \tau$

$$\frac{d}{d\tau}\left(e^{2a\xi}(+2\dot{\xi})\right) = 2a e^{2a\xi}(-\dot{\eta}^2 + \dot{\xi}^2)$$

$$\frac{d}{d\tau}\left(e^{2a\xi}(2\dot{\eta})\right) = 0$$

$$\mathcal{H} = -1 = e^{2a\xi}(-\dot{\eta}^2 + \dot{\xi}^2)$$

$$\dot{\eta} = \underset{\text{constant}}{A}\, e^{-2a\xi}$$

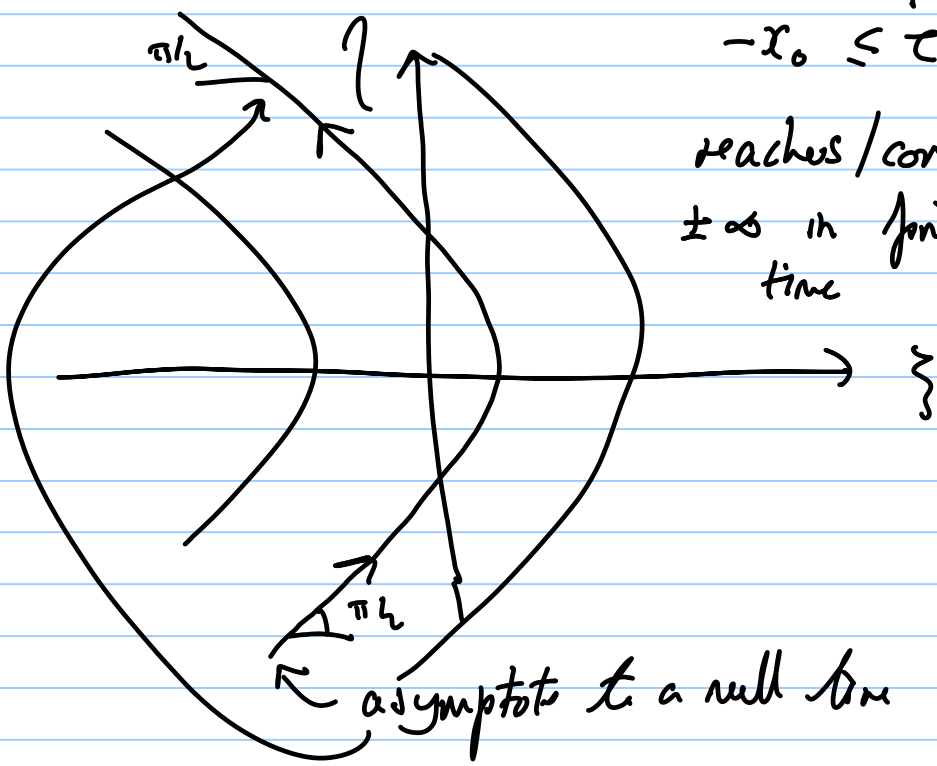
get an equation for ξ

Solutions :

$$\eta = \frac{1}{a}\tanh^{-1}(\tau/x_0)$$
$$\xi = \frac{1}{a}\log\left(a\sqrt{x_0^2-\tau^2}\right)$$

$-x_0 \leq \tau \leq x_0$

reaches / comes from $\pm\infty$ in finite proper time



asymptote to a null line

It looks like there are 2 "boundaries" this space in past and future null directions – can we extend over the boundary?