Note Title 23/10/2009

Spacetime: Lecture 2

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$$g(E_a, E_b) = \mathrm{diag}(-1,1,1,1)$$

$$d\tau^2 = -ds^2$$

Geodesics

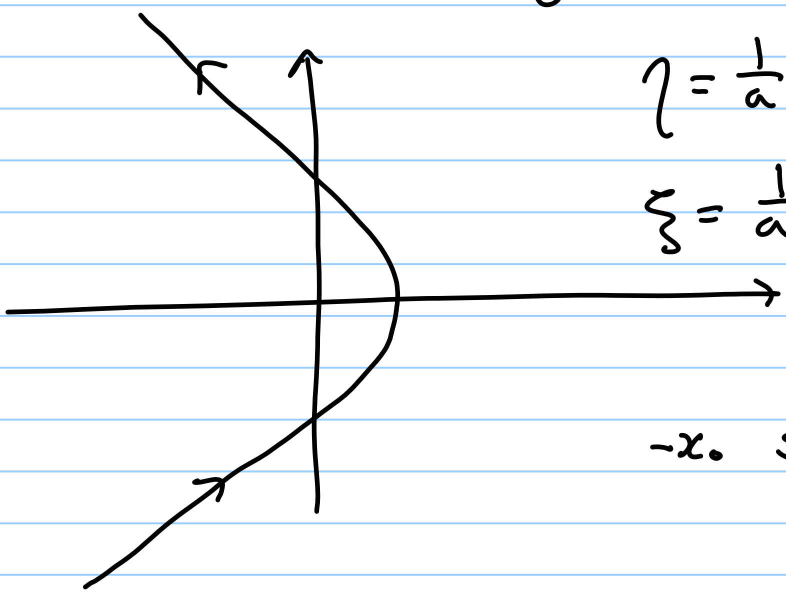
"Geodesic Lagrangian" $\quad H = g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu$

$$\frac{d}{d\tau}\left(\frac{\partial H}{\partial \dot{x}^\mu}\right) = \frac{\partial H}{\partial x^\mu}$$ GEODESIC EQS.

$$H = \begin{cases} -1 & \text{time-like} \\ 0 & \text{null} \end{cases}$$

conserved quantity

$$ds^2 = e^{-2a\xi}\left(-d\eta^2 + d\xi^2\right)$$



$$\eta = \frac{1}{a}\tanh^{-1}\left(\tau/x_0\right)$$

$$\xi = \frac{1}{a}\log\left(a\sqrt{x_0^2 - \tau^2}\right)$$

proper time

$$-x_0 \leq \tau \leq +x_0$$

This space is <u>not</u> GEODESICALLY COMPLETE

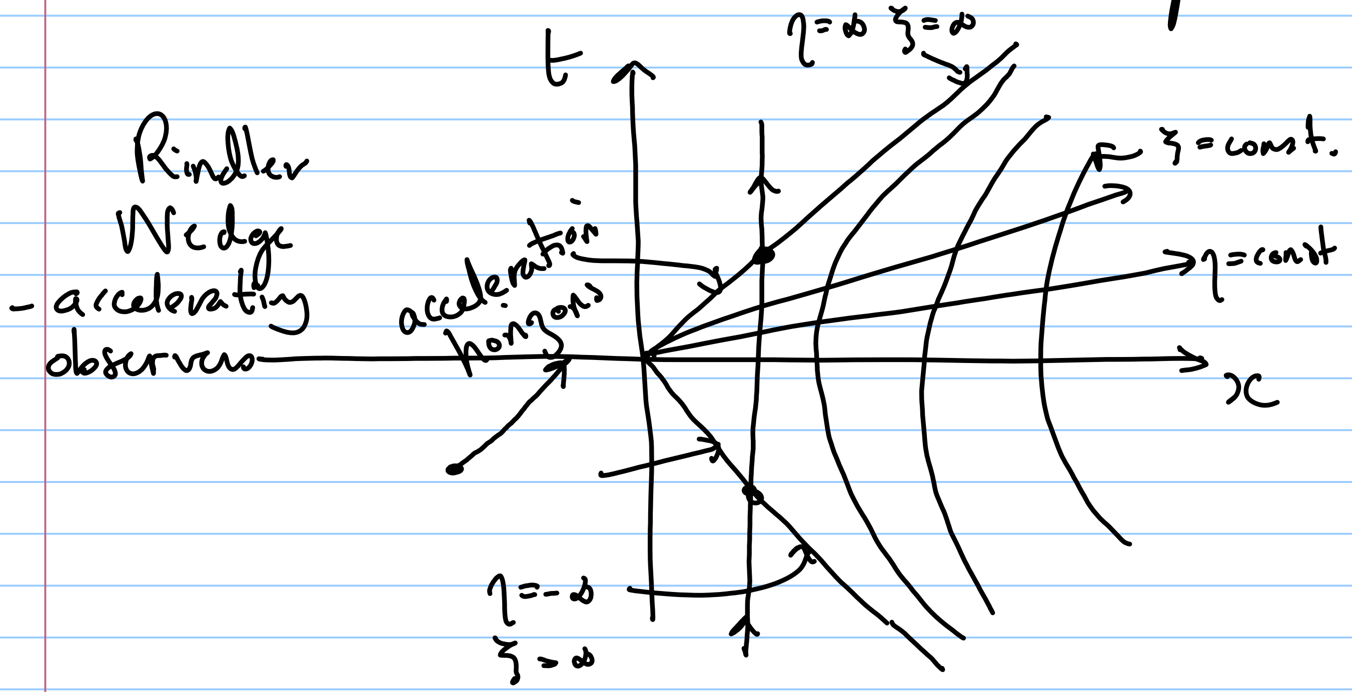
- can we extend the space?

Solve this problem by finding new coordinates

$$t = a^{-1} e^{a\xi} \sinh(a\eta)$$

$$x = a^{-1} e^{a\xi} \cosh(a\eta)$$

$$ds^2 = -dt^2 + dx^2 \quad \text{Minkowski space}$$



The structure of "infinity" plays a very important role in the global structure of the spacetime. We can bring ∞ 'in' to a finite distance using a change of coordinates

$$x^\mu \longrightarrow x'^\mu$$

e.g. 2d Minkowski Space

$$ds^2 = -dt^2 + dx^2 \qquad u = t - x \quad v = t + x$$

$$= -du\,dv$$

e.g. $-\infty \le u \le \infty \qquad -\infty \le v \le \infty$

$$u = \tan(u') \qquad v = \tan(v')$$

$-\infty \leftrightarrow +\infty \qquad -\pi/2 \leftrightarrow \pi/2$

$$du = \sec^2(u')\,du'$$

$$ds^2 = -\sec^2(u')\sec^2(v')\,du'dv'$$

$$\boxed{ds^2 = \Omega\, ds'^2}$$

Ω = conformal factor

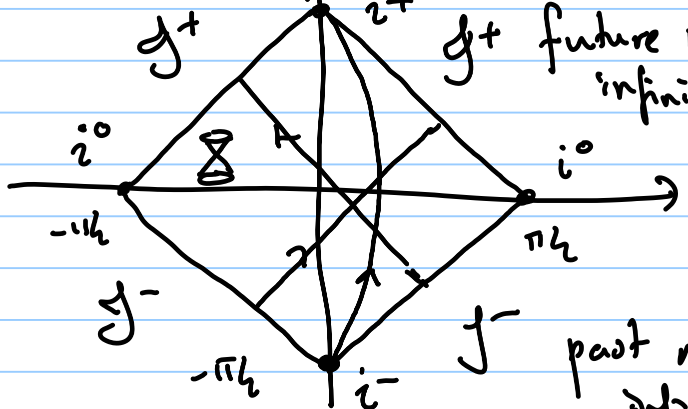
say g and g' are conformally equivalent

$ds'^2 = -du'dv'$ Minkowski space

$\sec^2(u')\sec^2(v')$ = conformal factor

i^+ = future time-like infinity

i^0 = spatial infinity



$\mathscr{I}^+$ future null infinity

$\mathscr{I}^-$ past null infinity

past time-like infinity

$u' = t' - x'$
$v' = t' + x'$

PENROSE DIAGRAM

for p+2 Minkowski space

solid angle on S^p

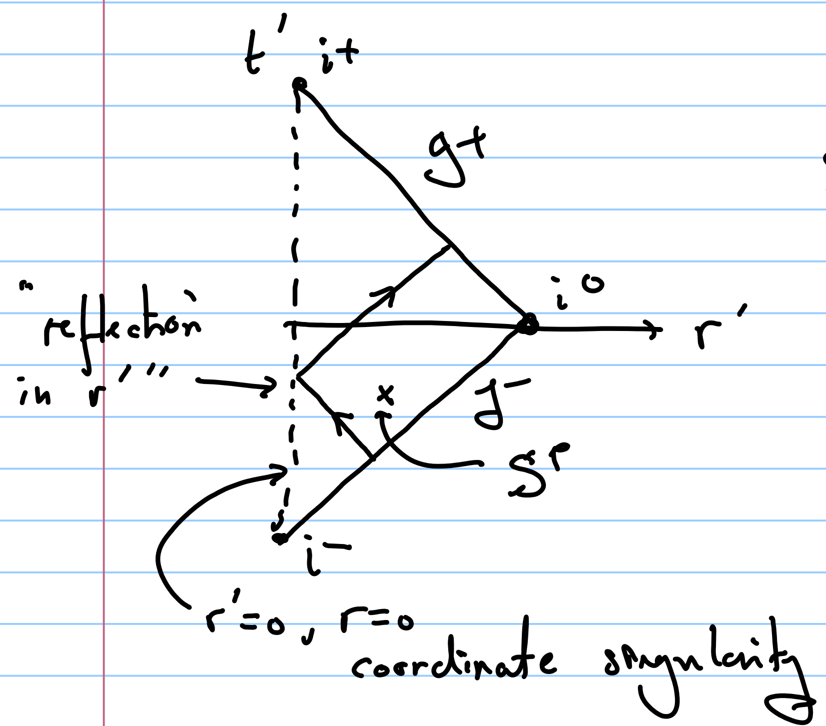
$$ds^2 = -\underbrace{dt^2 + dr^2}_{0 \le r \le \infty} + r^2 \underbrace{d\Omega_p^2}_{S^p}$$

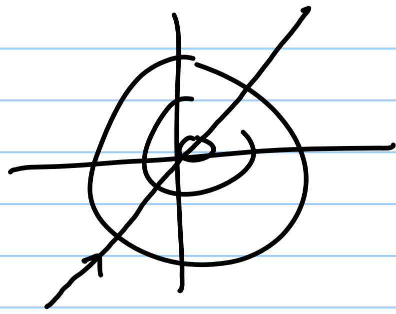
e.g. S^1 $\quad d\Omega_1^2 = d\phi^2$

S^2 $\quad d\Omega_2^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

S^3 $\quad d\Omega_3^2 = d\chi^2 + \sin^2\chi \, d\Omega_2^2$

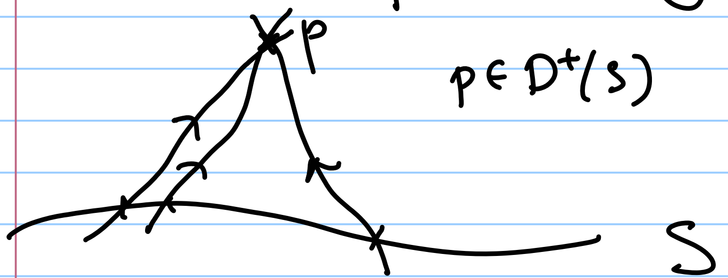
$\vdots$





Cauchy Surface

If S is a space-like surface then $p \in D^{\pm}(S)$ if all past (future) directed non-space-like curves (not necessarily geodesics) pass through S

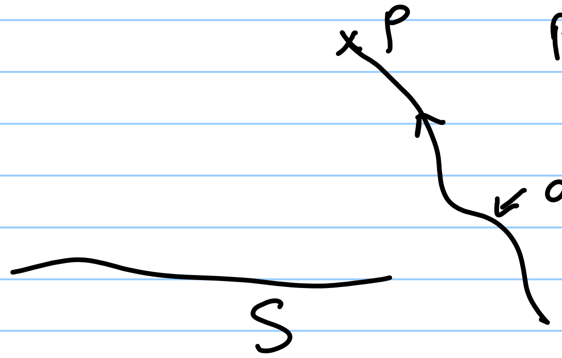


$p \in D^+(S)$

If $D^+(S) \cup D^-(S) = \mathcal{M}$ then S is a

Cauchy Surface

a "good" surface for an initial-value problem



$p \notin D^+(S)$

doesn't pass through S

The Field Equation for GR.

Recall $\quad R^\rho{}_{\nu\lambda\sigma}$ = second derivative of g

g = dynamical variable / field of GR

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}$$

cosmological constant

$R = g^{\mu\nu} R_{\mu\nu}$

$G = 1$

$T_{\mu\nu}$ = energy-momentum tensor of the matter fields

$$S = \int d^d x \sqrt{g} \left(R - 2\Lambda + \mathcal{L}_{\text{matter}} \right)$$

$$T_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}} \int d^d x \sqrt{g} \mathcal{L}_{\text{matter}}$$

Vacuum Solutions with Cosmological Constant

$\hookrightarrow T_{\mu\nu} = 0$

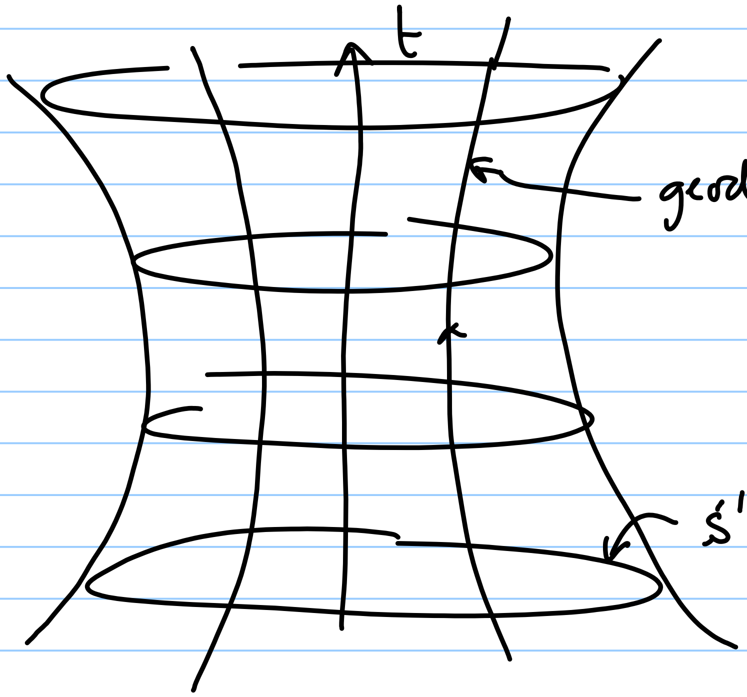
$$\Lambda = \begin{cases} > 0 & \text{de Sitter} \\ = 0 & \text{Minkowski} \\ < 0 & \text{anti-de Sitter} \end{cases}$$

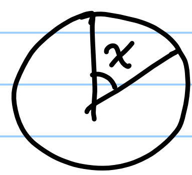
<u>De Sitter</u> in $p+2$ dimensions $\quad \Lambda = 3/\alpha^2$

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)\, d\Omega_{p+1}^2$$

$-\infty < t < \infty$

S^{p+1} with radius $\alpha \cosh(t/\alpha)$





$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)\left(d\chi^2 + \sin^2\chi\, d\Omega_p^2\right)$$

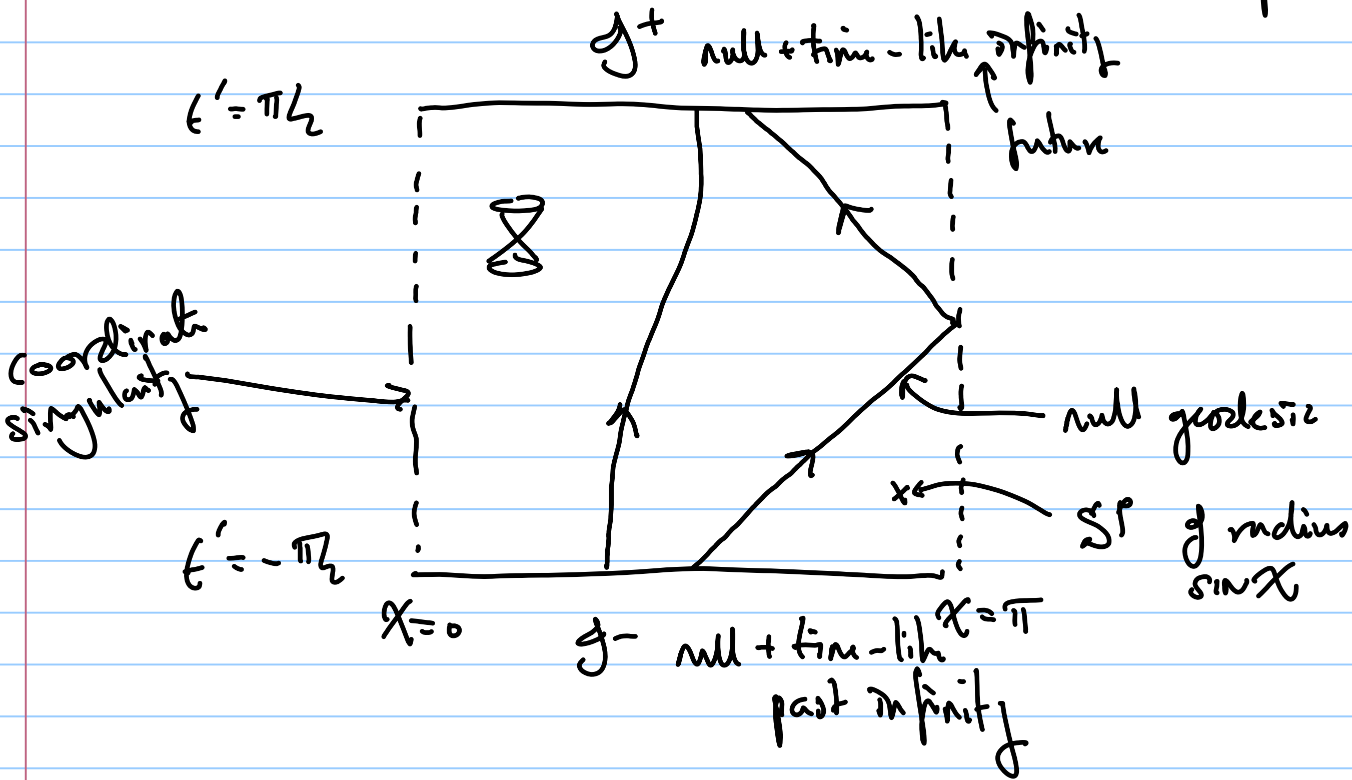
$0 \leq \chi \leq \pi$

To find the Penrose diagram

$$t' = 2\tan^{-1}(e^{t/\alpha}) - \pi/2$$

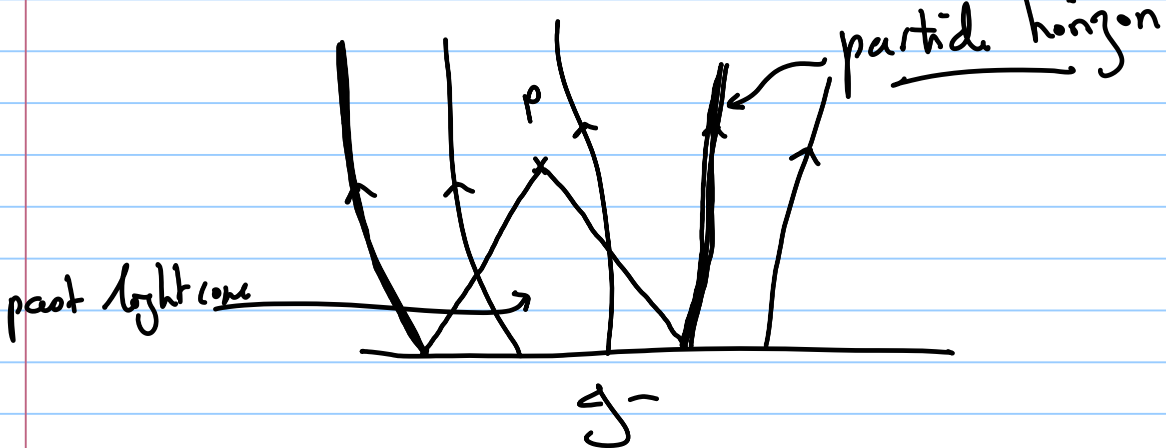
$$-\pi/2 \leq t' \leq \pi/2$$

$$ds^2 = \alpha^2 \cosh^2(t/\alpha)\left[\underbrace{-dt'^2 + d\chi^2 + \sin^2\chi \, d\Omega_p^2}_{\text{Minkowski space}}\right]$$



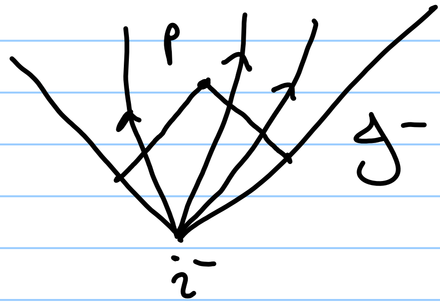
de Sitter space illustrates 2 new features:

(1) PARTICLE HORIZON: a family of massive particles (congruence of time-like geodesics)

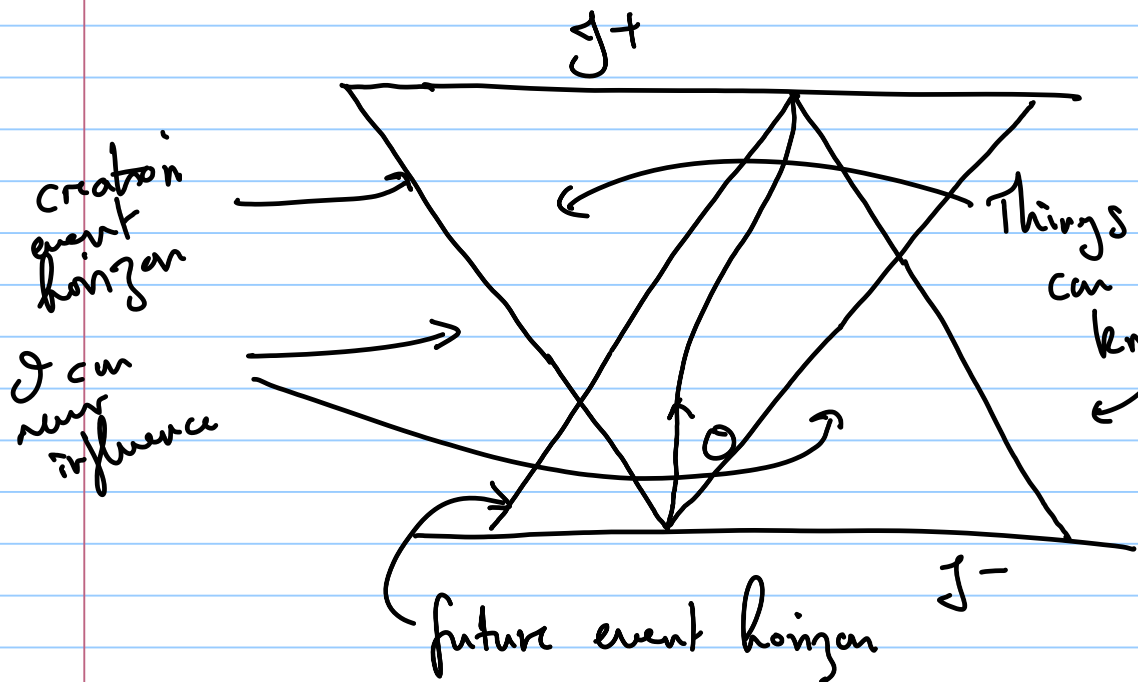


In Minkowski space

no particle horizon



(2) FUTURE / CREATION EVENT HORIZONS



In Minkowski there don't exist: