Vortices, Instantons, Monopoles + Kinks

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David Olive - Fest

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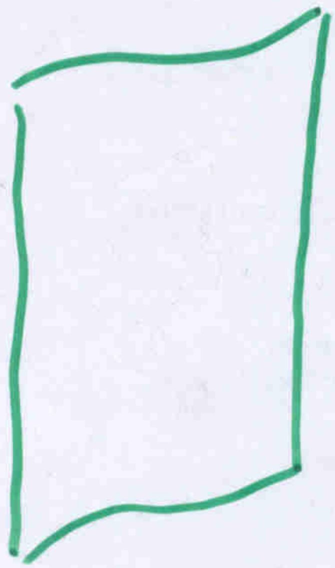
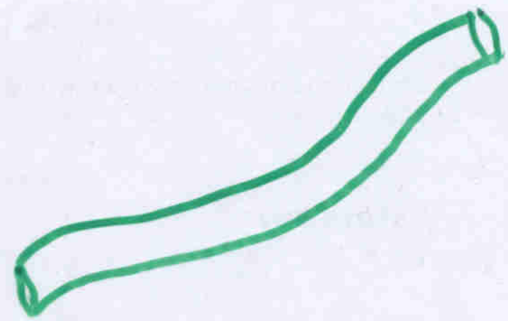
Domain walls



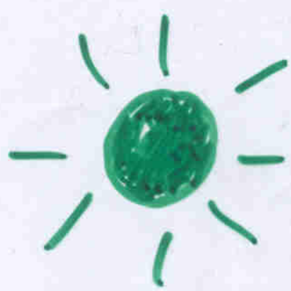
$$\partial\sigma = e^2(qq^\dagger - v^2)$$
$$\partial q = (\sigma - m_i)q$$

Vortices



$$F_{12} = e^2(qq^\dagger - v^2)$$
$$\mathcal{D}_z q = 0$$

Monopoles



$$B_i = \mathcal{D}_i\phi$$

Instantons

$$F_{\mu\nu} = {}^*F_{\mu\nu}$$

Starting Point

$d = 3+1$, $\mathcal{N} = 2$ $\quad$ $U(N_c)$ super Yang-Mills

with $N_f = N_c$ fundamental flavours

We'll focus on a subset of the fields

$$(A_\mu)^a{}_b \qquad q^a{}_i$$

$a = 1, \dots, N_c$ is colour index

$i = 1, \dots N_f$ is flavour index

with interactions

$$\mathcal{L} = \frac{1}{4e^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 - \frac{e^2}{2} \mathrm{Tr} \Big(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \Big)$$

Vacuum $\quad q^a{}_i = v \delta^a{}_i$ $\quad$ (unique with mass gap)

Symmetries $\quad U(N_c) \times SU(N_f) \longrightarrow SU(N)_{diag}$

(colour-flavour locking phase)

Note: Overall $U(1)$ broken $\Rightarrow$ vortices $\quad (\Pi_1(U(1)) \cong \mathbb{Z})$

Vortices

The first order Bogomol'nyi equations describing a vortex string in x^3-direction are

$$\boxed{\begin{aligned} (B_3)^a{}_b &= e^2\left(\sum_{i=1}^{N_f} q^a{}_i q^\dagger{}_{ib} - v^2\delta^a{}_b\right) \\ (\mathcal{D}_z q_i)^a &= 0 \end{aligned}}$$

which describes a vortex string with tension

$$\begin{aligned} T &= \mathrm{Tr}\, v^2 \int dx^1 dx^2\, B_3 \\ &= 2\pi v^2 k \end{aligned}$$

$k \in \mathbb{Z}$

Note: For $N_c = N_f = 1$, this is simply the usual Abrikosov-Nielsen-Olesen vortex in an abelian theory

What's known?

Solutions

- No analytic form of the solution is known (even for $k=1$)
- Existence of solutions proven by Taubes
- $$\boxed{\dim(\mathcal{V}_{k,N}) = 2kN}$$ (Hanany + Tong)

  moduli space of k vortices in $U(N)$ YMH

Moduli Space

Metric is

- smooth
- kähler
- unknown beyond asymptotic regime (Manton + Speight '02)

c.f. monopoles + Yang-Mills instantons

One Vortex

Suppose we have abelian vortex solution

$$B_* \quad , \quad q^*$$

Then we can embed this in non-abelian theory by

$$B = \begin{pmatrix} B_* & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad q = \overbrace{\begin{pmatrix} q^* & & \\ & v & \\ & & v \end{pmatrix}}^{\text{flavour}} \Big\} \text{colour}$$

Different embeddings $\Rightarrow$ moduli space

$$SU(N)_{diag} / SU(N-1) \times U(1) \cong \mathbb{CP}^{N-1}$$

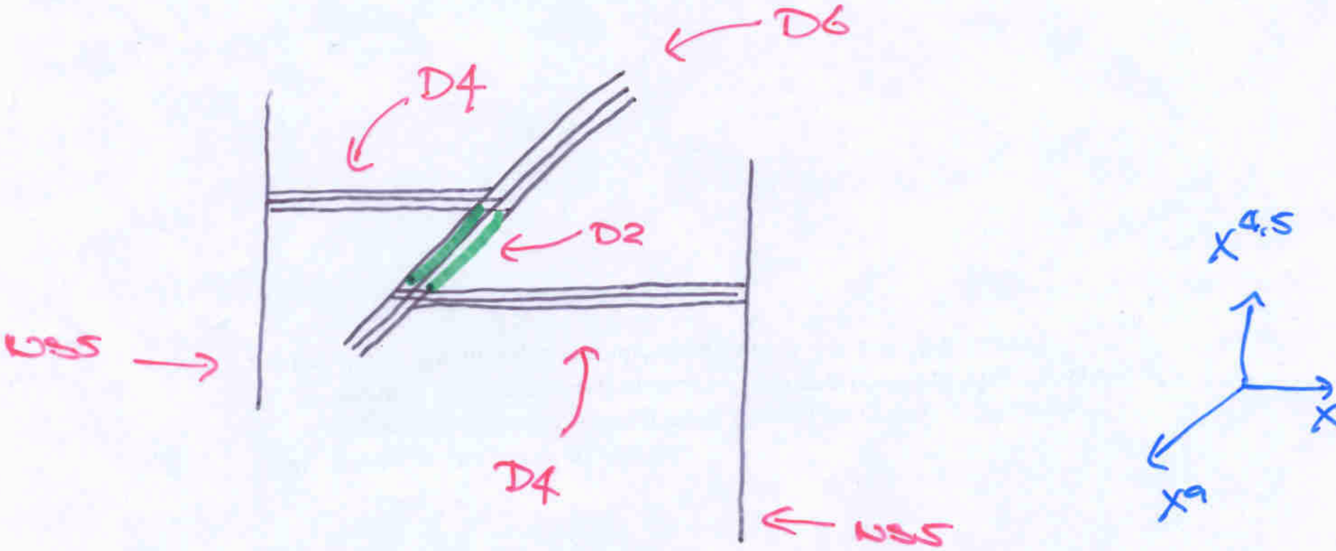
$$\Rightarrow \quad \boxed{V_{1,N} \cong \underset{\uparrow}{\mathbb{C}} \times \underset{\uparrow}{\mathbb{CP}^{N-1}}}$$

centre of mass ⌟   ⌞ internal embedding

Brane Construction

Idea: Use D-branes to understand dynamics of vortices (cf. D0-D4 system and Yang-Mills instantons)

(Jan de Boer talk)



Vortices ⟶ k x D2-branes 039

N_c x D4-branes 01236

N_f x D6-branes 0123789

2 x NS5-branes 012345

$$\boxed{\begin{array}{c}\Delta x^6 \sim 1/e^2 \\ \Delta x^9 \sim v^2\end{array}}$$

Vortex Theory

$d = 1+1$, $\mathcal{N} = (2,2)$

$U(k)$ vector multiplet

+ single adjoint chiral multiplet

$Z = X^4 + iX^5$

+ N_c fundamental chiral multiplets

$\psi_i \quad i = 1 \ldots N_c$

$$\mathcal{L}_{D2} \sim \mathrm{Tr}\,|\mathcal{D}Z|^2 + \sum_{i=1}^{N_c} |\mathcal{D}\psi_i|^2 - \frac{g^2}{2}\left(\sum_i \psi_i \psi_i^\dagger - [Z, Z^\dagger] - r\right)^2$$

with $g^2 \to \infty$ (ensures decoupling of 4d dynamics)

$$\boxed{r = 2\pi/e^2}$$

Check: One vortex $(k=1)$

Vortex Theory: $U(1)$ + neutral scalar Z

+ N_c charged scalars φ_i

$\Rightarrow$ Higgs branch $= \mathbb{C} \times \{\sum_i |\varphi_i|^2 = r\}/U(1)$

$\cong \mathbb{C} \times \mathbb{CP}^{N-1}$ as expected

with $r = 2\pi/e^2$

This method gives

- topology of $V_{k,N}$
- symmetries of $V_{k,N}$
- asymptotic metric of $V_{k,N}$
- singularity structure (in more complicated generalizations)

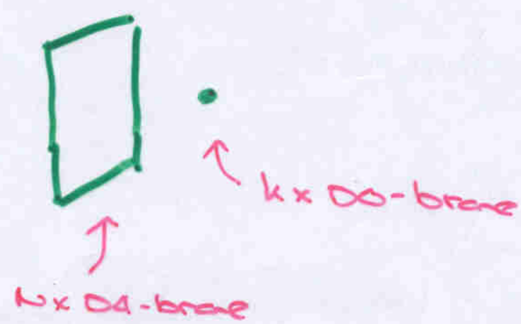
But does not give Kähler metric on $V_{k,N}$ in general

Note: Generalization to $N_f > N_c$, noncommutativity

Relationship to Instantons

ADHM construction

k instantons in $U(N)$ YM



$\mathcal{N} = (4,4)$ $U(k)$ gauge theory + adjoint hyper

+ N fund. hypers

$$\boxed{\dim \mathcal{I}_{k,N} = 4kN}$$

Vortex construction

k vortices in $U(N)$ Yang-Mills-Higgs

$\mathcal{N} = (2,2)$ $U(k)$ gauge theory + adjoint chiral

+ N fund chirals

$$\boxed{\dim \mathcal{V}_{k,N} = 2kN}$$

In fact

$$\boxed{\mathcal{V}_{k,N} \cong \mathcal{I}_{k,N}\Big|_{h=0}}$$

Killing vector rotating instantons in plane

with $r = \begin{cases} 2\pi/e^2 & \text{vortex coupling constant} \\ \theta & \text{instanton noncommutativity} \end{cases}$

(Nekrasov + Schwarz)

Conclusion

ADHM construction of instanton moduli space

$\Big\downarrow$ Throw away half the fields

Construction of vortex moduli space

Open Question

All at the level of moduli space

... what about at level of classical solitons?

Where do Monopoles + kinks fit in? (D. Tong 0307302)

Go back to original theory and change it slightly

$d=3+1$, $\mathcal{N}=2$ $U(N_c)$ super Yang-Mills
+ $N_f = N_c$ massive flavours (choose $\mathrm{Im}(m_i) = 0$)

Now truncate Lagrangian to A_μ, q_i and ϕ

adjoint scalar in vector multiplet ($\mathrm{Im}\,\phi = 0$)

$$\mathcal{L} = \frac{1}{4e^2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2e^2}(\mathcal{D}_\mu\phi)^2 + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2$$
$$- \frac{e^2}{2}\,\mathrm{Tr}\left(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\right)^2 - \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i$$

Vacuum $\phi = \mathrm{diag}(m_i)$, $q^i{}_a = v\,\delta^i{}_a$ unique with mass gap

Symmetries $U(N_c) \times SU(N_f) \xrightarrow{v^2} SU(N)_{\mathrm{diag}} \xrightarrow{m} U(1)^{N-1}_{\mathrm{diag}}$

How do these masses affect vortices ?

The old vortex solutions only remain as solutions if

$$\sum_{i=1}^{N_f} q_i^{+} (\phi - m_i)^2 q_i = 0$$

evaluated on the solution

$\Rightarrow$ only solutions are diagonal

$$B = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & B_* & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \qquad q^a{}_i = \begin{pmatrix} v & & & \\ & \ddots & & \\ & & q_* & \\ & & & \ddots \\ & & & & v \end{pmatrix} \quad \text{for } h=1$$

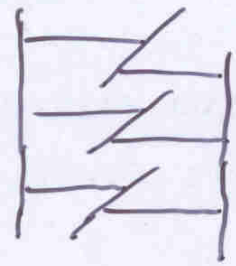
$\Rightarrow$ N_c different types of vortex string with

$$B \sim \mathrm{diag}(0, \ldots, 1, \ldots 0)$$

Note : Before we could sweep out $\mathbb{CP}^{N-1}$ moduli space with $SU(N)_{diag}$

But now $SU(N)_{diag} \xrightarrow{m} U(1)^{N-1}_{diag}$ $\Rightarrow$ moduli space lifted

The Vortex Theory — (look at k = 1 first)



$d = 1+1$, $\mathcal{N} = (2,2)$ $U(1)$ gauge theory

+ neutral scalar Z

+ N_c charged scalars ψ_i

$\boxed{\text{with twisted masses } m_i}$

$$\mathcal{L}_{vortex} = \frac{1}{2g^2} F_{03}^2 + \frac{1}{2g^2} (\partial\sigma)^2 + |\partial Z|^2 + \sum_{i=1}^{N_c} |\partial\psi_i|^2$$
$$- \frac{g^2}{2} \left(\sum_i^{N_c} |\psi_i|^2 - r \right)^2 - \sum_{i=1}^{N_c} (\sigma - m_i)^2 |\psi_i|^2$$

The theory has N_c vacua given by $\begin{cases} \sigma = m_i \\ |\psi_j|^2 = r\,\delta_{ij} \end{cases}$

But isolated vacua $\Rightarrow$ we can have kinks in vortex string

The kink in the vortex string

(choose $N_c = 2$ for simplicity
$m_1 = -m_2 = m$)

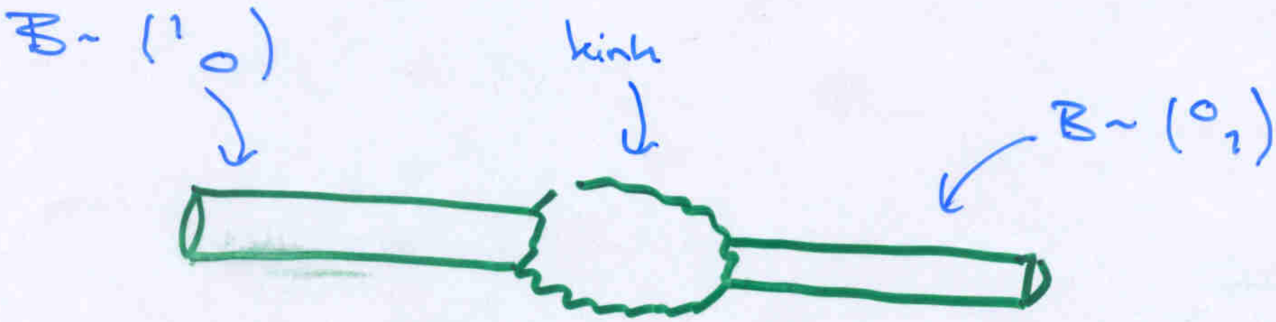
The vortex theory has a BPS kink satisfying

$$\partial \sigma = g^2 \left(\sum_{i=1}^{N_c} |\varphi_i|^2 - r \right)$$
$$\partial \varphi_i = (\sigma - m_i) \varphi_i$$

Subject to boundary conditions

$$\sigma \to \begin{cases} +m & x \to +\infty \\ -m & x \to -\infty \end{cases} \qquad |\varphi_i|^2 \to \begin{cases} r\delta_{i1} & x \to +\infty \\ r\delta_{i2} & x \to -\infty \end{cases}$$

The kink has mass
$$\boxed{M_{kink} = r\,|m_1 - m_2| = \frac{4\pi m}{e^2}}$$



The kink acts as a source of magnetic flux $\boxed{B \sim \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

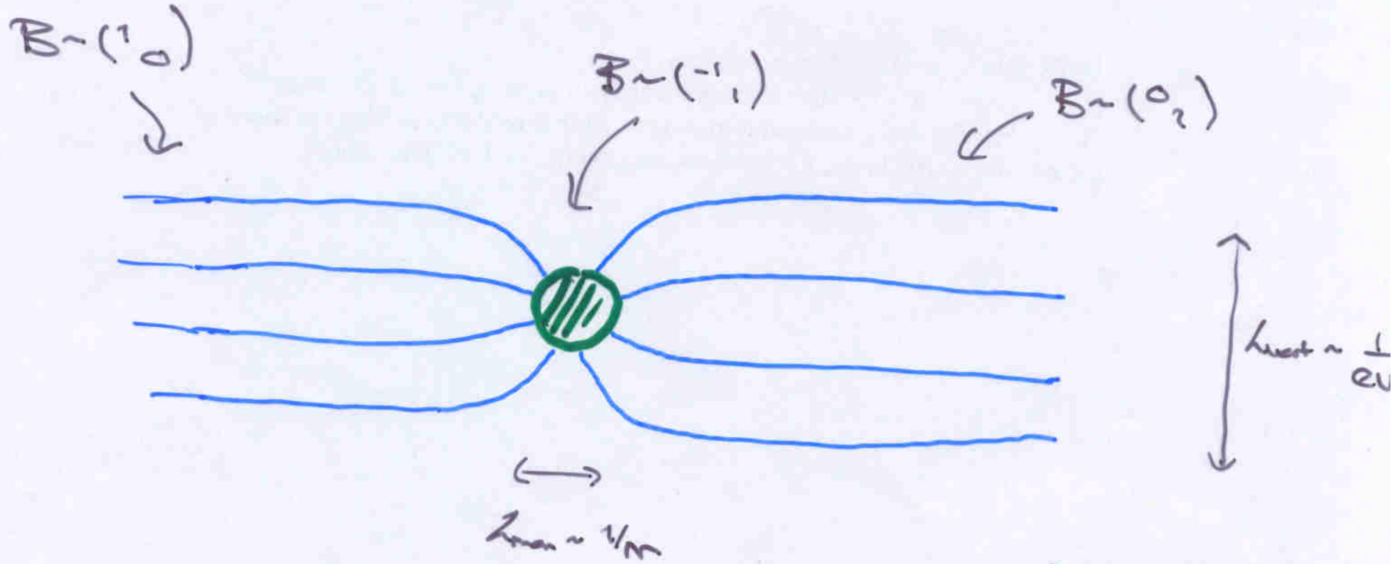
The kink in the vortex string is a magnetic monopole

The Monopole in the Higgs Phase

The $d = 3+1$ theory has symmetry breaking

$$U(N_c) \times SU(N_f) \xrightarrow{v^2} SU(N)_{diag} \xrightarrow{m} U(1)^{N-1}_{diag}$$

and a mass gap $\Rightarrow$ Meissner effect



The monopole is confined

$$M_{mon} = \frac{4\pi\langle\phi\rangle}{e^2} = \frac{4\pi m}{e^2} = M_{kink}$$

There are Bogomolnyi equations for this configuration

$$\boxed{\begin{array}{l} B_1 = \mathcal{D}_1\phi \quad , \quad B_2 = \mathcal{D}_2\phi \quad , \quad B_3 = \mathcal{D}_3\phi + e^2(qq^\dagger - v^2) \\ \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \end{array}}$$

Quantum Vortex Dynamics

(Hanany + Tong 0403158
also Shifman + Yung 0403149)

based on Dorey 9806056
Dorey, Hollowood, Tong 9902134

The result

$$\boxed{M_{kink} = M_{monopole}}$$

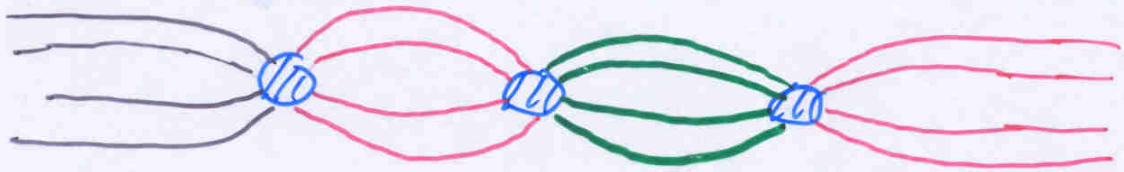
holds exactly in the quantum theory.

i.e. we can study two dimensional vortex theory to extract quantum information about four dimensional gauge dynamics

And theres more . . . .

- W-boson + quarks vs elementary string excitations
- 4d dyons vs 2d Q-kinks
- Theta angles + Witten effect
- $N_f > N_c$ version of story
- Vortex in vortex string is trapped instanton

Multi-Monopoles in Multi-Vortex Strings



Consider k vortex string: U(k) gauge theory, σ
+ adjoint scalar Z
+ N_c fundamental chirals ψ_i, mass m_i

$$V = \mathrm{Tr}\, \frac{g^2}{2} \left(\sum_{i=1}^{N_c} \psi_i \psi_i^\dagger - [Z, Z^\dagger] - r \right)^2 + \sum_{i=1}^{N_c} \psi_i^\dagger (\sigma - m_i)^2 \psi_i + |[\sigma, Z]|^2$$

The vacua are labelled in part by $\sigma = \mathrm{diag}(m_{i_1}, \ldots, m_{i_k})$

Index Thm

(Sakai, Tong, ... in progress)

Let $\mathrm{Tr}(\sigma_{+\infty} - \sigma_{-\infty}) = \vec{g} \cdot \vec{m}$

$\vec{g}$: characterizes topological charge; $\vec{m} = (m_1, \ldots, m_{N_c})$

and write $\vec{g} = \sum_a n_a \vec{\alpha}_a$

n_a: positive integers; $\vec{\alpha}_a$: simple roots of $SU(N_c)$

$$\boxed{\#\,\text{moduli of kink} = 2 \sum_a n_a = \tfrac{1}{2}\, \#\,\text{moduli of monopole}}$$

Conclusions

- Vortex theory (½ of ADHM instanton theory)
- Confined monopoles (kinks on vortex string)
- Quantum Dynamics (2d vs 4d theories)