Some aspects of affine Toda field theory

— Integrable field theory models in 1+1 —

- sine - Gordon   Skyrme

  - soliton, anti-soliton, breathers
  - non perturbative QFT
  - Duality SG ⟺ massive Thirring

    Coleman - Mandelstam
  - S-matrix Faddeev et al

    (Zamolodchikov)^2

  ⋮

- Toda theory   FPU

  Toda

— Chain

$$\mathcal{L} = \sum_n \left[\tfrac{1}{2}\dot{x}_n^2 - \left(e^{x_{n+1} - x_n} \right) \right]$$

— 'molecules' and field theory

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \underline{\phi} \cdot \partial^\mu \underline{\phi} - V(\underline{\phi})$$

$$V(\underline{\phi}) = \frac{m^2}{\beta^2} \sum_0^r n_i \, e^{\beta \underline{\alpha}_i \cdot \underline{\phi}}$$

$\underline{\alpha}_i \quad i = 1, 2, \ldots, r$ simple roots

- $\underline{\alpha}_0 = -\sum_1^r n_i \underline{\alpha}_i \qquad (n_0 = 1)$

  massive

- conformal $\qquad (n_0 = 0)$

Mikhailov, Olshanetsky, Perelomov

Olive – Turok & others

- Sine-Gordon (sh-Gordon) is the simplest $a_1^{(1)}$

$$V(\underline{\phi}) = \frac{m^2}{\beta^2}\left(e^{\beta\alpha\phi} + e^{-\beta\alpha\phi}\right)$$

- ϕ real — single scalar particle
- ϕ imaginary — solitons, breathers, etc

- Many interesting properties & remaining open questions for the rest, eg:

- $\underline{\phi}$ real — r scalar particles
  - S-matrix
  - $\beta \to \frac{4\pi}{\beta}$ duality

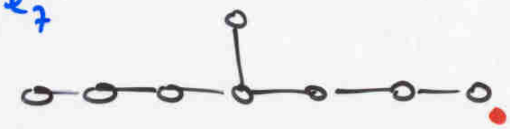
- $\underline{\phi}$ complex — solitons Hollowood, Olive et al
  - S-matrix Hollowood

⋮

Possibilities

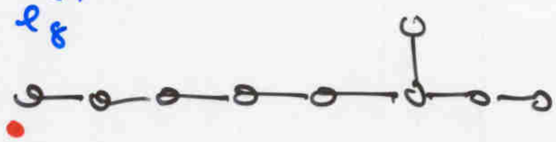
$a^{(1)}$

$e_7^{(1)}$

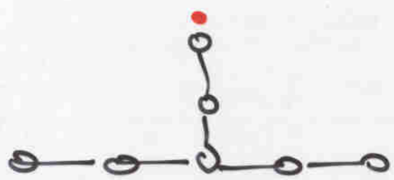


$d^{(1)}$

$e_8^{(1)}$



$e_6^{(1)}$



$a_{2n}^{(2)}$

$g_2^{(1)}$

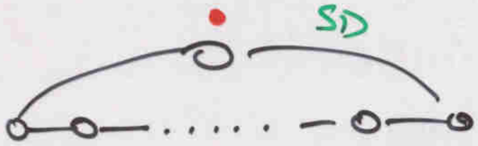
$d_4^{(3)}$

$f_4^{(1)}$

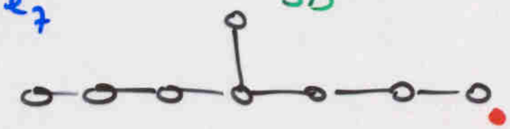
$e_6^{(2)}$

Possibilities

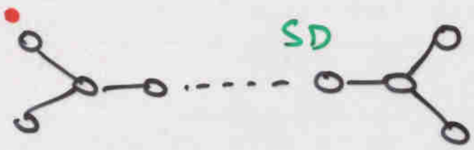
$a^{(1)}$ SD



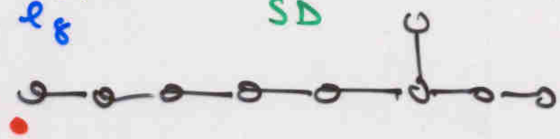
$e_7^{(1)}$ SD



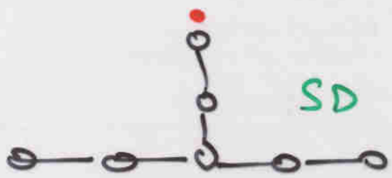
$d^{(1)}$ SD



$e_8^{(1)}$ SD



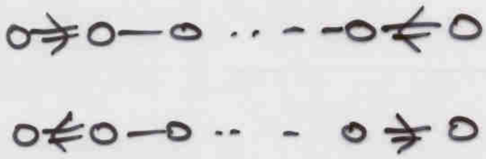
$e_6^{(1)}$ SD



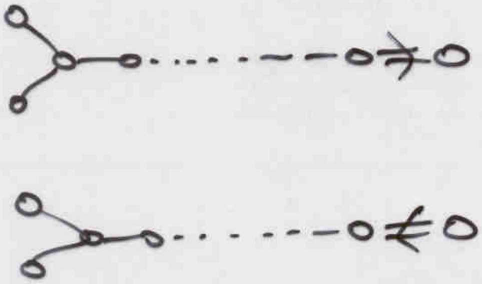
$$\underline{\alpha}_i \to \frac{2\underline{\alpha}_i}{\underline{\alpha}_i^2}$$

$a_{2n}^{(2)}$ SD

$$\beta \to \frac{4\pi}{\beta}$$



DP



DP

$g_2^{(1)}$

$d_4^{(3)}$

DP

$f_4^{(1)}$

$e_6^{(2)}$

DP

- duality ?

- missing solitons ?

- does complex Hamiltonian make sense ?

- needs more effort.

Note

- Classical set up - root data
  then Lax pair etc. needs Lie algebra
- QFT makes use of quantum groups
- exact calculations possible
- dualities
- source of inspiration

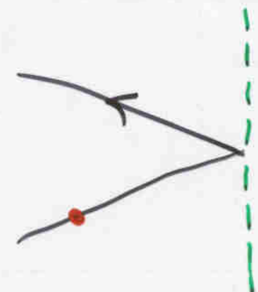
• Adding boundaries

Cherednik

Sklyanin

I

$$\partial_x \underline{\phi} = -\frac{\partial B}{\partial \underline{\phi}}$$



Integrability $\Rightarrow$ perfect reflection

$$S_{ab}(\theta_a - \theta_b)\ , \qquad R_a(\theta_a)$$

Fring - Köberle

Ghoshal - Zamolodchikov

EC - Dorey - Rietdijk - Sasaki

⋮

• For real $\underline{\phi}$ $\quad$ S all known, R some cases

— more to do

$$\mathcal{L} = \Theta(-x)\,\mathcal{L}_{TAN} - \delta(x)\,B(\underline{\phi})$$

- integrability requires

$$B(\underline{\phi}) = \frac{2m}{\beta}\sum_{0}^{r} c_i \sqrt{n_i}\, e^{\frac{\beta}{2}\underline{\alpha}_i\cdot\underline{\phi}}$$

$a_1^{(1)}$ c_i are free

$$\left.\begin{matrix} a_n^{(1)} \quad n \geq 2 \\ d_n^{(1)} \\ e_n^{(1)} \end{matrix}\right\} \quad c_i = \pm 1 \quad \text{or } \underline{\text{all}} \ 0$$

Bowcock, EC, Dorey Rietdijk.

other — mysterious

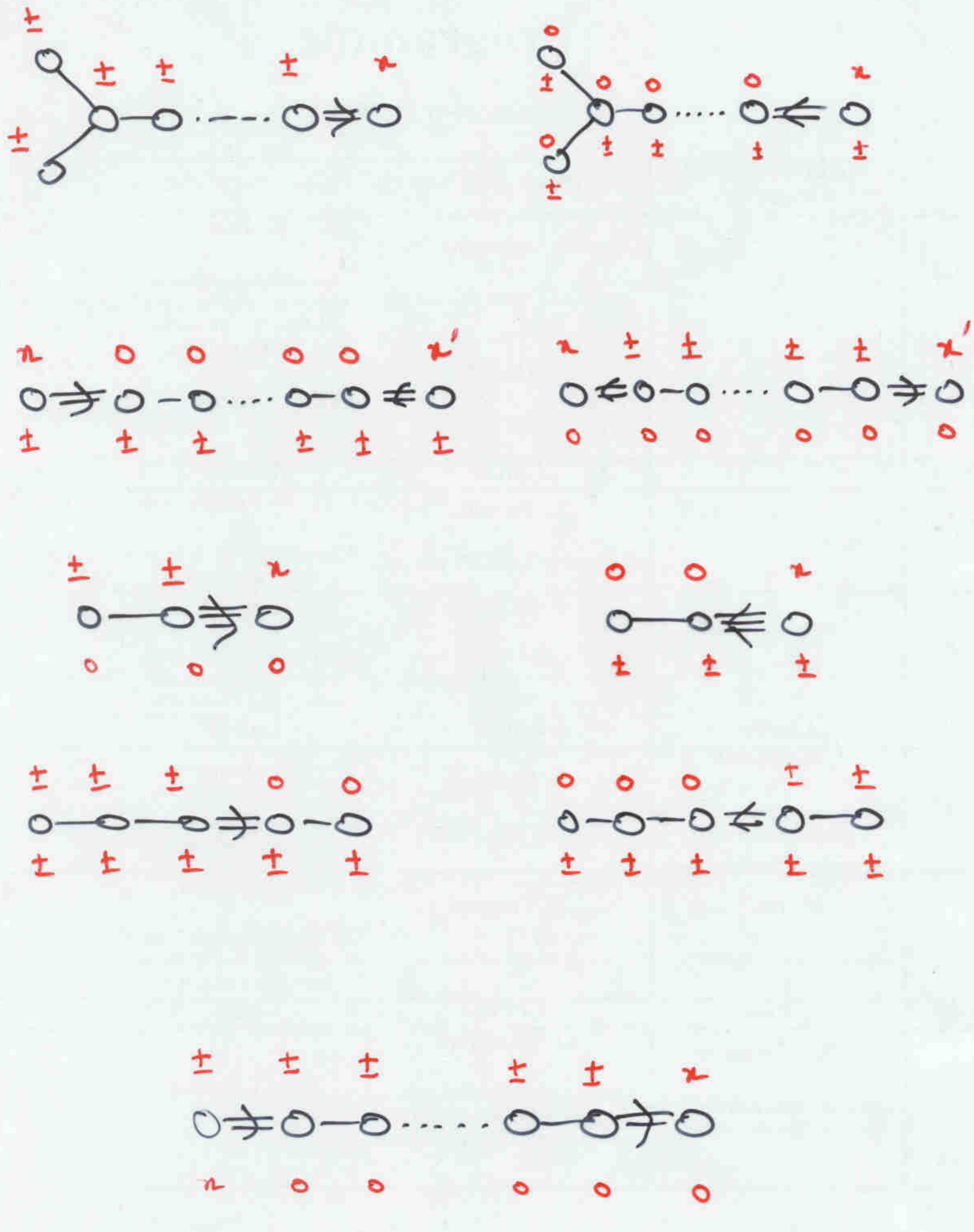
- $a_1^{(1)}$ $c_0 = \cos a_0\pi$ $c_1 = \cos a_1\pi$

$$(\beta, a_0, a_1) \longrightarrow \frac{4\pi}{\beta^2}(\beta, a_0, a_1)$$

is dual symmetry of both S & R

Chenaghlou-EC, EC-Taormina

Dual Pairs



- $B(\underline{\phi}) = \sum_{0}^{r} A_i \, e^{\underline{\alpha}_i \cdot \underline{\Phi}/2}$

II Two boundaries (or periodic)

$$\partial_x \phi = \mp \frac{\partial B_\pm}{\partial \phi} \qquad x = \pm L$$

— hard problem, not much to say yet for the general case

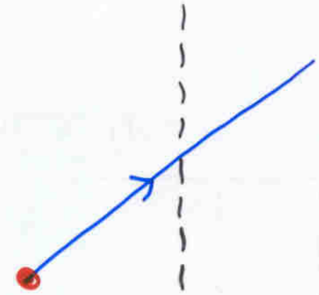
— needs 'N-zone' solutions suitably adapted

Mumford

Dubrovin - Natanzone

III

$$\phi\big|_{x_0} \neq \psi\big|_{x_0}$$



'defect'

- Integrability $\Rightarrow$ transmission with delay

— unfinished business

(EC + Bowcock + Zambon   hep-th/0305022
hep-th/0401020)

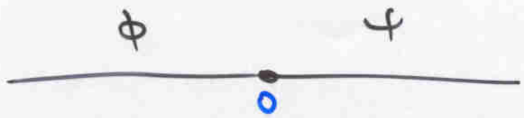
See also:

Delfino, Mussardo, Simonetti   NPB (1994)

Mintchev, Ragoucy, Sorba   PLB (2002)

Consistent set of relations between Transmission, Reflection and bulk S-matrix.

[eg discontinuity at $x=0$]



$$\mathcal{L}(\phi,\psi) = \theta(-x)\,\mathcal{L}(\phi) + \theta(x)\,\mathcal{L}(\psi) + \delta(x)\left(\frac{\phi\dot{\psi}-\psi\dot{\phi}}{2} - B(\phi,\psi)\right)$$

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial\phi)^2 - V(\phi)$$

$$\mathcal{L}(\psi) = \tfrac{1}{2}(\partial\psi)^2 - W(\psi)$$

$$\left.\begin{aligned} \partial_x\phi &= \partial_t\psi - \frac{\partial B}{\partial\phi} \\ \partial_x\psi &= \partial_t\phi + \frac{\partial B}{\partial\psi} \end{aligned}\right\} \quad x=0$$

Energy ✓ for any B

Momentum ✓ requires

$$\frac{\partial^2 B}{\partial\phi^2} - \frac{\partial^2 B}{\partial\psi^2} = 0 \qquad (\phi(0,t),\psi(0,t))$$

$$\& \quad \frac{1}{2}\left(\frac{\partial B}{\partial\phi}\right)^2 - \frac{1}{2}\left(\frac{\partial B}{\partial\psi}\right)^2 = V(\phi) - W(\psi)$$

Eg

$$V(\phi) = \frac{1}{2} m^2 \phi^2 , \quad V(\psi) = \frac{1}{2} m^2 \psi^2$$

$$\mathcal{B}(\phi,\psi) = \frac{m}{4} \left[\lambda (\phi+\psi)^2 + \frac{1}{\lambda} (\phi-\psi)^2 \right]_{x=0}$$

$$P_\pm = E \pm P + \frac{m}{2} \lambda^{\pm 1} (\phi \pm \psi)^2 \Big|_{x=0}$$

(translation invariance lost, but compensated)

Eg – ϕ, ψ both sine-Gordon

or Liouville

– ϕ free massless, ψ Liouville

– ϕ, ψ free massless, $\mathcal{B}$ non linear

(– integrability preserved)

– Generalise to other Toda, other integrable

$a_n^{(1)}$: Bowcock, EC, Zambon