QUANTUM SPACE-TIME FOAM

and topological STRINGS

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QUANTUM THEORY of GRAVITY

$$Z \sim \sum_{\substack{\text{geometries}\\ + \\ \text{topologies}}} \int \mathcal{D}g \, \exp(-S(g))$$

STRING THEORY

$Z = \exp\Big[\hbar^{-2}$  $+$  $+ \hbar^{2}$  $+ \ldots$

$+$ non-perturbative effects $\Big]$

$$\hbar = g_s = e^{\phi}$$

IS THERE A WAY TO

SEE fluctuating

GEOMETRY

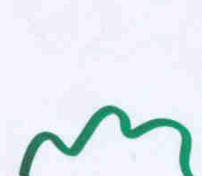
IN STRING PERTURBATION THEORY?

HARDLY, IF WE STAY AT

FINITE ORDER OF

STRING PERTURBATION

THEORY



gravitons – "small"
deviations
from background

TOPOLOGICAL STRING IS
THE ONLY EXAMPLE

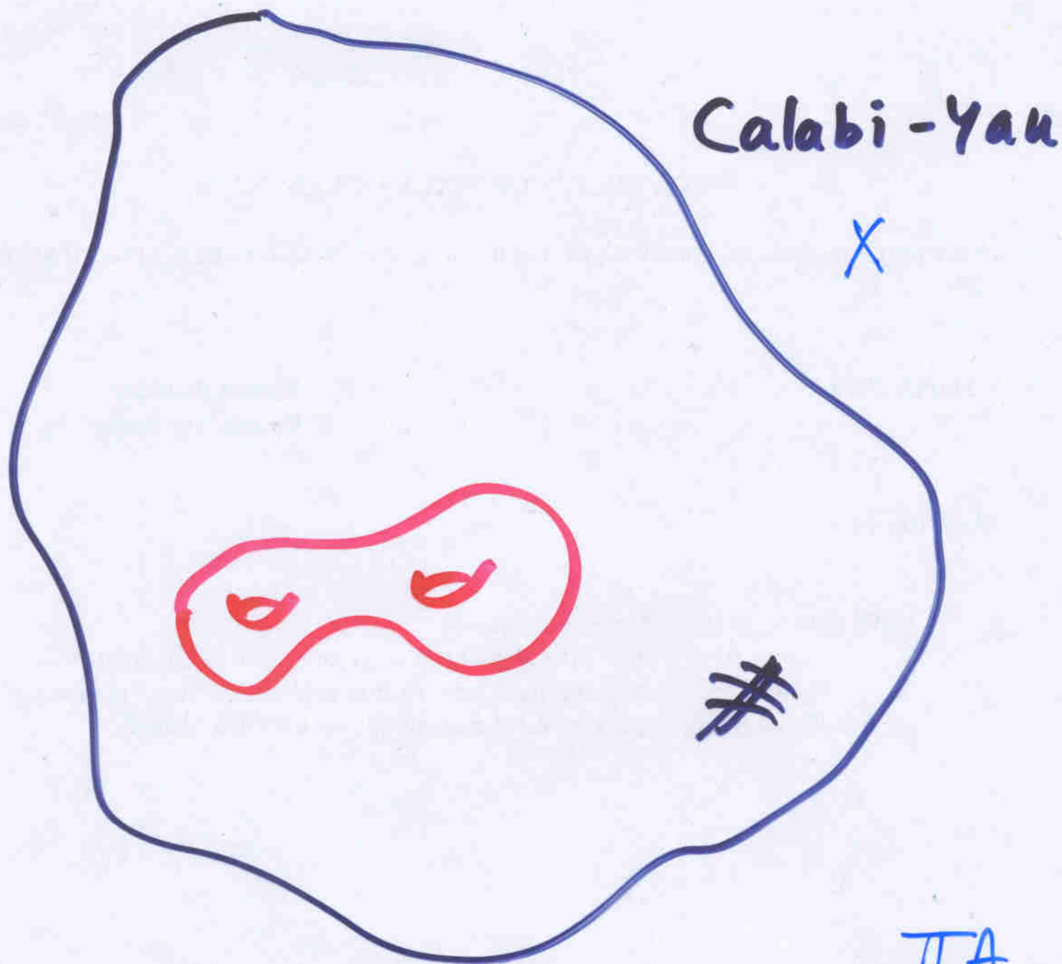
OF SUMMABLE STRING PERTURBATION THEORY (beyond 2 dim)

PHYSICALLY INTERESTING

TYPE II SUPERSTRING

COMPACTIFICATION ON CALABI-YAU SPACES

$\mathcal{N}=2$ FOUR DIMENSIONAL SUSY



IIA

VECTOR MULTIPLETS - KAHLER MODULI (a)
HYPER MULTIPLETS - COMPLEX MODULI

$$F_g(a) = \sum_{\substack{\text{genus } g \\ \text{worldsheet } C \xrightarrow{\varphi} X}} \exp\left(-\int_{\varphi(C)} a\right)$$

GEOMETRY OF VECTOR MULTIPLETS - - WS INSTANTONS

$\mathcal{F}_0(a)$ - GENUS ZERO PART

CALCULATES PREPOTENTIAL

OF LOW-ENERGY EFFECTIVE THEORY

$\mathcal{F}_g(a)$ $g>0$

GRAVIPHOTON COUPLINGS

CAN WE CALCULATE

ALL $\mathcal{F}_g$'s ?

\# of holom maps : $\Sigma_g \to X$

$[\Sigma_g] = \beta$

WORLDSHEET POINT OF VIEW:

$$\mathcal{F}_g(a) = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-\int_\beta a} N_{\beta,g}$$

TORIC CALABI-YAU MANIFOLDS

HAVE ENOUGH SYMMETRIES

CONSTRUCTION : linear σ-model
gauged susy

START WITH VECTOR SPACE

$$\mathbb{C}^{N+3} \quad \text{, coordinates}$$

$$Z^A \qquad A = 1, \ldots, N+3$$

QUOTIENT BY THE TORUS

$$\mathbb{T}_{\mathbb{C}} = (\mathbb{C}^{\times})^N \qquad Z^A \mapsto \left(\exp i Q^A_a \varphi^a\right) Z^A$$

$a = 1, \ldots, N$ $\qquad$ Q - $N+3 \times N$
$\mathbb{Z}$-valued
matrix

D-terms constraints :

$$\sum_A Q_a^A |Z^A|^2 = r_a = \text{const}$$

modulo $\quad Z^A \mapsto \exp i(Q\cdot\varphi)^A \; Z^A$

$$\varphi^a \in \mathbb{R}/2\pi\mathbb{Z}$$

i.e. $\quad \mathcal{T} = U(1)^N$

THE RESULTING SPACE

$$X_3 = \mathbb{C}^{N+3}/(\mathbb{C}^\times)^N$$

ADMITS

CALABI-YAU (RICCI FLAT KÄHLER)

METRIC iff

$$\sum_A Q^A{}_a = 0 \qquad a=1,\dots,N$$

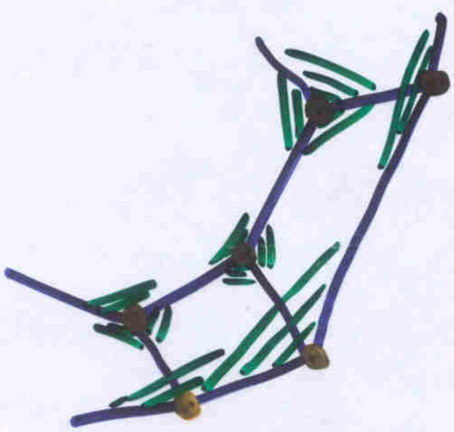
GEOMETRIC IMAGE OF TORIC MANIFOLD

NEWTON POLYHEDRON

$$\Delta_X = X_3 / U(1)^3 \qquad \text{(CONVEX)}$$

$$\Delta_X \subset \mathbb{R}^{N+3}_{\geq 0}$$

$$= \left\{ (p_1, \ldots, p_{N+3}) \,\middle|\, \sum_A Q^A_a\, p_A = r_a \;,\; p_A \geq 0 \right\}$$



ORBITS OF T^3

OVER INTERIOR POINTS OF Δ_X – $U(1)^3$

OVER FACES $U(1)^2$

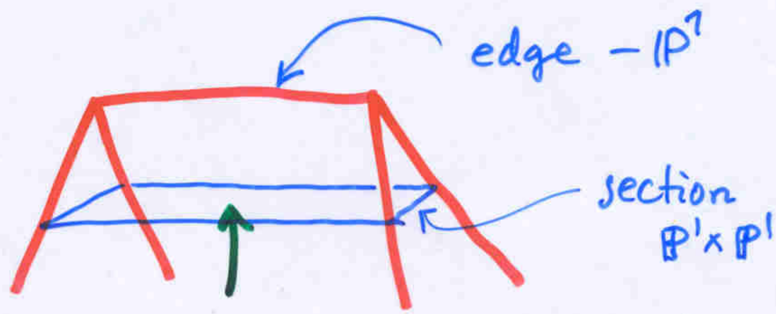
EDGES $U(1)$

VERTICES •

EXAMPLES :

$X_3 = \mathbb{C}^3$ $\qquad \Delta_X =$  $\mathbb{R}^3_{\geq 0}$

$$X_3 - \text{local } \mathbb{P}^1 \equiv \begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

$\Delta_X =$ 

house under the roof

$p_1 + p_2 - p_3 - p_4 = r$, $p_A \geq 0$

TOPOLOGICAL STRING OF TYPE A ON X_3

$$S = \int p_{iz}\partial_{\bar z}X^i + p_{\bar i\bar z}\partial_z X^{\bar i} +$$

$$+ \chi_{iz}\nabla_{\bar z}\psi^i + \chi_{\bar i\bar z}\nabla_z\psi^{\bar i} +$$

$$\alpha'\left(G^{i\bar i}p_{iz}p_{\bar i\bar z} + R_{\mu\nu}{}^{i\bar j}\psi^\mu\psi^\nu\chi_{iz}\chi_{\bar j\bar z}\right)$$

$$+ \chi\bar\chi\partial v +$$

$$+ \chi_{iz}\bar\chi_{\bar j\bar z}\Gamma^i_{\mu j}G^{j\bar j}V^\mu(\varepsilon) + \text{c.c.}$$

$$+\frac{1}{\alpha'}\Big(G_{\mu\nu}V^\mu(\bar\varepsilon)V^\nu(\bar\varepsilon)$$

$$+ \psi\partial\psi \ldots\Big)$$

$$\varepsilon \in \mathrm{Lie}\,\mathbb{T}^3_{\mathbb{C}}$$

$$\psi$$

$\bar\varepsilon$ – auxiliary (nothing depends on $\bar\varepsilon$, or α')

IN THE LINEAR SIGMA MODEL APPROACH

ε - TWISTED MASSES

RESULTS :

$$X_3 = \mathbb{C}^3$$

$$Z(\hbar, \varepsilon_1, \varepsilon_2, \varepsilon_3) = M(q)^{-\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}}$$

$$= \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(\varepsilon)$$

$$\frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \int_{\mathcal{M}_g} c_{\mathbb{H}}(\varepsilon_1)\, c_{\mathbb{H}}(\varepsilon_2)\, c_{\mathbb{H}}(\varepsilon_3)$$

Hodge bundle Chern polynomial

$H^0(\Omega^1_\Sigma)$

Here $M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$

$q = -e^{i\hbar}$

$$M(q) \sim \exp\left(\frac{\zeta(3)}{\hbar^2} + \sum_{g>1} \frac{|B_{2g}||B_{2g-2}|\hbar^{2g-2}}{2g(2g-2)(2g-2)!} \right)$$

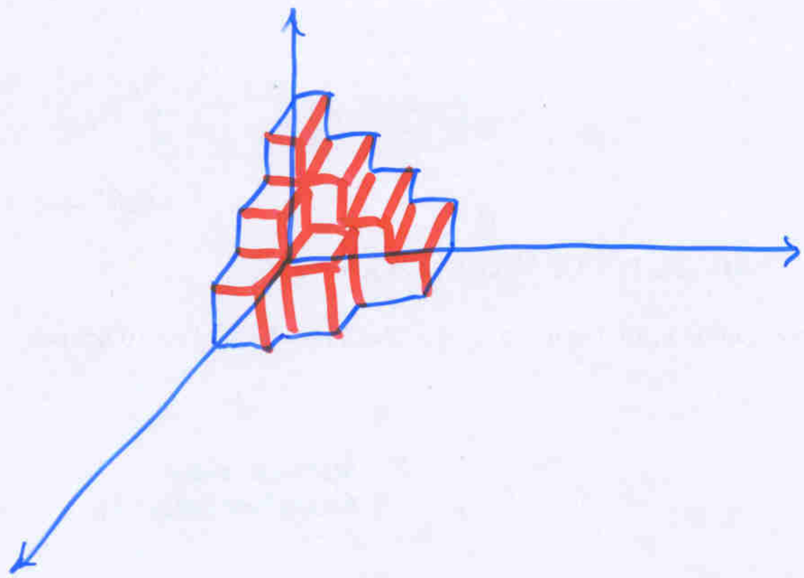
asymptotic series from top. string

$M(q)$ – MacMahon function

$$\overset{\shortparallel}{\sum_{\pi:\ \text{3d partitions}}} q^{|\pi|} \overset{\text{also}}{=}$$

M. Green's function

$$= \exp \sum_{k=1}^{\infty} k q^k \underset{\sum_{d|k} \frac{1}{d^2}}{\underset{\shortparallel}{\mu(k)}}$$



Why 3d partitions?

$\pi \longleftrightarrow \mathcal{I}_\pi$ – monomial ideals in $\mathbb{C}[z_1, z_2, z_3]$

$$\mathcal{I}_\pi = \left\{ z_1^{i-1} z_2^{j-1} z_3^{k-1} \mid (i,j,k) \notin \pi \right\}$$

Why ideals? Why monomial? Why in $\mathbb{C}^3$?

see below

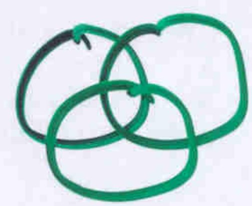
invariance under T^3

STRING ON $\mathbb{C}^3$

WE WANT to COUNT HOLOMORPHIC CURVES IN X_3

CURVES ARE DESRIBED BY HOLOMORPHIC FUNCTIONS (LOCALLY, POLYNOMIALS)

$X_3 = \bigcup_\alpha \mathcal{U}_\alpha$



$\Sigma \subset X_3$ -holomorphic curve

$\mathcal{I}_\alpha \subset \mathcal{O}_\alpha = \mathbb{C}[\mathcal{U}_\alpha]$

all polynomials in $\mathcal{U}_\alpha$, which vanish on $\Sigma \cap \mathcal{U}_\alpha$

COMPACT CURVES IN

$\mathbb{C}^3$ — POINT-LIKE

TOGETHER WITH INVARIANCE UNDER

TORUS (GAUGING) THIS

ESTABLISHES 1-1 CORRESPONDENCE

WITH 3D PARTITIONS

WHY SHOULD WE COUNT

THEM SO SIMPLY

— ANOTHER STORY

(CALABI-YAU
CONDITION
IS IMPORTANT)

In some sense, toric X is built out of several $\mathbb{C}^3$'s

one copy per fixed point of $T^3 = U(1)^3$ action on X $\Longleftrightarrow$ vertex of Δ_X

$\mathrm{Tr}_{\text{Holomorphic functions}_X}\, q$

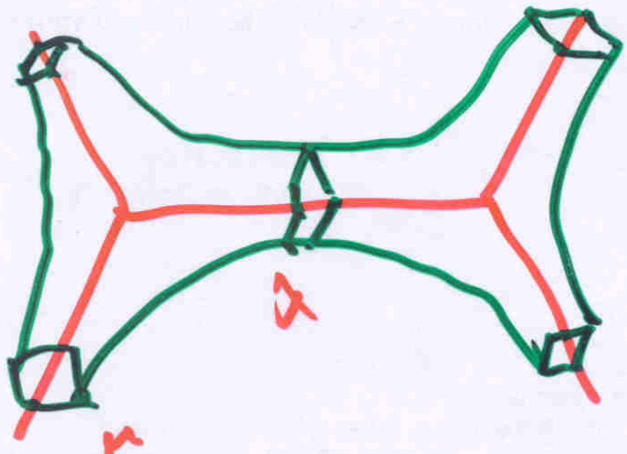
$\parallel$

$$Ch_X = \sum_{\substack{\vec{n} \in \mathbb{Z}^{N+3}_{\geq 0} \\ Q \cdot n = 0}} \vec{q}^{\,\vec{n}} = \quad \leftarrow \text{twisted Witten index of SQM on } X$$

$$= \sum_{v \in \mathrm{Vert}\,\Delta_X} \frac{1}{(1-q_{1v})(1-q_{2v})(1-q_{3v})}$$

$q_1, q_2, q_3|_v$ – local weights of T^3 action on $T_v X$

GW theory of toric X



Δ_X

also — PANTS DECOMPOSITION FOR WFT (ask Robbert Dijkgraaf)

AT EACH VERTEX — $\mathbb{C}^3$ story

but, with non-trivial asymptotics along the edges

2d partition

Kähler moduli

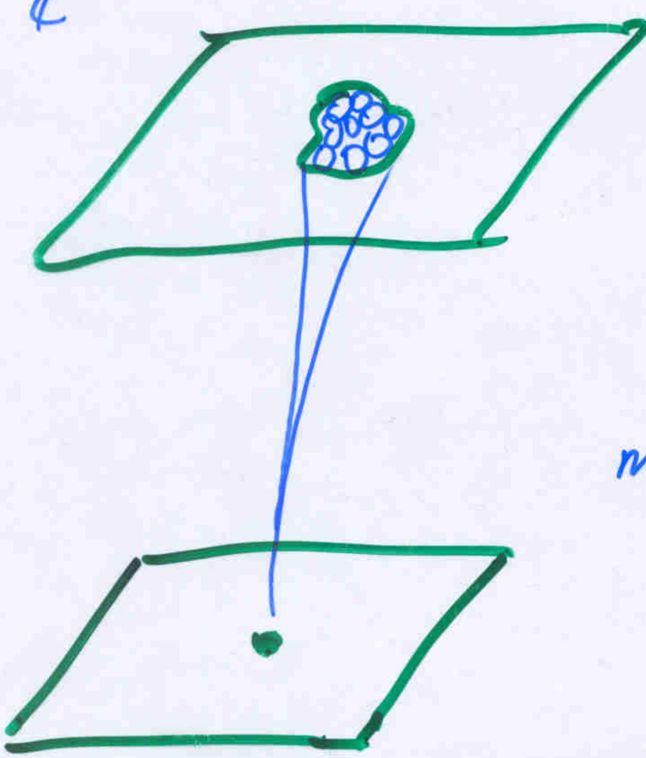
$$Z_X(\hbar)[t_e] = \sum_{\pi_v,\,\lambda_e} \pm\, q^{|\pi_v|}\, e^{-t_e|\lambda_e|}$$

"CRYSTALS" (also: fluctuating geometry)

GO BACK TO $\mathbb{C}^3$ EXAMPLE

WE NOW GIVE ANOTHER INTERPRETATION TO THE CRYSTAL PICTURE

CONSIDER ALL TORIC MANIFOLDS $\begin{array}{c} X_3 \\ \downarrow \\ \mathbb{C}^3 \end{array}$ WHICH PROJECT TO



1-1 EXCEPT AT 0

(multiple BLOWUPS AT 0)

NOW, PICK UP THE PIECES

TOPOLOGICAL STRING ON $\mathbb{C}^3$ (LARGE VOLUME LIMIT OF ANY CALABI-YAU)

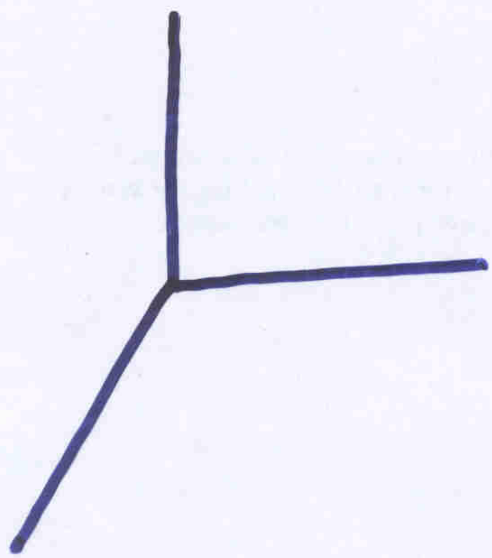
$\|$

IDEALS OF POINTS ON $\mathbb{C}^3$

$\|$ TARGET SPACE THEORY

$=$ (TORIC) KÄHLER GEOMETRIES OBTAINED BY BLOWUPS (FOAM)

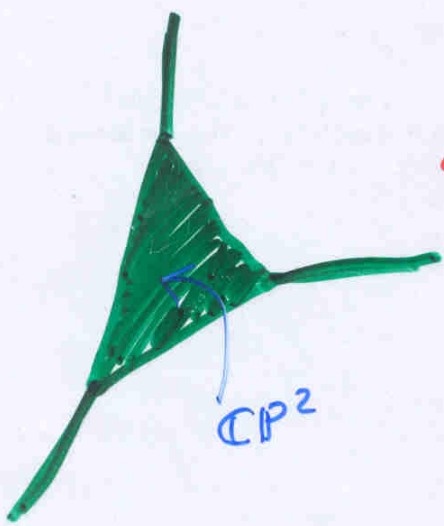
$\mathbb{R}^3_{\geq 0}$



$\mathbb{C}^3$

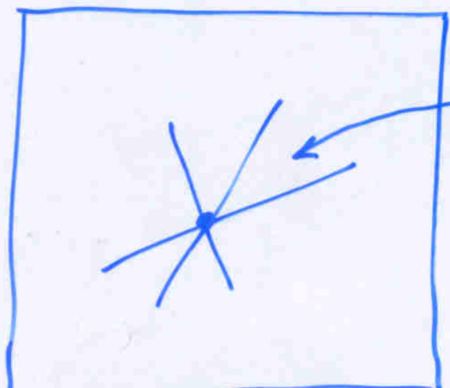
Δ_X is obtained from

$\mathbb{R}^3_{\geq 0}$ by cutting vertices



single blowup $\widetilde{\mathbb{C}^2}$

$$|z^1|^2 + |z^2|^2 + |z^3|^2 = $$
$$= r + |z^4|^2$$
$$r > 0$$

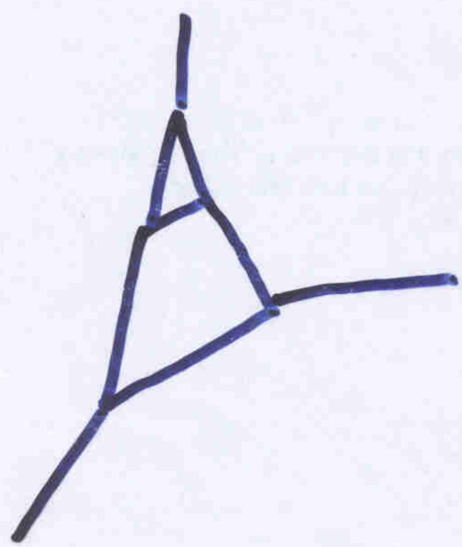


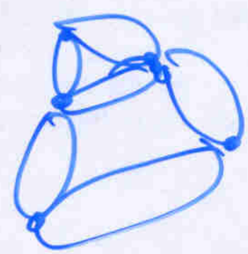
$\widetilde{\mathbb{CP}^2}$ - space of lines

$$\widetilde{\mathbb{C}^2} = \{ (\ell, x) \mid x \in \mathbb{C}^3$$

ℓ - line, passing through 0 and x $\}$

DOUBLE BLOWUPS





CONTAINS NON-CONTR.
2-SPHERES

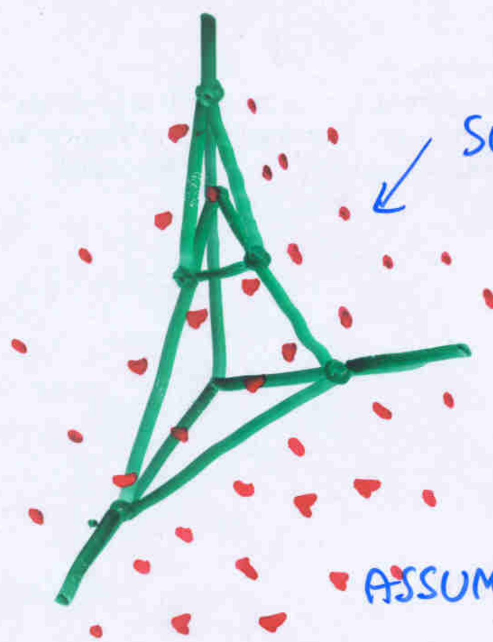
CONTINUE ....

FROM BLOWN UP GEOMETRY

$\downarrow$

3D PARTITIONS

GEOMETRIC QUANTIZATION



SOME TORIC SPACE

X_3,

IT HAS KÄHLER FORM ω

ASSUME $[\omega] \in \hbar H^2(X, \mathbb{Z})$

THEN QUANTUM STATES FORM THE IDEAL $\rightarrow$

INTEGRAL POINTS INSIDE

ΔX

THUS WE LEARN:

Z_X^A PARTITION FUNCTION OF TYPE A TOP. STRING

$$= \exp \sum_g \hbar^{2g-2} \mathcal{F}_g(a)$$

IIA BPS

||

SUM OVER 3D CRYSTALS, EMBEDDED INTO Δ_X

another picture! "Guiness"

$\approx$

SUM OVER RANDOM TORIC "GEOMETRIES"

MORE 3d PARTITIONS THEN SMOOTH TORIC'S

$\approx$

LOCALIZED PATH INTEGRAL IN SOME GRAVITY THEORY ON

X

IS THIS PICTURE REASONABLE?

WHAT SORT OF GRAVITY THEORY DO WE EXPECT TO LIVE IN THE TARGET SPACE OF TOPOLOGICAL CLOSED STRING?

B MODEL IS SIMPLER

(CARES ABOUT COMPLEX STRUCTURE ONLY)

Kodaira - Spencer F.T.

$$\int A' \frac{1}{\partial} \bar{\partial} A' + \int A' \wedge A' \wedge A' \, \Omega^2$$

A MODEL IS TRICKIER

(KÄHLER STRUCTURES)

$$\int \underset{k \frac{1}{d^c} d k}{\text{KIN. TERM}} \; + \; \int k \wedge k \wedge k \; + \dots$$

IN OUR COUNTING WE SEE

$$k = k_0 + \hbar F$$

quantized $\in H^2(X, \mathbb{Z})$

ORIGINAL KÄHLER FORM on X_3

LOOKS LIKE DISCRETIZAT'N

(cf. 2d MATRIX MODELS)

EACH CRYSTAL

(mathematically they correspond to the so-called torsion free sheaves of rank 1)

is characterised by:

$$ch_2 \sim F \wedge F$$

$$ch_3 \sim F \wedge F \wedge F$$

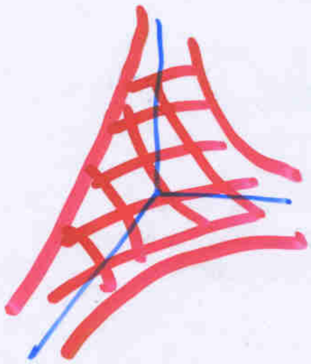
CRYSTAL FORMULA

$$\sum \exp \int_X k_0 \wedge \underset{\begin{array}{c}\downarrow \\ \int k^3\end{array}}{ch_2} + \hbar \int_X ch_3$$

REMARKS

1. $\hbar \to 0$ CLASSICAL GEOMETRY

(NOT THE ORIGINAL ONE)



LIMIT SHAPE - MIRROR OF X_3

SW CURVES

2. IN PARTICULAR, FOR

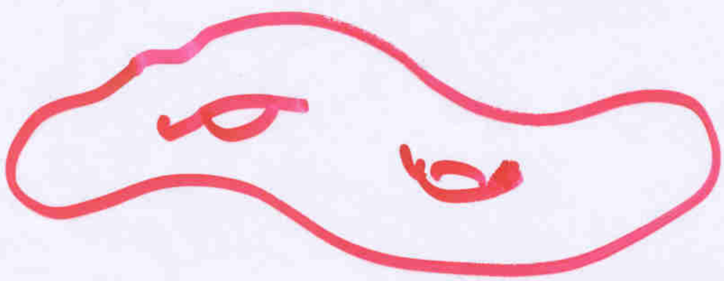
$$X_3 = \widetilde{\mathbb{C}^2/\Gamma}$$
$$\downarrow$$
$$\mathbb{P}^1$$

IN 4d $\mathcal{N}=2$ SYM WITH GAUGE GROUP A, D, E

CRYSTAL SUM
$\|$
4d INST. SUM

3. S-duality

VERY DIFFERENT PACKAGING OF INSTANTON + PERT. SUM



holomorphic curve in X_3

A model perturbation theory

B model-like D-instanton sum

$$\hbar = g_A^s = \frac{1}{g_B^s}$$

$$\sum_{g,m} e^{-m\,\mathrm{Area}(C)} N_{g,m}\, \hbar^{2g-2} = \log\left(\sum_J e^{-\hbar k^J} \tilde{N}_J \, e^{-\int_C \mathrm{ch}_2 J \wedge k_0}\right)$$

5. Duality continues to hold in more complicated backgrounds (non-CY)

SAY, $U(N+1)\times U(N)$ $\mathcal{N}=2$ gauge theory

AT SPECIAL POINT ON THE COULOMB BRANCH

(RELATED TO B model on $\mathbb{P}^1 \times \mathbb{C}^2$, S-dual to A model + top. gravity sector turned on)

$$\mathrm{Tr}_1\phi^k - \mathrm{Tr}_2\phi^k \leftrightarrow \sigma_{k-1}(\omega)$$

CONCLUSIONS

- WE SEE GLIMPSES OF FLUCTUATING SPACE-TIME GEOMETRY (FOAM) IN THE FULL ALL-GENUS TOPOLOGICAL STRING PARTITION FUNCTION
  - SO FAR ON TORIC MANIFOLDS (BUT THE CONJECTURE MAKES SENSE ON ANY CALABI-YAU)
- S-duality
- Kodaira-Spencer + self-dual tensor ?
- M-TOPOLOGICAL THEORY ?
  G_2 ? E_{10} ? $\mathbb{CP}^{3/4}$?