The Hagedorn/Deconfinement Phase Transition in Weakly Coupled Large N Gauge Theories

or

A tale of four phase transitions

Ofer Aharony

Strings, Gauge Fields and Duality

David Olive Fest

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Aharony, Marsano, Minwalla, Papadodimas and van Raamsdonk, hep-th/0310285 + to appear

Outline

- Review of confinement, the deconfinement and Hagedorn phase transitions and their relation in large N gauge theories.
- The partition function of free large N SU(N) Yang-Mills theories at finite volume as a unitary matrix model.
- Solution of the matrix model and the free Yang-Mills phase diagram.
- Generalization to weakly coupled theories and possible extrapolations to strong coupling.

Confinement and deconfinement

- Confinement : all finite-energy states carry no "color" charge, they are singlet "bound states" (glueballs, mesons, hadrons). Area law.
- An experimental property of the strong interactions.
- Seen on lattice (numerically), hard to prove theoretically, though there are various models (monopole condensation, Abelian dominance, string theory, …).
- Seems to be a property of a large class of non-Abelian asymptotically free gauge theories. We will focus on theories with adjoint fields.
- Related to strong coupling at low energies, so expected to disappear at high temperatures (compared to the QCD scale ~200 MeV) where the coupling is weak, in a deconfinement phase transition.

- Order parameters :
- Euclidean SU(N) gauge theory on $R^3 \times S^1$ has a Z_N symmetry related to "large gauge transformations" on the thermal circle. The simplest charged object is the Polyakov loop

$$\mathcal{P} = \frac{1}{N} tr(P \exp(-\oint A))$$

which can be related to the free energy of an external quark by $< \mathcal{P} >= \exp(-\beta F_q)$.

Thus, it vanishes in a confined phase, so it is an order parameter for deconfinement, and the deconfinement transition breaks the Z_N symmetry.

- Another order parameter appears in the large N limit : in confined phase the free energy $F(T) \sim \mathrm{O}(1)$ while in deconfined phase $F(T) \sim \mathrm{O}(N^2)$ so we can use as an order parameter $\lim_{N \to \infty} F_{SU(N)}(T) / N^2$, which also vanishes in the confined phase.

- The phase transition could theoretically be of either first order or second order, in QCD it seems to be of first order.

- Finite volume :

  We will be interested in gauge theories at finite volume because :

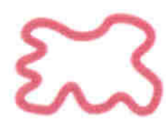
1) At small volumes compared to the QCD scale the theory is weakly coupled.
2) Can study phase transitions in conformal gauge theories which have AdS/CFT duals.

- At finite volume phase transitions are smoothed out. Correspondingly, the Polyakov loop always vanishes since a single quark violates Gauss' law.

  However, can still have phase transitions at large N. These can be characterized by our second order parameter, even though they are smoothed out for any finite N.

Hagedorn behaviour

- An asymptotically free theory with n degrees of freedom has a high-energy density of states $\rho(E) \sim \exp(n^{1/4} E^{3/4})$.
- On the other hand, a free string theory has at high energies $\rho(E) \sim \exp(E / M_s)$, because there are more and more oscillators available at high energies. Thus, free string theory has a maximal temperature $T_H \sim M_s$ called the Hagedorn temperature, where the partition function diverges (at finite volume) as $\ln(Z(T)) \propto \ln(T_H - T)$.



- At finite coupling the behaviour at high energies changes. It has been conjectured that the theory undergoes a Hagedorn phase transition near the Hagedorn temperature, into a new phase. In Euclidean space – a winding mode becomes tachyonic and condenses.

- In this transition the free energy goes from being $F(T) \sim O(1)$ to being $F(T) \sim O(g_s^{-2})$.
- Such a transition seems unlikely in Minkowski space, where the high-energy density of states grows even faster due to Schwarzschild black holes. However, it could potentially take place in string theories in curved space…

Deconfinement = Hagedorn ?

- We believe that many gauge theories are equivalent to string theories. This was originally motivated by confinement.
- This relation works best in 't Hooft's large N limit with fixed $\lambda = g_{YM}^2 N$, in which $g_s \sim 1/N$.

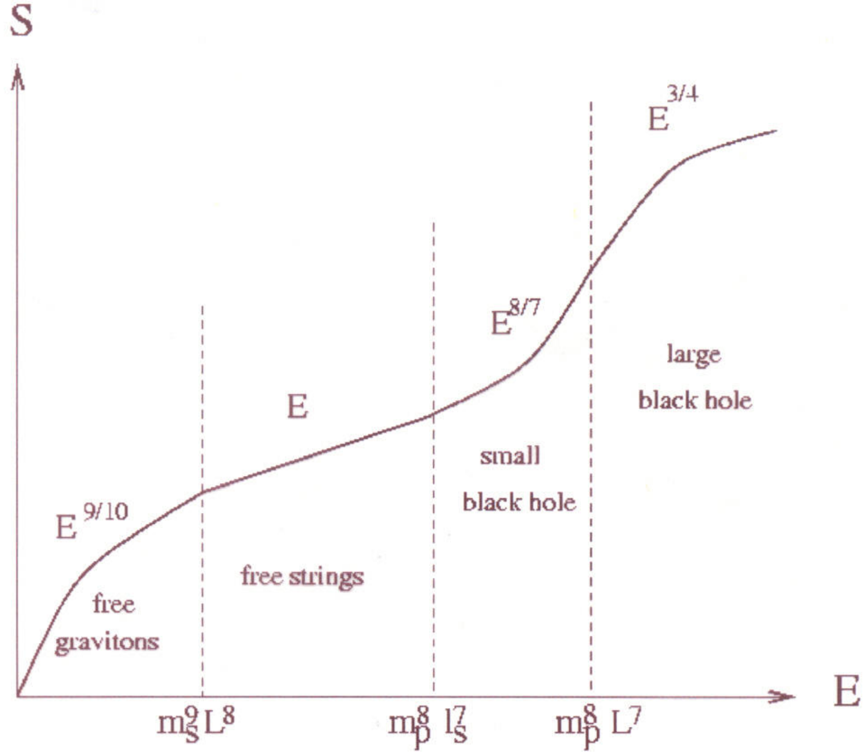
- In this limit the deconfinement transition is similar to the Hagedorn transition :

* Low-energy phase has stringy spectrum of "glueballs" on Regge trajectories (high-energy phase has gluons),
* Free energy jumps from order one to order $N^2 \sim g_s^{-2}$.

- So, it is natural to ask whether the two phase transitions are the same, or whether deconfinement happens below the Hagedorn temperature ?

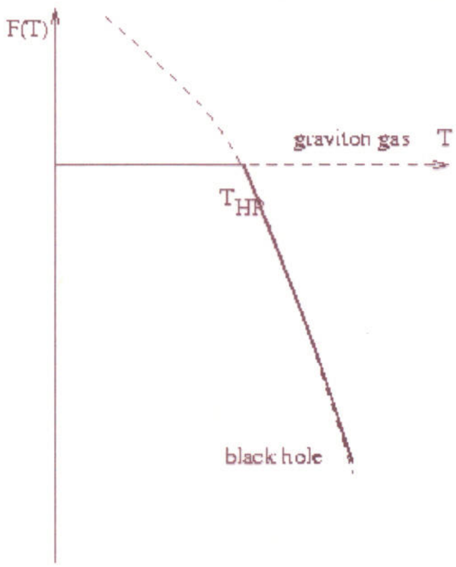
- The best understood relation is in the AdS/CFT correspondence between type IIB string theory on $AdS_5 \times S^5$ and the N=4 supersymmetric SU(N) Yang-Mills theory. In this case we can analyze the strongly coupled gauge theory compactified on a 3-sphere of radius R using string theory at small curvature, and we find :

- In microcanonical ensemble :



- In canonical ensemble have Hawking-Page transition at

$$T_{HP} = \frac{3}{2\pi R} = \frac{0.477465}{R} << T_H \propto \frac{\lambda^{1/4}}{R} :$$



- This is a deconfinement transition happening well below the Hagedorn temperature (censorship).
- What happens at weak coupling ? Can we understand transitions ?

The partition function of free Yang-Mills theory

(also derived by Sundborg, hep-th/9908001)

- Free Yang-Mills theory at finite volume is non-trivial, because we still have the Gauss law constraint – the total charge must vanish, all states are singlets of the gauge group.
- We will compute the partition function of the free Yang-Mills theory on a 3-sphere (with various matter fields) and see that it exhibits a Hagedorn behaviour of the density of states, and that there is a deconfinement transition occurring precisely at the Hagedorn temperature.
- The free theory is conformal, so we can count the states on a 3-sphere by counting operators :

$$Z(T) = \sum_{states} e^{-\beta E} = \sum_{operators} e^{-\beta \Delta / R} \equiv \sum_{operators} x^{\Delta}$$

- In a theory with only adjoint fields Φ_i , the gauge-invariant operators take the form $tr(\partial\cdots\partial\Phi_{i1}\partial\cdots\partial\Phi_{i2}\partial\cdots\partial\Phi_{i3}\cdots\partial\cdots\partial\Phi_{in})$. At large N, even with only two quantum mechanical modes A and B of dimension one, this gives a Hagedorn density of states of the form $tr(A\cdots AB\cdots BA\cdots AB\cdots B\cdots)$, since the number of operators of dimension Δ obeys $2^\Delta/\Delta \le n(\Delta) \le 2^\Delta$ due to the different inequivalent orderings of A and B. With more fields and with derivatives, the growth is even faster, but still exponential.
- Denote the single-particle density of states by z(x). For example, for a single scalar field of dimension one, the operators are all derivatives of Φ modulu the equation of motion $\partial^2\Phi = 0$ (=spherical harmonics in symmetric traceless SO(4) representations), giving
$$z(x) = \frac{x+x^2}{(1-x)^3}$$
and similarly for other fields.

- A simple combinatorical counting of the different trace operators gives

$$Z_{ST}(x) = -\sum_{q=1}^{\infty} \frac{\varphi(q)}{q} \ln(1 - z_B(x^q) + (-1)^q z_F(x^q)),$$

for single-trace operators, and

$$Z(x) = \prod_{k=1}^{\infty} \frac{1}{1 - z_B(x^k) + (-1)^k z_F(x^k)}$$

for the full theory.

- These expressions diverge at the temperature corresponding to $x_H = e^{-1/T_H R}$ given by $z_B(x_H) + z_F(x_H) = 1$. The divergence is precisely as in free string theory, $Z_{ST}(T) \sim \ln(T_H - T)$,

$$Z(T) \sim 1/(T_H - T),$$

corresponding to a density of states $\rho(E) \sim \exp(E/T_H)$. So, free Yang-Mills theories have a Hagedorn spectrum !

- In the free N=4 SYM theory (for example) we find

$$T_H = -1/R\ln(7 - 4\sqrt{3}) = 0.379663/R.$$

- In the counting we performed above we ignored the relations between traces and multi-traces when we have more than N fields in a trace, so it was only exact for energies small compared to N. Above this scale, because of trace dependences, the density of states is smaller.
- We have two ways to compute the exact partition function :
  - We can sum over all states in the Fock space of our particles, and then impose a projection onto singlets. This is most easily done by associating to a particle in some representation R a factor of the character $\chi_R(U)$, and then an integration over the gauge group imposes the projection onto singlets. We end up with the exact formula

$$Z(x) = \int [dU] \exp\{\sum_R \sum_{m=1}^{\infty} \frac{1}{m} [z_B^R(x^m) - (-1)^m z_F^R(x^m)] \chi_R(U^m)\}$$

Easy to generalize to other spaces.

- The same formula may be derived by computing the one-loop path integral of the gauge theory, in the gauge

$$\partial_i A^i = 0, \partial_t \alpha = 0, \qquad \alpha \equiv \frac{1}{vol(S^3)} \int_{S^3} A_0$$

The mode α is the only zero mode of the gauge theory on the three-sphere, so we can integrate out all other modes and write an effective action for $U \equiv e^{i\beta\alpha}$, which is the untraced Polyakov loop. This effective action turns out to be exactly the same as our previous expression

$$Z(x) = \int [dU] \exp\{\sum_R \sum_{m=1}^{\infty} \frac{1}{m} [z_B^R(x^m) - (-1)^m z_F^R(x^m)] \chi_R(U^m)\}$$

but the new derivation allows us also to compute averages of the Polyakov loop by inserting factors of tr(U) into the integral.

- In any case, we end up with an exact unitary matrix model for the free SU(N) Yang-Mills theory.

Solution of the unitary matrix model

- As usual, to solve the matrix model we change variables to the eigenvalues of U,

$$\{e^{i\alpha_i}\}; i=1,\cdots,N; -\pi \le \alpha_i \le \pi,$$

and to the eigenvalue distribution $\rho(\alpha)$ which becomes continuous in the large N limit.

- When all fields are in the adjoint representation, we have

$$Z(x) = \int [dU] \exp\{\sum_R \sum_{m=1}^{\infty} \frac{1}{m}[z_B(x^m) - (-1)^m z_F(x^m)] \cdot tr(U^m) tr((U^+)^m)\} =$$

$$= \int [d\alpha_i] \exp\{-\sum_{i<j} V(\alpha_i - \alpha_j)\}$$

$$V(\theta) = -\ln(\sin(\theta/2)) - \sum_{n=1}^{\infty} \frac{1}{n}[z_B(x^n) - (-1)^n z_F(x^n)] \cos(n\theta)$$

with a repulsive force coming from the measure, and an attractive force growing with the temperature.

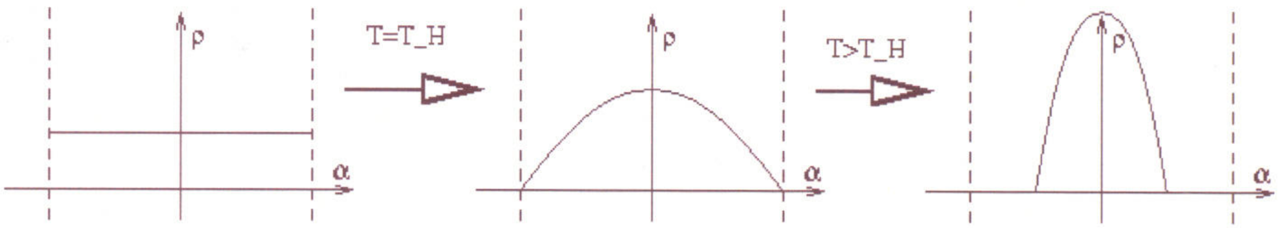
- This matrix model can be solved exactly, by similar methods to those used to study the Gross-Witten phase transition of two dimensional lattice gauge theories.
- The analysis of the low-temperature phase is simplest in the (constrained) variables

$$\rho_n \equiv \int \rho(\alpha) \exp(in\alpha) d\alpha = tr(U^n)/N,$$

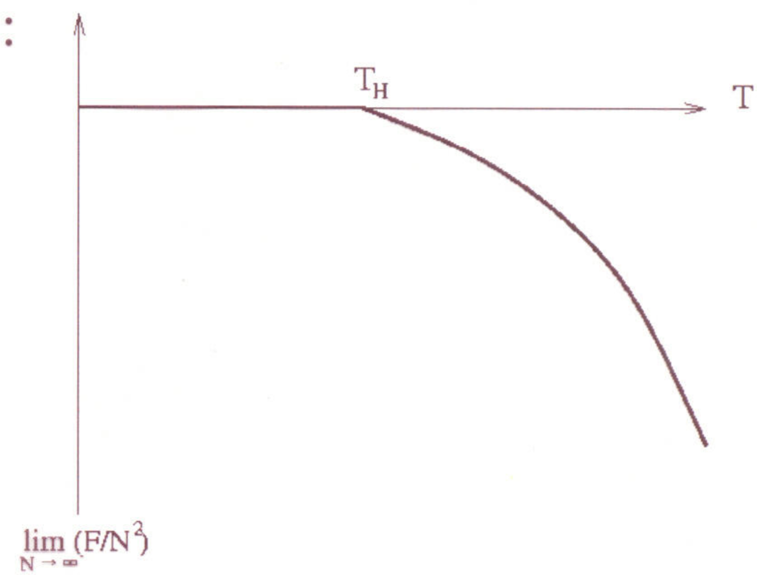
in which the "effective action" becomes

$$S = N^2 \sum_{n=1}^{\infty} |\rho_n|^2 \frac{1}{n}[1 - z_B(x^n) + (-1)^n z_F(x^n)].$$

- At low temperatures all coefficients are positive, the large N saddle point is a constant distribution $\rho_n = 0$, and the integration gives precisely our Z(x) found above.
- At the Hagedorn temperature, ρ_1 (which is precisely the Polyakov loop) becomes tachyonic and condenses, and at higher temperatures the higher modes of the distribution condense as well :



- We find a (weakly) first-order phase transition, which is a deconfinement transition according to our two order parameters ! ($<\rho_1>=0$ in both phases but for different reasons). The transition happens precisely at the Hagedorn temperature, and for a given field content we can compute all thermodynamical quantities (though the high-temperature expressions are complicated). For example, the free energy :



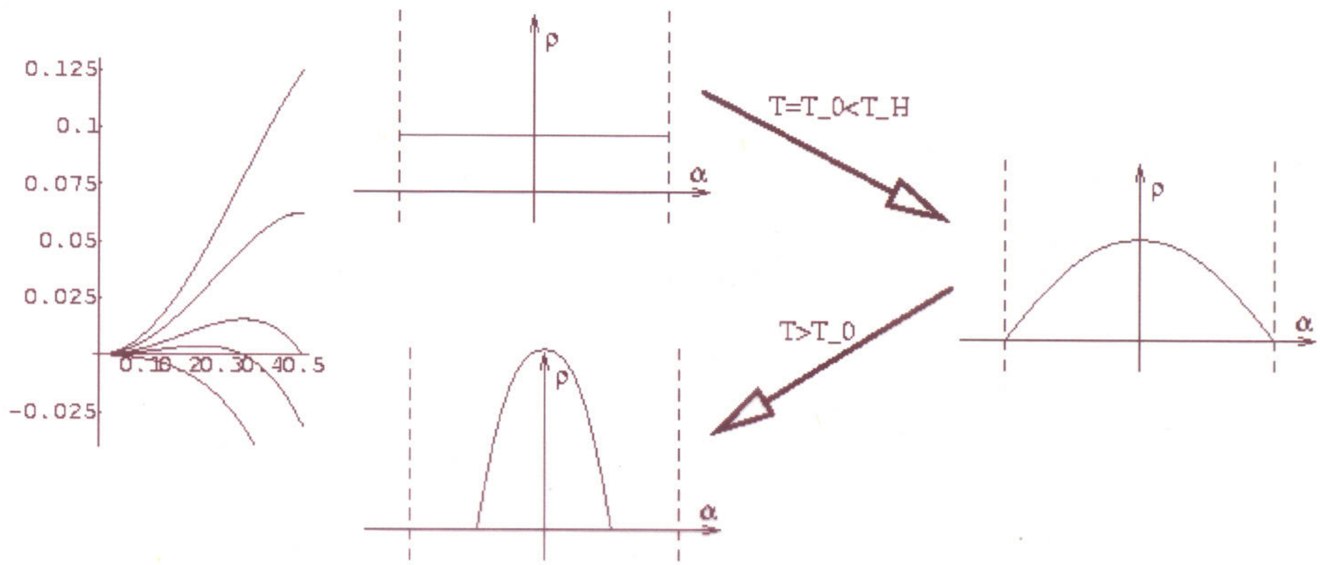
Weak coupling behaviour

- At weak coupling our one-loop action gets perturbative corrections. At k-loop order these take the form $\lambda^{k-1} tr(U^{n_1}) tr(U^{n_2}) \cdots tr(U^{n_{k+1}})$ with computable coefficients.
- The behaviour near the phase transition is dominated by the light mode $\rho_1 \sim tr(U)/N$ and the effective Lagrangian for this mode near the transition takes the form

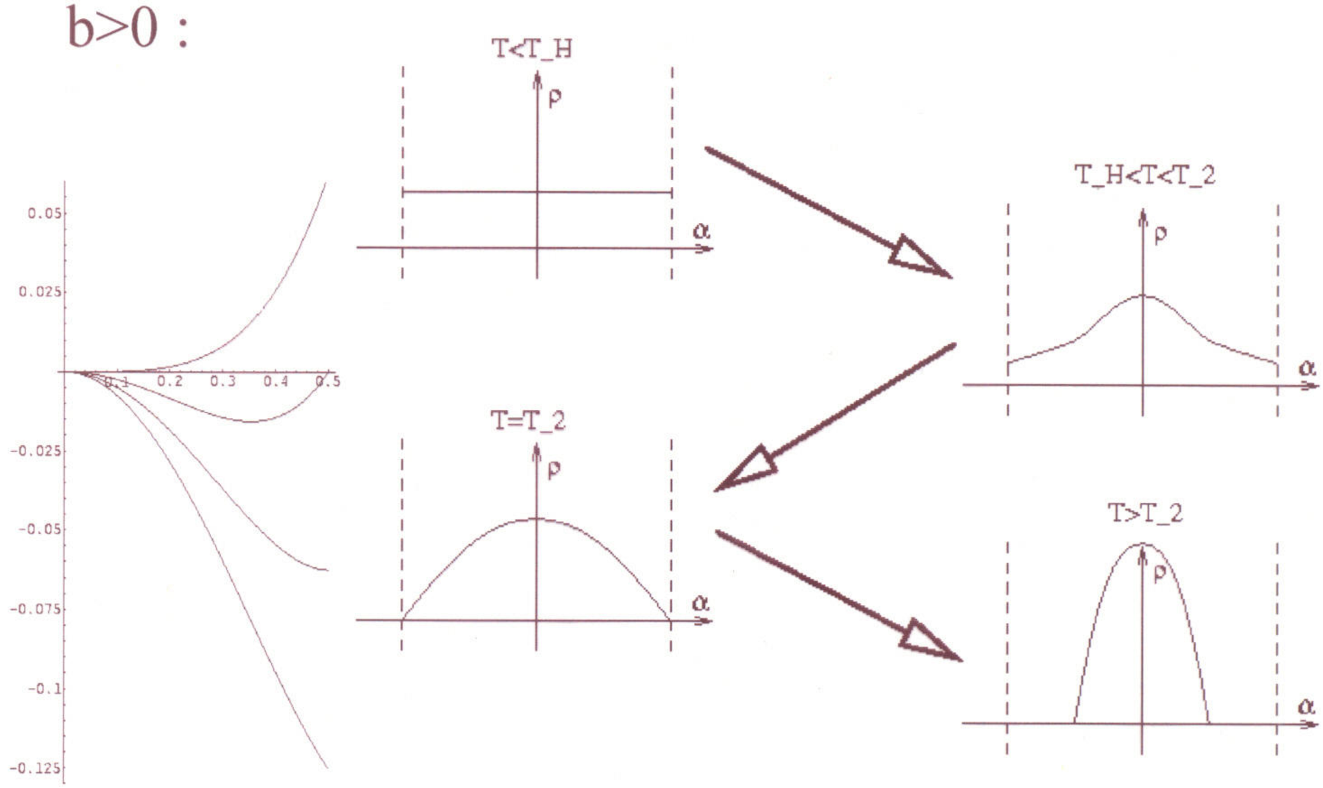
$$S_{eff} = N^2 [a(T_H - T) |\rho_1|^2 + b\lambda^2 |\rho_1|^4]$$

  where a and b can be computed from two-loop and three-loop diagrams in the theory.
- As usual, the order of the phase transition depends on the sign of b. When $b<0$ it is first order, and occurs below the Hagedorn temperature. When $b>0$ there are two continuous transitions, the first at T_H and the second at a higher temperature.

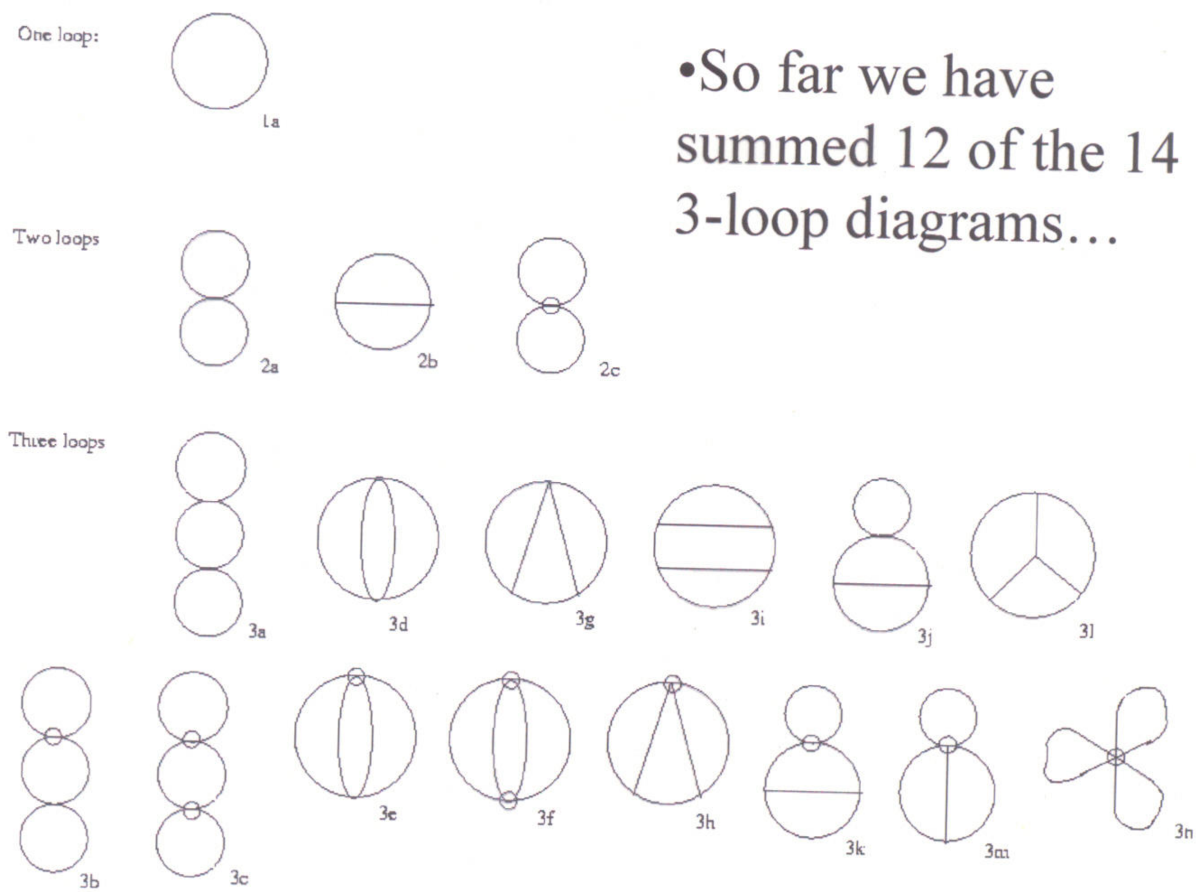
b<0 :


0.125
0.1
0.075
0.05
0.025
-0.025
ρ
α
T=T_0<T_H
T>T_0

b>0 :


0.05
0.025
0.1
0.2
0.3
0.4
0.5
-0.025
-0.05
-0.075
-0.1
-0.125
T<T_H
ρ
α
T_H<T<T_2
T=T_2
T>T_2

- Unfortunately we must sum all 3-loop vacuum diagrams which contribute to this term in order to know the order of the deconfining phase transition, and whether it occurs at or below the Hagedorn temperature.
- In the case of pure Yang-Mills theory, in a convenient choice of gauge (and after integrating out some fields), the following diagrams contribute :



•So far we have summed 12 of the 14 3-loop diagrams…

Summary

- Weakly coupled large N gauge theories on compact spaces :

* Exhibit a Hagedorn spectrum (for finite energies in the large N limit),

* Have a deconfinement phase transition at at a temperature inversely related to the size of the space.

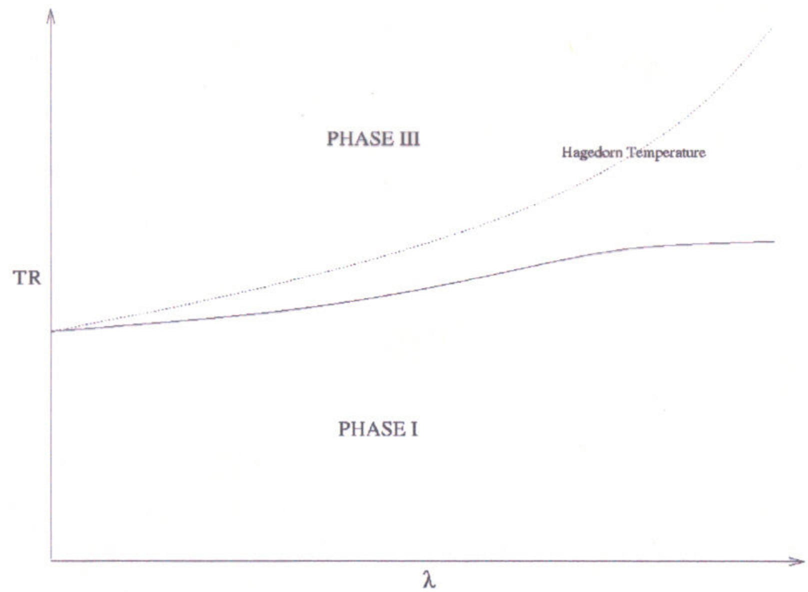
- The deconfinement transition is either :

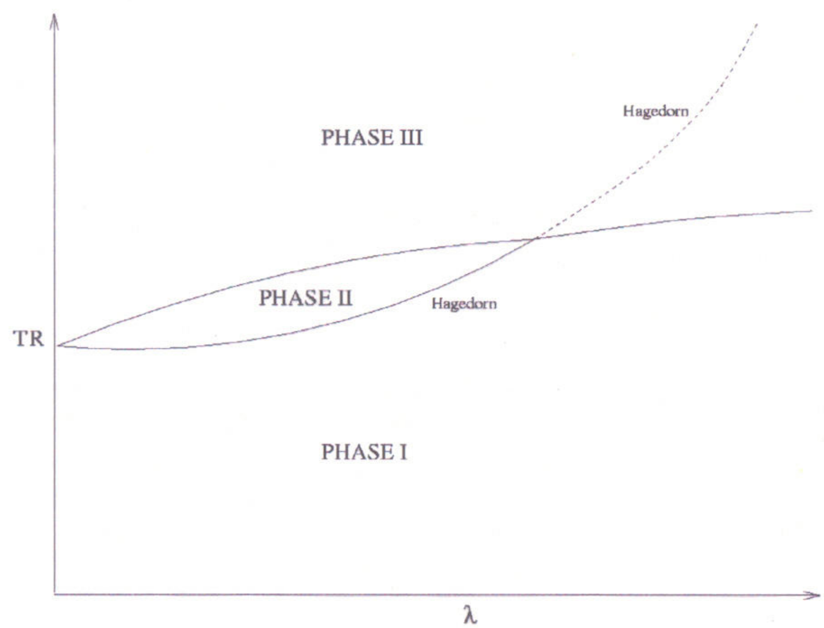
* A first order transition below the Hagedorn temperature, or

* A second order transition at the Hagedorn temperature, followed by another third order phase transition.

- The properties of the transition (and of the stringy spectrum) can be computed in perturbation theory, through a unitary matrix model.

- These results apply (for instance) to pure Yang-Mills on a very small 3-sphere, or to the N=4 SYM theory at weak coupling. These theories have a first order transition at strong coupling, so their phase diagram could perhaps be one of :





- Future directions :
- Compute the value of b (=the order of the transition) in several interesting cases (pure YM, N=4 SYM). When adding a large number of fundamentals the transition is always smooth since there is a potential for single eigenvalues.
- Try to understand the free string theory corresponding to the free large N (supersymmetric) Yang-Mills theory. We know the exact spectrum of single-string and multi-string states ! Zero size AdS ? Not tensionless strings…
- Understand better the “intermediate phase” which sometimes appears. Is this related to “small black holes” ?
- Generalize to cases with zero modes (torus).
- Study integrability properties of the large N limit. Can study anomalous dimensions.

- Can we extrapolate from weak coupling to strong coupling, and get a good model for "realistic" deconfinement ? Are the regimes continuously related ?

"We have dared to present our speculations, as yet unproved, because we feel that they do succeed in relating previously uncorrelated facts and would be, if true, of some importance for the further unveiling of the secrets of quantum gauge field theories."

(Montonen+Olive, 1977)