Electric-magnetic duality and all that

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diofest

-
- a simple duality
- two papers and a talk
- $M + \Phi$
- E - diagrams

S-matrix $\longleftrightarrow$ CFT

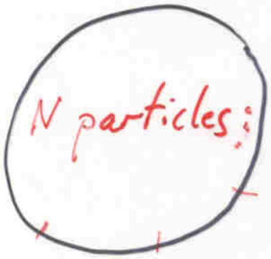
Consider an integrable QFT_2 on a circle of circumference L.

S-matrix $\infty \xleftarrow{\quad L \quad} 0$ CFT

Bethe ansatz for the wave function of two particles of rapidities θ_1, θ_2.

$$\psi(x_1, x_2) = \begin{cases} e^{ik_1x_1 + ik_2x_2} & \text{for } x_1 \ll x_2 \\ \underbrace{e^{i\delta(\theta_1-\theta_2)}}_{\text{S-matrix}} e^{ik_1x_1+ik_2x_2} & x_1 \gg x_2 \end{cases}$$

for $\theta_1 > \theta_2$



$$\sum_{j \neq i} \delta(\theta_i - \theta_j) + k_i L = 2\pi n_i \qquad i = 1,..,N$$

For $L \to 0$, left- and right-movers decouple.
Energy of right movers $\approx$ momentum of right movers.

Momenta are quantized and easy to calculate.
$\Longrightarrow$ CFT energies are easy to calculate ?!

Two particles:

$$\delta(\theta_1-\theta_2) + k_1 L = 2\pi n_1$$
$$\delta(\theta_2-\theta_1) + k_2 L = 2\pi n_2$$

$$\delta(\theta_1-\theta_2) + \delta(\theta_2-\theta_1) = -2\pi A \qquad A \in \mathbb{Z}$$

N particles:

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \delta(\theta_i-\theta_j) + L\sum_1^N k_i = 2\pi \sum_1^N n_i$$

$$L\sum k_i = 2\pi \sum n_i + 2\pi \binom{N}{2} A$$

assume Bose statistics, $n_i \geq 6$

$$\mathrm{tr}_{\mathcal{H}}\, q^{L_0} = \sum_{N=0}^{\infty} \frac{q^{\binom{N}{2}A + 6N + 6}}{(q)_N} \Rightarrow \text{central charge, } \ldots$$

Application for non-unitary models:

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{H}_\pm$ graded by L_0, metric positive definite on $\mathcal{H}_+$, negative on $\mathcal{H}_-$.

For single particle states in $\mathcal{H}_-$ (as in the (2,5) minimal model)

$$\mathrm{tr}_{\mathcal{H}_+}\, q^{L_0} - \mathrm{tr}_{\mathcal{H}_-}\, q^{L_0} = \sum_{N=0}^{\infty} (-)^N \frac{q^{\binom{N}{2}A + 6N + 6}}{(q)_N}$$

(2,5) model: $A = 2$

Gauge theory and magnetic charge

P. Goddard, J. Nuyts, D.I. Olive, Dec. 1976

gauge field, Higgs field φ

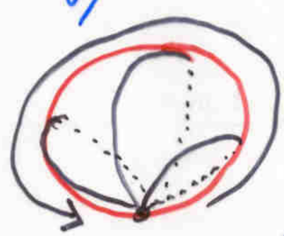
gauge group G broken to $H \subset G$

$\varphi(S^2_\infty)$ in $\underbrace{\text{single orbit}}_{G/H}$ of G

$$\pi_2(G) \longrightarrow \pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G)$$

$\overset{\shortparallel}{0}$

$$\underbrace{\pi_2(G/H)}_{\text{Higgs topology}} \simeq \underbrace{\pi_1(H)_G}_{\text{gauge field topology}}$$



Replace H by maximal torus $T = \mathbb{R}^r/\Lambda$

$\pi_1(T) = \Lambda$ monopoles (modulo Weyl group)

Representations of $H \Rightarrow$ repr. of T

$\Lambda^\vee$ electric charges (modulo Weyl group)

Conjecture: There is a magnetic gauge group $H^\vee$ with weight lattice Λ.

Kernel of $\pi_1(T) \longrightarrow \pi_1(H)$: unstable monopoles.

Preference for representations for which the weights form a single orbit of the Weyl group (minimal representations).

Words in quotation marks:

"Weyl group", "Weyl reflection"

"section" (ref. Wu, Yang)

"maximal torus"

"centre"

"universal covering group"

"weight", "root"

Not in quotation marks:

Cartan metric, Killing form

Speculations:

- True symmetry group is $H \otimes H^\vee$.
- analogy to Thirring model ~ Sine-Gordon theory

  topological charge, current $\longleftrightarrow$ Noether charge, current

- monopoles ~ quarks (confined)

  $H = SU(3)_{colour}$

  $H^\vee = SU(3)/Z_3$ electroweak (citation H. Fritzsch + P. Minkowski)

Magnetic monopoles as gauge particles

C. Montonen, D. Olive, Oct. 1977

- $qg \in 2\pi\hbar\mathbb{Z}$ (Dirac)

  $(q,g) \longmapsto (g,-q)$

- BPS mass formula $M \sim \sqrt{q^2+g^2}$
- forces in BPS limit (Manton)
- moduli space: charge lattices / $SO(2)$

[now: $SL(2,\mathbb{Z}) \backslash SL(2,\mathbb{R}) / SO(2)$]

Speculation: Spin $= 1\hbar$.

Higgs: Goldstone boson of $SO(2) \subset SL(2,\mathbb{R})$.

E. Witten and D. Olive (1978):

BPS mass formula as consequence of supersymmetry.

Mass $\sim$ central charge

Magnetic monopoles and electromagnetic duality conjectures.

D. Olive, talk at Monopole Meeting, Trieste Dec. 1981 (no citations)

- are there local monopole fields?
  (cf. ϕ^4 solitons versus $\cos\beta\phi$ solitons)

- susy $N=2$ or $N=4 \Rightarrow$ spin 1

"The [fermionic] zero modes .. indicate what sort of multiplet the monopoles lie in. The fermionic zero modes are represented by Dirac gamma matrices indicative of the varying spin content".

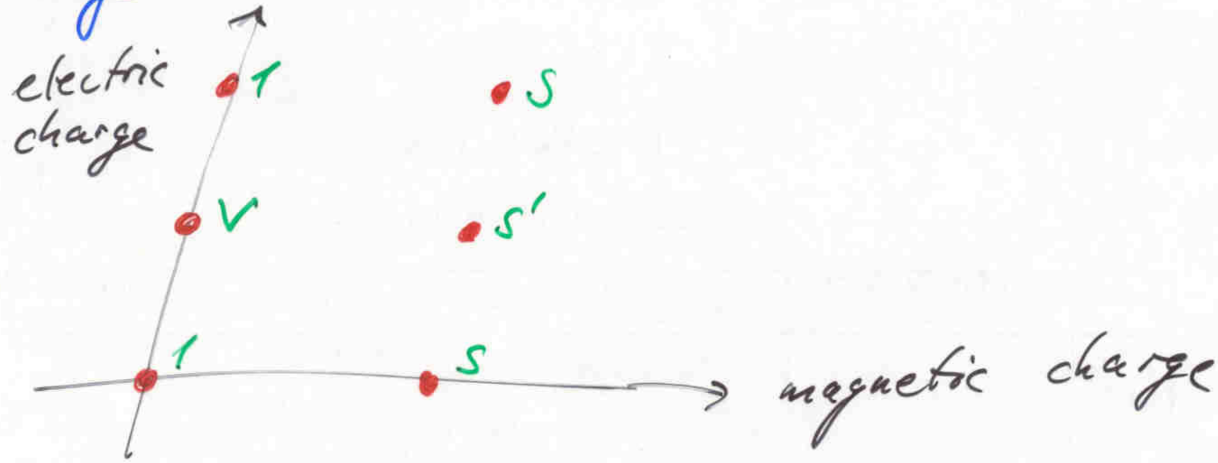
- renormalization of $q \cdot g$

$$q(\Lambda)\cdot g(\Lambda) \in 2\pi\hbar\mathbb{Z} \Longrightarrow \text{no renormalization, } \beta = 0$$
$$q(\Lambda) \sim g(\Lambda)$$

"... the $N=4$ supersymmetric extension .. seems to be the only theory for which the Montonen Olive conjecture could be valid and .. its validity is reasonably plausible at least for $G = SU(2)$".

Why did we overlook $N_f = 4$?

charge lattice for $N_f = 4$ (massless)



$SO(2N_f)$ representations:

v vector

s, s' half-spinors

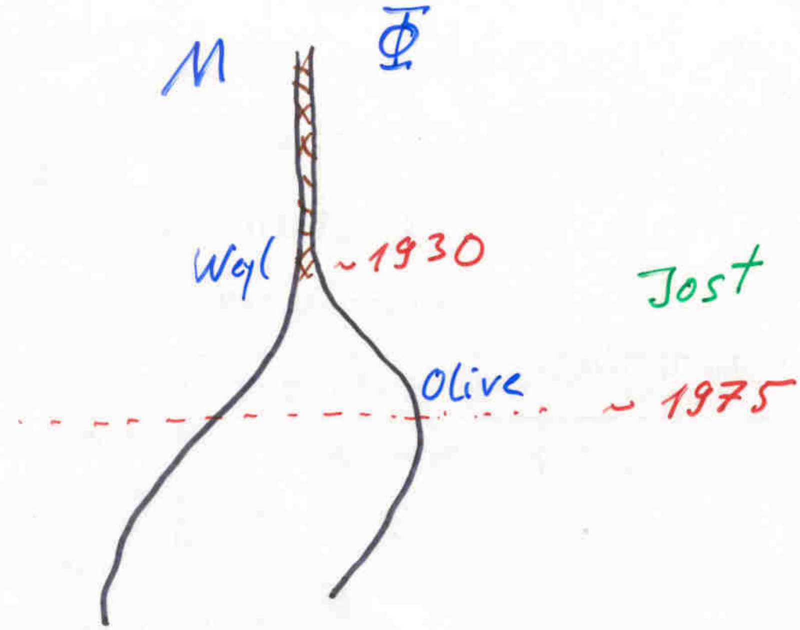
Full $SL(2,\mathbb{Z})$ electric-magnetic duality by using $SO(8)$ triality.

For $N_f = 0, 2, 3$:

finite index subgroup of $SL(2,\mathbb{Z})$

$N_f = 1$: Argyres-Douglas singularity in moduli space $\Longrightarrow$ double cover.

"in the $N=2$ theory the monopoles lie in a multiplet not containing spin $\frac{1}{2}$."



QFT + geometry

solitons

19th century

Skyrme

Coleman

't Hooft, Polyakov

gauge theory

Weyl

Dirac

Yang

susy

GSO

Witten, Olive

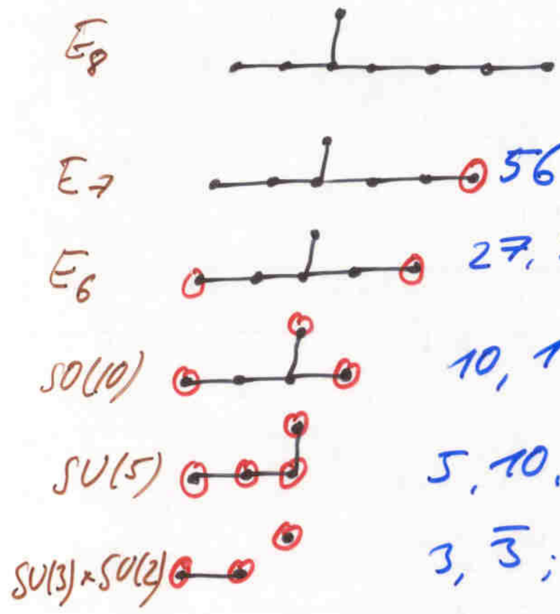
'76 Goddard, Nuyts, Olive

'77 Montonen, "

duality

'94 Sen

Seiberg, Witten



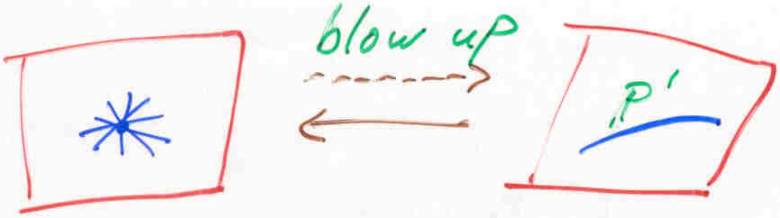
"... the groups E_8 to E_3 (with possible $U(1)$ factors) are candidates for grand unified groups of the fundamental interactions with the fundamental fermions assigned to the minimal representation".

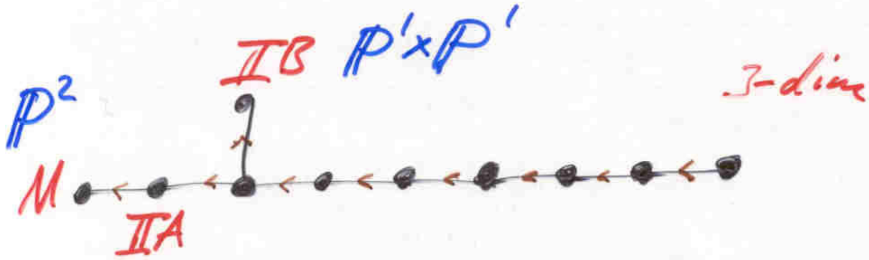
"Mathematically there is nothing very special about the groups $SU(3)\times SU(2)$, $SU(5)$ or $SO(10)$ per se, but they do have a special significance as the lowest members of the sequence of E_n groups"

An E-series also appears in the torus compactification of 11-dim supergravity (E. Cremmer, B. Julia). The $SL(2,\mathbb{R})$ electric-magnetic duality group of the sourceless Maxwell equations can arise in this way.

E_{11-d} for compactification to d dimensions.

Relation to Del Pezzo surfaces (Iqbal, Neitzke, Vafa).





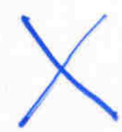
10 Del Pezzo surfaces

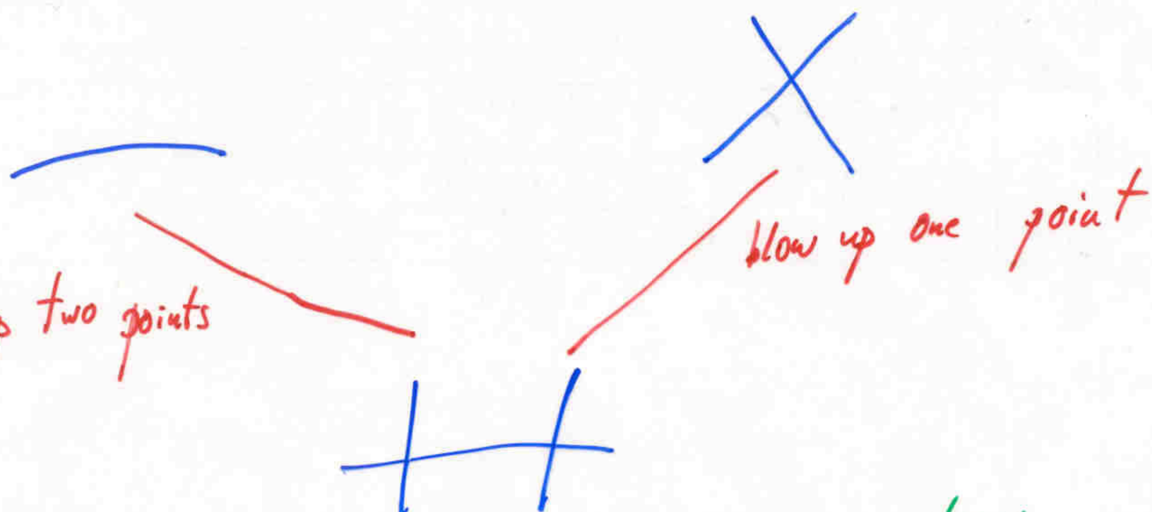
$\mathbb{P}^2$: Affine $\mathbb{C}^2 + \mathbb{P}^1$ at ∞.

$\mathbb{P}^1\times\mathbb{P}^1$: $xy = uv$ $\quad (x,y,u,v)\in\mathbb{P}^3$

For $x=1$: affine $\mathbb{C}^2 \ni \{u,v\}$

$x=0$: two lines ($u=0$ or $v=0$).





Most elementary algebraic geometry yields E-diagrams.