Real-time dynamics of heavy quark systems at high temperature

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Y.A., A.Rothkopf, PRD85(2012),105011 (arXiv:1110.1203[hep-ph])
Y.A., in preparation

Recent progress in heavy quarkonium

- From complex to stochastic potential
- Schroedinger equation $i\frac{\partial}{\partial t}\Psi(X,t)=\left(2M-\frac{\nabla_1{}^2+\nabla_2{}^2}{2M}+V(r)\right)\Psi(X,t),\ X=(x_1,x_2)$
- Complex potential $V(r)=V_{\mathrm{Re}}(r)+iV_{\mathrm{Im}}(r)$

  pQCD (Laine, et al. '07)
  lattice (Rothkopf, et al. '11)
- Stochastic potential

$$\Psi(X,t)=U_{\Theta}^{(X)}(\Delta t\,|\,0)\Psi(X,0),\ U_{\Theta}^{(X)}(\Delta t\,|\,0)=\exp\left[-\frac{i}{\hbar}\Delta t\{H(X)+\Theta(X,t)\}\right],$$

$$\langle\Theta(X,t)\rangle=0,\ \ \langle\Theta(X,t)\Theta(X',t')\rangle=\hbar\Gamma(X,X')\delta_{tt'}/\Delta t,$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi(X,t)=\left\{H(X)-\underbrace{\boxed{\frac{i}{2}\Gamma(X,X)}}_{\text{Complex potential}}+\underbrace{\boxed{\Xi(X,t)}}_{\text{noise}}\right\}\Psi(X,t).$$

$$\Xi(X,t)\equiv\Theta(X,t)-\frac{i\Delta t}{2\hbar}\left\{\Theta(X,t)^2-\langle\Theta(X,t)^2\rangle\right\},\ \langle\Xi(X,t)\rangle=0$$

Akamatsu, Rothkopf '12

Recent progress in heavy quarkonium

- Corresponding classical system
- Stochastic Hamiltonian = Brownian motion w/o friction

⬇

- Diffusion equation in momentum space

⬇

$$\left(\partial_t - D\nabla_p^2\right) f(p) = 0$$

c.f. Fokker-Planck equation $\left(\partial_t - \nabla_p\left(\Gamma p + D\nabla_p\right)\right) f(p) = 0$

- Stationary solution = uniform in momentum space

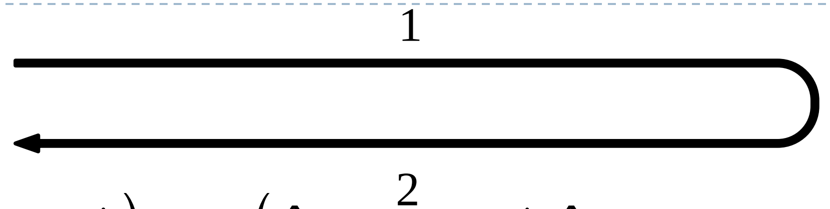
Without friction, energy rises forever …

(Quantum version: Ehrenfest relation)

How can we describe friction quantum mechanically?

Path integral on CTP

- Closed-time path



$$Z[j_1, j_2] = \mathrm{Tr}\left(\hat{U}(\infty,0;j_1)\hat{\rho}\hat{U}(\infty,0;j_2)^\dagger\right) = \mathrm{Tr}\left(\hat{U}(\infty,0;j_2)^\dagger\hat{U}(\infty,0;j_1)\hat{\rho}\right)$$

$$\sim \int D\varphi_{1,2}\,\rho[\varphi_1^{\mathrm{ini}},\varphi_2^{\mathrm{ini}}]\exp\left(iS(\varphi_1) - iS(\varphi_2) + i\int j_1\varphi_1 - i\int j_2\varphi_2\right)$$

$$\prod_i \frac{\delta}{\delta j_i(x_i)} \ln Z[j_1, j_2]\bigg|_{j_{1,2}=0} \propto \left\langle \mathrm{T}_{\mathrm{C}} \prod_i \hat{\varphi}_i(x_i)\right\rangle_{\mathrm{conn}}$$

$$\Rightarrow \begin{cases} \dfrac{\delta^2}{\delta j_1(x_1)\delta j_1(x_2)} \ln Z[j_1, j_2]\bigg|_{j_{1,2}=0} \propto \left\langle \mathrm{T}\hat{\varphi}(x_1)\hat{\varphi}(x_2)\right\rangle_{\mathrm{conn}} = G^F(x_1, x_2) \\ \dfrac{\delta^2}{\delta j_1(x_1)\delta j_2(x_2)} \ln Z[j_1, j_2]\bigg|_{j_{1,2}=0} \propto \left\langle \hat{\varphi}(x_2)\hat{\varphi}(x_1)\right\rangle_{\mathrm{conn}} = G^<(x_1, x_2) \\ \cdots \end{cases}$$

Path integral on CTP

- Application to QCD (qA + ψ)

$$Z[\eta_1,\eta_2] \sim \int D[\psi \underline{qA_{1,2}}]\rho[\psi qA_1^{\mathrm{ini}},\psi qA_2^{\mathrm{ini}}]\exp\left(iS_1^{\psi}-iS_2^{\psi}+i\int\psi_1\eta_1-i\int\psi_2\eta_2\right)$$
$$\underline{\times\exp\left(iS_1^{qA}-iS_2^{qA}+i\int j_1A_1-i\int j_2A_2\right)}$$
$$\rho=\rho_{\mathrm{eq}}^{E}\otimes\rho^{S}\rightarrow\rho[\psi qA_1^{\mathrm{ini}},\psi qA_2^{\mathrm{ini}}]=\underline{\rho_{\mathrm{eq}}^{E}[qA_1^{\mathrm{ini}},qA_2^{\mathrm{ini}}]}\cdot\rho^{S}[\psi_1^{\mathrm{ini}},\psi_2^{\mathrm{ini}}]$$

E:Environment
S:System

* Ghost and FP term omitted for simplicity

Influence functional

$$=Z^{qA}[j_1,j_2]$$
$$=\exp\left(-1/2\int j_1G^{F}j_1+j_2G^{\tilde{F}}j_2-j_1G^{>}j_2-j_2G^{<}j_1\right)$$
$$\times\exp\left(\int G_3\,jjj+G_4\,jjjj+\cdots\right)$$

Path integral on CTP

- Approximations
- Expansion in j up to 2nd order
  → An approximation best satisfied in weak-coupling
  → However, let us also focus on the *structure* of dynamics.

- Non-relativistic limit (Foldy-Wouthuysen transformation)

$$\psi = (Q, Q_c^\dagger),$$

$$S_\psi = Q^\dagger\left(i\partial_0 - M + \nabla^2/2M\right)Q + Q_c^\dagger\left(i\partial_0 - M + \nabla^2/2M\right)Q_c + O\left(p^4/M^3\right)$$

$$j_0 = Q^\dagger Q - Q_c^\dagger Q_c$$

$$\vec{j} = Q^\dagger\left(\vec{\nabla}/iM\right)Q - Q_c^\dagger\left(\vec{\nabla}/iM\right)Q_c + O\left(p^3/M^3\right)$$ → Truncate (NR approx.)

$$+ Q^\dagger\vec{\sigma}Q_c^\dagger + Q_c\vec{\sigma}Q + O\left(p^2/M^2\right)$$ → Neglect (quenched approx.)

Path integral on CTP

- Instantaneous approx. (~ladder approx. in Bethe-Salpeter equation)

  Time scale for j is slow ⇔ Time scale for G is fast

  → Satisfied best in weak-coupling (cross ladder = higher order)

e.g.

$$G^F(x,y) \approx \overline{G}^F(\vec{x},\vec{y})\delta(t_x - t_y) \quad \Leftrightarrow \quad \overline{G}^F(\vec{x},\vec{y}) = \int_{-\infty}^{\infty} dt_x G^F(x,y)$$

$$\int_{x,y} j_1(x) G^F(x,y) j_1(y) \approx \int_{t,\vec{x},\vec{y}} j_1(\vec{x},t)\overline{G}^F(\vec{x},\vec{y}) j_1(\vec{y},t)$$

$$\overline{G}^F(\vec{x},\vec{y}) = -i\overline{G}^R(\vec{x},\vec{y}) + \overline{G}^<(\vec{x},\vec{y})$$

$$\overline{G}^{\tilde{F}}(\vec{x},\vec{y}) = i\overline{G}^R(\vec{x},\vec{y}) + \overline{G}^>(\vec{x},\vec{y})$$

$$\Rightarrow \boxed{V(\vec{x},\vec{y}) \equiv -\left\{\overline{G}^R(\vec{x},\vec{y}) + i\overline{G}^>(\vec{x},\vec{y})\right\}} \quad \text{Complex potential}$$

$$\boxed{D(\vec{x},\vec{y}) \equiv -\overline{G}^>(\vec{x},\vec{y})} \quad \text{Dissipation}$$

Path integral on CTP

- Influence functional

j: NR approx.
V,D: Simultaneous (ladder)
Up to j^2: 2-body interaction

$$Z^{qA}[j_1, j_2]$$
$$\approx \exp\left(-i/2\int_{t,\vec{x},\vec{y}} j_1 V j_1 - j_2 V^* j_2 - iD(j_1 j_2 + j_2 j_1)\right)\times\cdots$$

- (Bare) Green functions

$$\boxed{\overline{G}^{R}_{ab,\mu\nu}(\vec{x},\vec{y}) = g^2\int_0^\beta d\tau\left\langle \hat{A}_{a\mu}(\vec{x},-i\tau)\hat{A}_{b\nu}(\vec{y},0)\right\rangle}$$

=Real potential

$$\boxed{\overline{G}^{>}_{ab,\mu\nu}(\vec{x},\vec{y}) = \lim_{\omega\to 0}\frac{T}{\omega}\sigma_{ab,\mu\nu}(\omega,\vec{x},\vec{y})}$$

=Imaginary potential and dissipation

$$\sigma_{ab,\mu\nu}(\omega,\vec{x},\vec{y}) = g^2\int dt\, \mathrm{e}^{i\omega t}\left\langle\left[\hat{A}_{a\mu}(\vec{x},t), \hat{A}_{b\nu}(\vec{y},0)\right]\right\rangle$$

Effective Hamiltonian and renormalization

- Why Hamiltonian and renormalization necessary?

  In order to discuss density matrix.

  (It will be clear in the next section)

- 4-fermi interaction for $\psi_{1,2}$ on a single time axis

$$Z[\eta_1,\eta_2] \sim \int D[\psi_{1,2}]\,\rho^S[\psi_1^{\mathrm{ini}},\psi_2^{\mathrm{ini}}]$$

$$\times \exp\left(iS_1^{\psi} - iS_2^{\psi} + i\int \psi_1\eta_1 - i\int \psi_2\eta_2\right) Z^{qA}[j_1, j_2]$$

$$Z^{qA}[j_1, j_2]$$

$$\approx \exp\left(-i/2\int_{t,\vec{x},\vec{y}} j_1Vj_1 - j_2V^* j_2 - iD(j_1 j_2 + j_2 j_1)\right)$$

Effective Hamiltonian and renormalization

- Time arguments at t
- Which order in Hamiltonian?

$$\overline{\psi}\psi, \psi\overline{\psi}, N[\overline{\psi}\psi]??$$

- Fermion bilinears

Order of fermions → time arguments in path integral

For later purpose, define $\tilde{\psi}_2 \equiv \psi_2^\dagger, \tilde{\psi}_2^\dagger \equiv \psi_2, \tilde{\psi}_2^\dagger(t+\varepsilon)\cdots\tilde{\psi}_2(t-\varepsilon)$

➡ $\boxed{\psi^*{}_1(t+\varepsilon)\cdots\psi_1(t\text{-}\varepsilon) \text{ and } \tilde{\psi}^*{}_2(t+\varepsilon)\cdots\tilde{\psi}_2(t\text{-}\varepsilon)}$

- In instantaneous interaction

→ Symmetric in all possible order in time

Effective Hamiltonian and renormalization

- Effective Hamiltonian
- Inserting fermionic complete sets on a single time path
  → Fermions are time-ordered
- Time evolution of what?

$$\left\langle \psi_1^{*\mathrm{fin}} \right| \mathrm{Tr}_{\mathrm{E}}\left(\hat{U}(t,0)\hat{\rho}\hat{U}(t,0)^\dagger \right) \left| \tilde{\psi}_2^{*\mathrm{fin}} \right\rangle$$

$$= \int d\psi_1^{*\mathrm{ini}} d\tilde{\psi}_2^{*\mathrm{ini}} \int_{\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}}^{\psi_1^{*\mathrm{fin}},\tilde{\psi}_2^{*\mathrm{fin}}} D[\psi_{1,2}]\,\rho^S[\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}]\exp\left(iS_1^\psi - iS_2^\psi\right) Z^{qA}[j_1, j_2]$$

$$= \left\langle \psi_1^{*\mathrm{fin}},\tilde{\psi}_2^{*\mathrm{fin}} \right| \exp\left[-i\int dt\hat{H}_{\mathrm{eff}} \right] \left| \Psi_{\mathrm{ini}} \right\rangle, \quad \left\langle \psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}} \middle| \Psi_{\mathrm{ini}} \right\rangle = \rho^S[\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}]$$

★Coherent state built on **empty Dirac sea** for HQ $\neq$ HQ **vacuum**

Effective Hamiltonian and renormalization

- Change basis to

$$\left\langle Q_1^*, Q_{1c}^* \right| \equiv \left\langle \Omega \right| \exp\left[-\int_x \hat{Q} Q_1^* + \hat{Q}_c Q_{1c}^* \right]$$

$$\left| \tilde{Q}_2^*, \tilde{Q}_{2c}^* \right\rangle \equiv \exp\left[-\int_x \tilde{Q}_2^* \hat{Q}^\dagger + \tilde{Q}_{2c}^* \hat{Q}_c^\dagger \right] \left| \Omega \right\rangle$$

$$\Downarrow$$

$$\boxed{\left\langle Q_{1(c)}^{*\mathrm{fin}} \right| \mathrm{Tr}_{\mathrm{E}} \left(\hat{U}(t,0) \hat{\rho} \hat{U}(t,0)^\dagger \right) \left| \tilde{Q}_{2(c)}^{*\mathrm{fin}} \right\rangle}$$

← This is what we want

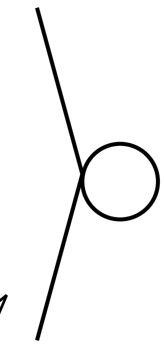
$$= \int dQ_{1(c)}^{*\mathrm{ini}} d\tilde{Q}_{2(c)}^{*\mathrm{ini}} \int_{Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}}^{Q_{1(c)}^{*\mathrm{fin}}, \tilde{Q}_{2(c)}^{*\mathrm{fin}}} D[Q_{1(c),2(c)}^{(*)}] \, \rho^S [Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}] \exp\left(iS_1^\psi - iS_2^\psi \right) Z^{qA}[j_1, j_2]$$

$$= \boxed{\left\langle Q_{1(c)}^{*\mathrm{fin}}, \tilde{Q}_{2(c)}^{*\mathrm{fin}} \right| \exp\left[-i \int dt \hat{H}_{\mathrm{eff}} \right] \left| \Psi_{\mathrm{ini}} \right\rangle}, \quad \left\langle Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}} \middle| \Psi_{\mathrm{ini}} \right\rangle = \rho^S [Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}]$$

Convenient to have H_{eff} in **normal order** for Q_1 and $\tilde{Q}_2$ (not Q_2)

Effective Hamiltonian and renormalization

- Results



$$
\begin{aligned}
\hat{H}_{\mathrm{kin}} &= \int d^3x Z_M M \left[\hat{Q}_1^\dagger \hat{Q}_1 + \hat{Q}_{1c}^\dagger \hat{Q}_{1c} + \hat{\hat{Q}}_2^\dagger \hat{\hat{Q}}_2 + \hat{\hat{Q}}_{2c}^\dagger \hat{\hat{Q}}_{2c} \right] \\
&+ \int d^3x Z_\psi \left[\begin{array}{c} \hat{Q}_1^\dagger \left(-\frac{\nabla^2}{2M}\right) \hat{Q}_1 + \hat{Q}_{1c}^\dagger \left(-\frac{\nabla^2}{2M}\right) \hat{Q}_{1c} \\ +\hat{\hat{Q}}_2^\dagger \left(-\frac{\nabla^2}{2M}\right) \hat{\hat{Q}}_2 + \hat{\hat{Q}}_{2c}^\dagger \left(-\frac{\nabla^2}{2M}\right) \hat{\hat{Q}}_{2c} \end{array} \right] + \text{const.}, \\
\hat{H}_{\mathrm{int}}^{(11)} &= \frac{1}{2} \int d^3x d^3y Z_g^2 V_{\mu\nu}(\vec{x}-\vec{y}) N \left\{ \hat{j}_{1,\mathrm{NR}}^{a\mu}(\vec{x}) \hat{j}_{1,\mathrm{NR}}^{a\nu}(\vec{y}) \right\} \\
& \boxed{+\frac{1}{2} \int d^3x Z_g^2 C_{\mathrm{F}} \left[\begin{array}{l} V_{\mathrm{E}}(\hat{Q}_1^\dagger \hat{Q}_1 + \hat{Q}_{1c}^\dagger \hat{Q}_{1c}) \\ -V_{\mathrm{M}} \left\{ \hat{Q}_1^\dagger \left(-\frac{\nabla^2}{2M^2}\right) \hat{Q}_1 + \hat{Q}_{1c}^\dagger \left(-\frac{\nabla^2}{2M^2}\right) \hat{Q}_{1c} \right\} \end{array} \right]} + \text{const.}, \quad (37) \\
\hat{H}_{\mathrm{int}}^{(22)} &= -\frac{1}{2} \int d^3x d^3y Z_g^2 V_{\mu\nu}^*(\vec{x}-\vec{y}) N \left\{ \hat{j}_{2,\mathrm{NR}}^{a\mu}(\vec{x}) \hat{j}_{2,\mathrm{NR}}^{a\nu}(\vec{y}) \right\} \\
& \boxed{+\frac{1}{2} \int d^3x Z_g^2 C_{\mathrm{F}} \left[\begin{array}{l} V_{\mathrm{E}}^*(\hat{\hat{Q}}_2^\dagger \hat{\hat{Q}}_2 + \hat{\hat{Q}}_{2c}^\dagger \hat{\hat{Q}}_{2c}) \\ -V_{\mathrm{M}}^* \left\{ \hat{\hat{Q}}_2^\dagger \left(-\frac{\nabla^2}{2M^2}\right) \hat{\hat{Q}}_2 + \hat{\hat{Q}}_{2c}^\dagger \left(-\frac{\nabla^2}{2M^2}\right) \hat{\hat{Q}}_{2c} \right\} \end{array} \right]} + \text{const.}, \quad (38) \\
\hat{H}_{\mathrm{int}}^{(12)} &= -\frac{i}{2} \int d^3x d^3y Z_g^2 D_{\mu\nu}(\vec{x}-\vec{y}) N \left\{ \hat{j}_{1,\mathrm{NR}}^{a\mu}(\vec{x}) \hat{j}_{2,\mathrm{NR}}^{a\nu}(\vec{y}) + \hat{j}_{2,\mathrm{NR}}^{a\mu}(\vec{x}) \hat{j}_{1,\mathrm{NR}}^{a\nu}(\vec{y}) \right\}, \quad (39)
\end{aligned}
$$

V_{E}, V_{M} Complex!

Effective Hamiltonian and renormalization

- UV divergence and renormalization
- Divergence

$$V_{ab,\mu\nu}(\vec{x}-\vec{y}) = \delta_{ab} V_{\mu\nu}(\vec{x}-\vec{y}),\ D_{ab,\mu\nu}(\vec{x}-\vec{y}) = \delta_{ab} D_{\mu\nu}(\vec{x}-\vec{y})$$

$$V_{\mu\nu}(0) = \left(V^{(\mathrm{E})} = V_{\mathrm{vac}} + V_{\mathrm{med}}^{(\mathrm{E})}, -V^{(\mathrm{M})} = -V_{\mathrm{vac}} - V_{\mathrm{med}}^{(\mathrm{M})}, -V^{(\mathrm{M})}, -V^{(\mathrm{M})}\right) \quad \text{(diagonal)}$$

$\rightarrow V_{\mathrm{vac}}$ = divergent, $D = \mathrm{Im}(V) = \mathrm{Im}(V_{\mathrm{med}})$ = finite

- Rernormalization of vacuum contribution

- correct kinetic term at $k \sim (MT)^{1/2}$~typical HQ momentum

- correct potential at $r \sim 1/T \sim$(typical exchanged momentum)$^{-1}$

$$\Rightarrow \begin{cases} Z_\psi(T) - \dfrac{V_{\mathrm{vac}} C_{\mathrm{F}}}{M} Z_g(T)^2 = 1, \quad Z_M(T) + \dfrac{V_{\mathrm{vac}} C_{\mathrm{F}}}{2M} Z_g(T)^2 = 1. \\ \text{Perturbatively, } Z_g(T) = g(T)/g_{\mathrm{bare}} \text{ or just } Z_g(T)=1,\ g=g(T) \end{cases}$$

Effective Hamiltonian and renormalization

- Renormalized effective Hamiltonian

$$
\begin{aligned}
\hat{H}_{\mathrm{eff}} &= \int d^3x \left\{ \begin{array}{l} \left(1+\frac{V_{\mathrm{med}}^{(\mathrm{E})}C_{\mathrm{F}}}{2M}Z_g(T)^2\right) M\left[\hat{Q}_1^\dagger\hat{Q}_1+\hat{Q}_{1c}^\dagger\hat{Q}_{1c}\right] \\ +\left(1-\frac{V_{\mathrm{med}}^{(\mathrm{M})}C_{\mathrm{F}}}{M}Z_g(T)^2\right)\left[\hat{Q}_1^\dagger\left(-\frac{\nabla^2}{2M}\right)\hat{Q}_1+\hat{Q}_{1c}^\dagger\left(-\frac{\nabla^2}{2M}\right)\hat{Q}_{1c}\right] \end{array} \right\} \\
&+\int d^3x \left\{ \begin{array}{l} \left(1+\frac{V_{\mathrm{med}}^{(\mathrm{E})*}C_{\mathrm{F}}}{2M}Z_g(T)^2\right) M\left[\hat{\tilde{Q}}_2^\dagger\hat{\tilde{Q}}_2+\hat{\tilde{Q}}_{2c}^\dagger\hat{\tilde{Q}}_{2c}\right] \\ +\left(1-\frac{V_{\mathrm{med}}^{(\mathrm{M})*}C_{\mathrm{F}}}{M}Z_g(T)^2\right)\left[\hat{\tilde{Q}}_2^\dagger\left(-\frac{\nabla^2}{2M}\right)\hat{\tilde{Q}}_2+\hat{\tilde{Q}}_{2c}^\dagger\left(-\frac{\nabla^2}{2M}\right)\hat{\tilde{Q}}_{2c}\right] \end{array} \right\} \\
&+\frac{1}{2}\int d^3x d^3y Z_g(T)^2 \left\{ \begin{array}{l} V_{\mu\nu}(\vec{x}-\vec{y})N\left[\hat{j}_{1,\mathrm{NR}}^{a\mu}(\vec{x})\hat{j}_{1,\mathrm{NR}}^{a\nu}(\vec{y})\right] \\ -V_{\mu\nu}^*(\vec{x}-\vec{y})N\left[\hat{j}_{2,\mathrm{NR}}^{a\mu}(\vec{x})\hat{j}_{2,\mathrm{NR}}^{a\nu}(\vec{y})\right] \\ -iD_{\mu\nu}(\vec{x}-\vec{y})N\left[\hat{j}_{1,\mathrm{NR}}^{a\mu}(\vec{x})\hat{j}_{2,\mathrm{NR}}^{a\nu}(\vec{y})+\hat{j}_{2,\mathrm{NR}}^{a\mu}(\vec{x})\hat{j}_{1,\mathrm{NR}}^{a\nu}(\vec{y})\right] \end{array} \right\}. \qquad (43
\end{aligned}
$$

Density matrix

- Generating functional of density matrix

$$\rho_S\left[Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^*, t\right] = \langle Q_1^*, Q_{1c}^*|\hat{\rho}_S(t)|\tilde{Q}_2^*, \tilde{Q}_{2c}^*\rangle = \langle Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^*|\Psi(t)\rangle,$$

$$\langle Q_1^*, Q_{1c}^*| \equiv \langle\Omega|\exp\left[-\int d^3x\left\{\hat{Q}(\vec{x})Q_1^*(\vec{x}) + \hat{Q}_c(\vec{x})Q_{1c}^*(\vec{x})\right\}\right],$$

$$|\tilde{Q}_2^*, \tilde{Q}_{2c}^*\rangle \equiv \exp\left[-\int d^3x\left\{\tilde{Q}_2^*(\vec{x})\hat{Q}^\dagger(\vec{x}) + \tilde{Q}_{2c}^*(\vec{x})\hat{Q}_c^\dagger(\vec{x})\right\}\right]|\Omega\rangle.$$

- Time evolution

  In analogy to Schroedinger wave equation

$$\boxed{\begin{aligned} & i\frac{\partial}{\partial t}\rho^Q\left[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*, t\right] \\ & = H_{\mathrm{eff}}\left[Q_{1(c)}^*, Q_{1(c)} = \frac{\delta}{\delta Q_{1(c)}^*}, \tilde{Q}_{2(c)}^*, \tilde{Q}_{2(c)} = -\frac{\delta}{\delta\tilde{Q}_{2(c)}^*}, t\right]\rho^Q\left[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*, t\right] \end{aligned}}$$

Density matrix

- Density matrix and master equation
  - Single HQ

$$\begin{aligned}\rho_{\mathrm{S}}(\vec{x},\vec{y},t) &\propto \langle\Omega|\hat{Q}(\vec{x})\hat{\rho}_{\mathrm{S}}(t)\hat{Q}^{\dagger}(\vec{y})|\Omega\rangle \\ &= -\frac{\delta}{\delta Q_1^*(\vec{x})}\frac{\delta}{\delta\tilde{Q}_2^*(\vec{y})}\rho_{\mathrm{S}}\left[Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^*, t\right]\Big|_{Q_{1(c)}^*=\tilde{Q}_{2(c)}^*=0}\end{aligned}$$

$$\begin{aligned}i\frac{\partial}{\partial t}\rho_S(\vec{x},\vec{y},t) &= \left\{a - a^* + b\left(-\frac{\nabla_x^2}{2M}\right) - b^*\left(-\frac{\nabla_y^2}{2M}\right) + id(\vec{x},\vec{y})\right\}\rho_S(\vec{x},\vec{y},t) \\ a &\equiv M + \frac{1}{2}V_{\mathrm{med}}^{(\mathrm{E})}C_{\mathrm{F}}Z_g(T)^2, \quad b \equiv 1 - \frac{V_{\mathrm{med}}^{(\mathrm{M})}C_{\mathrm{F}}}{M}Z_g(T)^2, \\ d(\vec{x},\vec{y}) &\equiv Z_g(T)^2 D_{\mu\nu}(\vec{x}-\vec{y})\left[t^a \otimes (1, \vec{\nabla}/iM)^{\mu}\right]_x \left[-t^{a*}\otimes(1, -\vec{\nabla}/iM)^{\nu}\right]_y\end{aligned}$$

★This master equation is unitarity

Summary

- Starting from non-equilibrium field theory on a closed-time path, we derive renormalized effective Hamiltonian on a single time.
- The Hamiltonian has complex kinetic term, mass term, and potential.
- Master equation and forward correlator for any heavy quark system can be derived from the effective Hamiltonian.
- Above, we took non-relativistic approximation, adopted ladder approximation, and truncated many-body (>2) interaction among heavy quarks.