Effective potential in the boundary effective theory framework

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1 Introduction: Boundary Effective Theory

- Boundary Effective Theory (BET) is a non-perturbative method to calculate the partition function of quantum systems in thermal equilibrium [1].
- The functional space of β-periodic fields is sliced in sectors with fixed boundary field:

$$Z = \int [\mathcal{D}\varphi_0(\boldsymbol{x})] \int_{\varphi(0,\boldsymbol{x})=\varphi_0(\boldsymbol{x})}^{\varphi(\beta,\boldsymbol{x})=\varphi_0(\boldsymbol{x})} [\mathcal{D}\varphi(\tau,\boldsymbol{x})] e^{-S_E[\varphi]}$$

- The innermost integral is dominated by configurations in the vicinity of the classical solution φ_c :

$$\Box_E\, \varphi_c(\tau,\boldsymbol{x}) + U'(\varphi_c(\tau,\boldsymbol{x})) = 0,$$
$$\varphi_c(0,\boldsymbol{x}) = \varphi_c(\beta,\boldsymbol{x}) = \varphi_0(\boldsymbol{x}).$$

- Saddle-point approximation in the first integral: effective theory for the boundary fields.
  $\rightarrow$ The related action is naturally dimensionally reduced
- Intricate dependence on $\varphi_0(\boldsymbol{x})$: encoded in $\varphi_c(\tau,\boldsymbol{x})$.

- BET one-loop effective action for a general single-well potential $U(\varphi)$:

$$\beta\Gamma[\varphi_0(\boldsymbol{x})] = S_E[\varphi_c] + \frac{1}{2}\mathrm{Tr}\,\ln(\Delta_F^{-1} + U''(\varphi_c)).$$

- **NB:** simple relationship between effective actions at zero and finite temperature.
  $\rightarrow$ Natural bridge between correlations and renormalization conditions, at zero and finite temperature [1].

For more details on BET, see the poster by André Bessa.

2 Effective potential for massless $\lambda\varphi^4$ theory

- Dependence of $\Gamma[\varphi_0(\boldsymbol{x})]$ on $\varphi_0(\boldsymbol{x})$: solution of the classical equations of motion for arbitrary boundary configurations.
- Boundary field with arbitrary $\boldsymbol{x}$ dependence: not feasible.
- *Effective Potential* – given by the effective action evaluated for uniform boundary fields [2]:

$$V_{eff}(\varphi_0) = \frac{\Gamma[\varphi_0] - \Gamma[0]}{V}, \quad \text{for constant } \varphi_0$$

and $-\Gamma[0]/V = \pi^2 T^4/90$ (free pressure) [1].

- The Tracelog is computed using the technique of Ref. [3]:

$$\frac{1}{2}\mathrm{Tr}\,\ln(\Delta_F^{-1} + U''(\varphi_c)) = \frac{V}{2}\int^{\Lambda} \frac{d^3\boldsymbol{k}}{(2\pi)^3} \ln[2(\eta(\beta,\boldsymbol{k}^2) - 1)],$$

$\eta(\beta,\boldsymbol{k}^2)$ is the solution of the equation for small field perturbations propagating on top of $\varphi_c(\tau)$:

$$[\partial_\tau^2 - k^2 + U''(\varphi_c(\tau))]\eta(\beta,\boldsymbol{k}^2) = 0,$$
$$\eta(0,\boldsymbol{k}^2) = 1, \qquad \frac{d\eta}{d\tau}(0,\boldsymbol{k}^2) = 0.$$

- Nonrenormalized expression for the effective potential:

$$\beta V V_{eff}(\varphi_0) = S_E[\varphi_c] + \frac{V}{2}\int^{\Lambda} \frac{d^3\boldsymbol{k}}{(2\pi)^3} \ln[2(\eta(\beta,\boldsymbol{k}^2) - 1)] - \beta\Gamma[0].$$

- Renormalization: standard one-loop counterterms and subtraction of the zero-point energy (not obvious).

- Classical solutions for $\lambda\varphi^4$ theory – Jacobi elliptic functions:

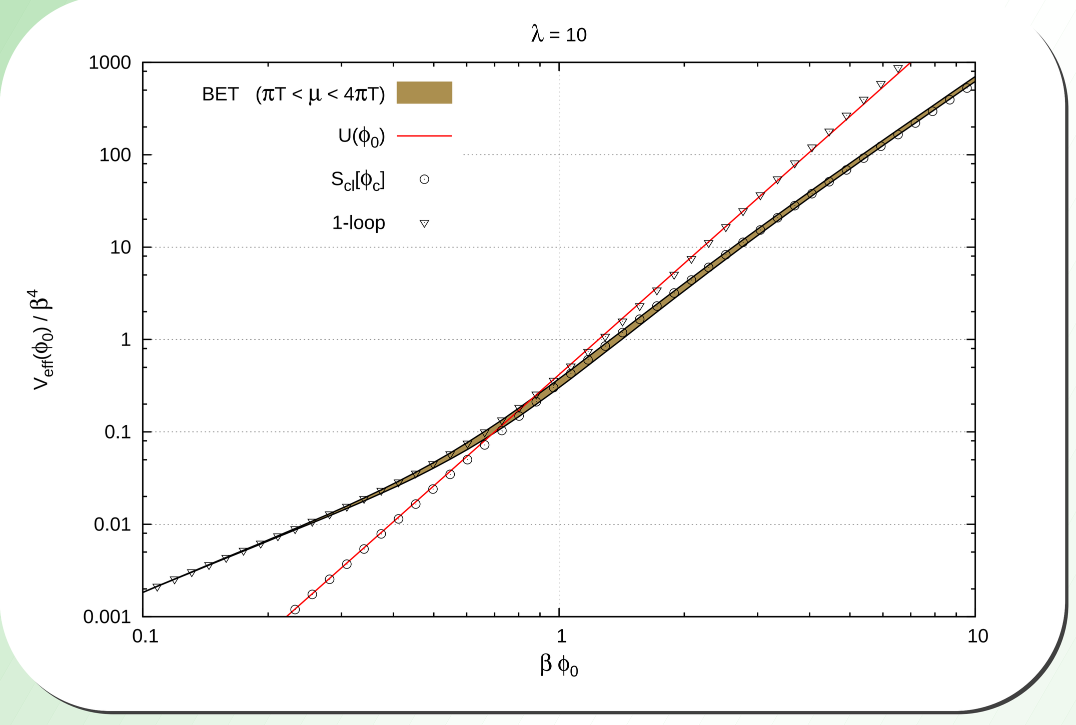
$$\varphi_c(\tau) = \sqrt{\frac{6}{\lambda}}\,\varphi_t\,\mathrm{nc}\left[\varphi_t(\tau - \beta/2), 1/\sqrt{2}\right],$$

where φ_t is chosen to ensure that $\varphi_c(0) = \varphi_c(\beta) = \varphi_0$:

$$\varphi_0 = \sqrt{\frac{6}{\lambda}}\,\varphi_t\,\mathrm{nc}\left[\varphi_t\beta/2, 1/\sqrt{2}\right].$$

2.1 Results and discussion

The plot shows the BET effective potential as a function of φ_0 for $\lambda = 10$. The result is compared to the standard one-loop calculation [4], with the classical action associated to φ_c and also with the classical potential evaluated at φ_0.



The effective potential obtained with BET reproduces the standard one-loop result for small fields, as expected, since the BET effective action contains the effect of the thermal mass. For large fields, BET goes beyond by incorporating nonlinear corrections that are not captured by the standard one-loop calculation.

3 Double-well potentials: bifurcations and the problem of multiple classical solutions

- A natural extension of the previous result: effective potential in the case of a double-well potential, allowing for the study of thermally-driven symmetry restoration and implications for phase transitions [9].
- This extension is not trivial: difficulties are related to caustics and complex trajectories in the calculation of the semiclassical density matrix [5, 6].
- We start with a simpler case: the semiclassical partition function for a double-well in quantum mechanics [8].

3.1 Semiclassical partition function for the double-well potential in quantum mechanics

- Semiclassical expansion for the density matrix diagonal elements, $\rho[q_0, q_0]$: expanding the related path integral around $q_c(\tau)$, path of a classical particle under the influence of the inverted potential $-U(q)$.
- Boundary conditions: $q(0) = q(\Theta) = q_0$.
- Semiclassical partition function:

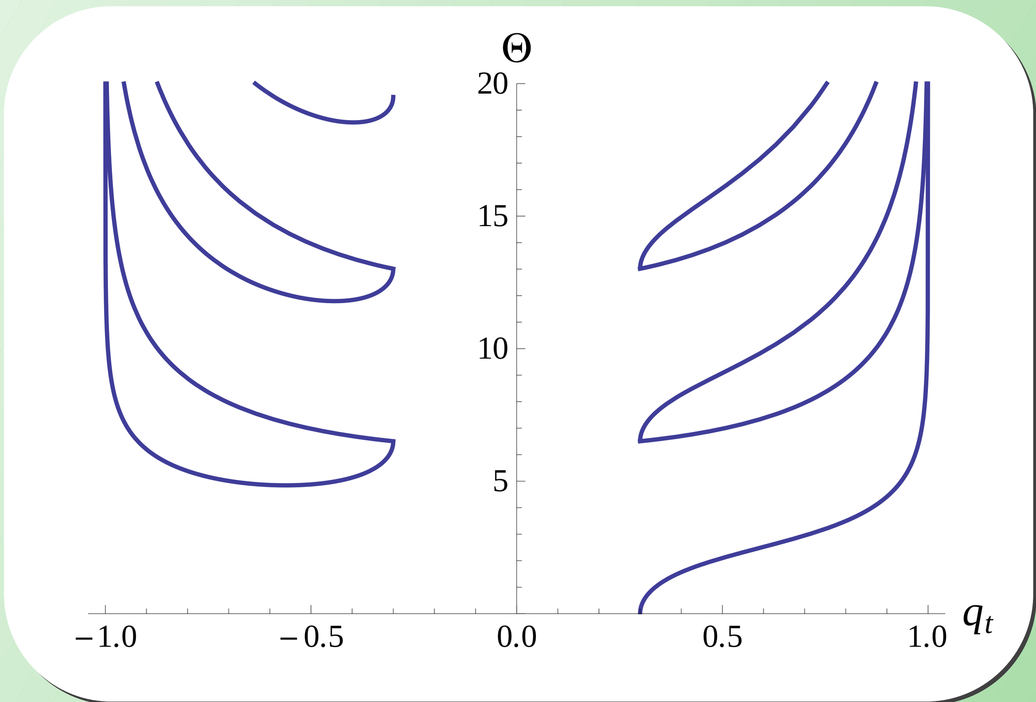
$$Z \approx \sum_i \int_{-\infty}^{\infty} dq_0\, \Delta_i^{-1/2} e^{-S_E[\bar{q}_c^i]}\,.$$

Δ_i denotes the determinant of the fluctuation operator,

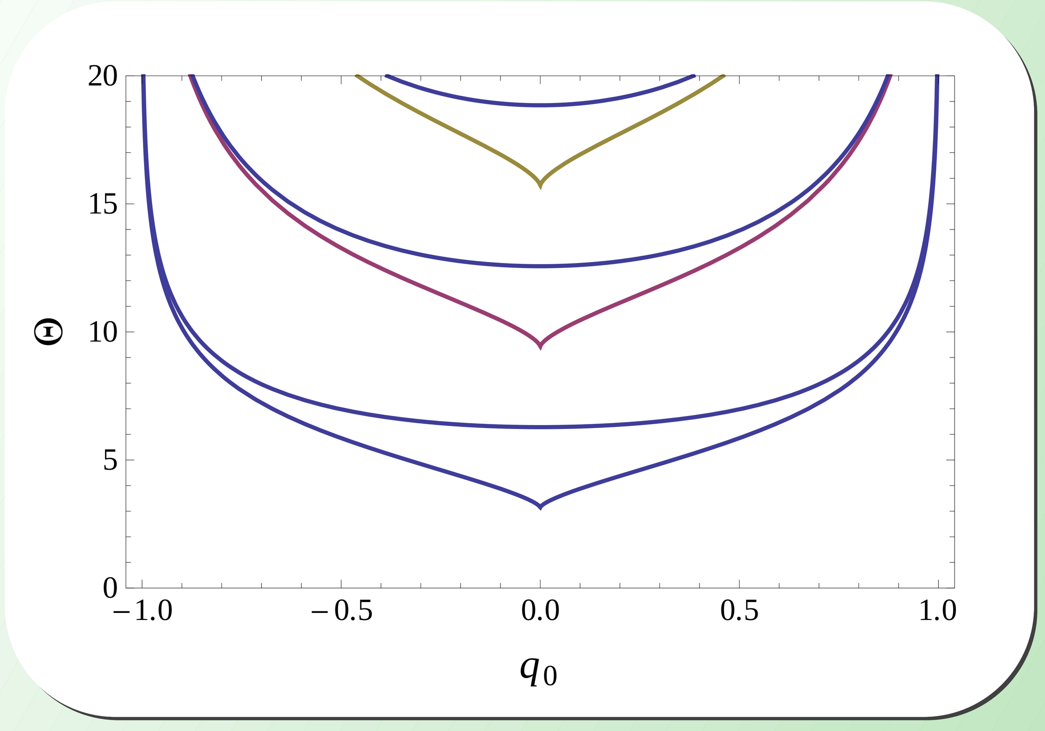
$$\Delta_i = \det \hat{F}[\bar{q}_c^i] = \det\left[-\frac{d^2}{d\theta^2} + U''(\bar{q}_c^i)\right],$$

and $\bar{q}_c^{\,i}$ are the *minima* of the euclidean action.

- Multiple classical solutions arise due to the existence of a region of bounded motion.
  $\rightarrow$ For fixed q_0 in this region, solution bifurcation is observed for increasing Θ.
- Quartic double-well potential, $U(q) = -q^2/2 + q^4/4$ (below).



- The (q_0, Θ) plane is divided into regions with different numbers of classical solutions.
- The frontiers between two such regions are called caustics. On the caustics Δ vanishes!
- Whenever a caustic is crossed two solutions are either created or annihilated (bifurcations).



- Following [5, 6], we use the framework of catastrophe theory to identify the solutions that minimize the action.
- We only have to keep the minima present after crossing the first caustic. No more than two solutions have to be considered [8].
- These are single turning point trajectories, written as:

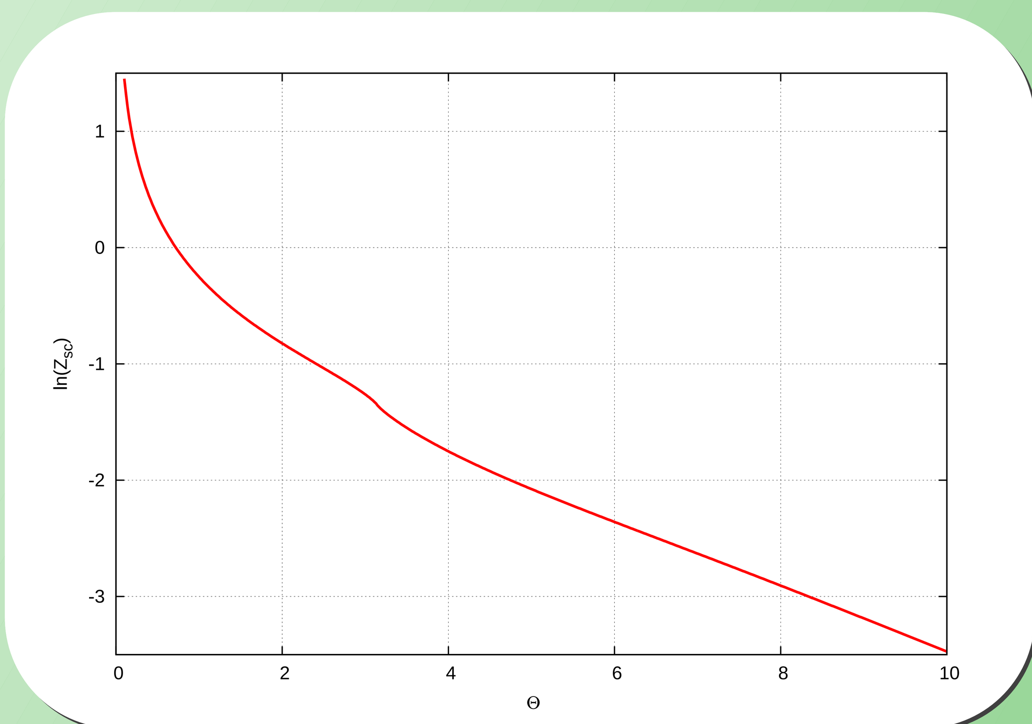
$$q(\theta) = q_t\,\mathrm{cd}\left[\sqrt{1 - q_t^2/2}(\theta - \Theta/2), \frac{q_t}{\sqrt{2 - q_t^2}}\right].$$

- For this class of solutions the expression for the determinant Δ is known [7]:

$$\Delta = \frac{4\pi g[U(q_t) - U(q_0)]}{U'(q_t)}\left(\frac{\partial\Theta}{\partial q_t}\right)_{q_0}.$$

3.2 Preliminary results and discussion

The plot shows $\log Z$ obtained using the semiclassical approximation. One classical solution is used before the first caustic and two after.



The result shows a clear concavity change around $\Theta = \pi$. This reflects the fact that the semiclassical method, although being able to deal with multiple solutions, is not a good approximation when solutions coalesce.

It is not enough to simply sum over the two minima of the action – a better treatment of the caustics, as was performed in Ref. [6], is necessary.

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