Effective lattice theory for hot and/or dense QCD

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Introduction

- Lattice: Nonperturbative method to investigate QCD thermodynamics; successful at vanishing quark chemical potential μ
- But: Fermionic sign problem, i.e. complex quark determinant at finite chemical potential forbids to simulate directly at $\mu \neq 0 \longrightarrow$ Need new ideas to simulate lattice QCD at finite chemical potential, especially at $\mu > T$
- Here: Solve the sign problem by going to an effective 3d theory, which drastically reduces the number of degrees of freedom
- Calculate effective couplings analytically, then solve the effective theory with simulations and map back to original theory

Setup

- Lattice QCD with Wilson action goes to continuum action for lattice spacing $a \to 0$ and reads

$$Z = \int [dU_\mu d\bar{\psi} d\psi] e^{-S_g - S_f} = \int [dU_\mu d\bar{\psi} d\psi] \exp\left[\frac{\beta}{3}\sum_p \mathrm{Re}\,\mathrm{Tr}\, U_p + \bar{\psi} Q(\kappa_f, \mu_f)\psi\right] , \qquad \beta = \frac{6}{g^2}$$

- Finite temperature: Compact temporal direction and (anti-)periodic boundary conditions
- Effective theory by integrating over spatial link variables $\longrightarrow$ yields infinitely many terms

$$Z = \int [dU_\mu d\bar{\psi} d\psi] e^{-S_g - S_f} \equiv \int [dU_0] e^{-S_{\mathrm{eff}}(\beta,\kappa_f)} , \qquad T = \frac{1}{aN_\tau}$$

$$-S_{\mathrm{eff}} = \sum_{i=1}^{\infty} \lambda_i(\beta, \kappa_f, N_\tau) S_i^s - 2\sum_f \sum_{i=1}^{\infty} \left[h_i(\beta, \kappa_f, \mu, N_\tau) S_i^a + \bar{h}_i(\beta, \kappa_f, \mu, N_\tau) S_i^{a,\dagger}\right] .$$

- Two kinds of effective interaction terms: S_i^s are $Z(3)$-symmetric, S_i^a asymmetric

Pure gauge theory

- Pure gauge theory on the lattice is described by Wilson action for vanshing κ_f

$$\int [dU_i] \exp\left[-S_g(\beta)\right] = \exp\left[-S_{\mathrm{eff}}(\beta)\right] = \exp\left[\sum_i \lambda_i(\beta, N_\tau) S_i^s\right]$$

- Integrating out the spatial link variables U_i can be done in an expansion in β and leads to a $Z(3)$-symmetric effective action formulated in terms of Polyakov loops $L_i = \mathrm{Tr} W_i = \mathrm{Tr}\prod_{\tau=1}^{N_\tau} U_0(\tau, \vec{x}_i)$

$$\exp\left[-S_{\mathrm{eff}}(\beta)\right] = \prod_{ij}\left[1 + \lambda_1\left(L_i L_j^* + L_i^* L_j\right)\right]\left[1 + \ldots\right] , \qquad \lambda_1 = \left(\frac{\beta}{18}\right)^{N_\tau}\left(1 + \ldots\right) .$$

- Determination of λ_c in the effective action allows to map back to $\beta_c(N_\tau)$ for the $(3+1)d$-theory
- Very important to include corrections in order to get accurate results; λ_1 is known up to $\mathcal{O}(\beta^{N_\tau+10})$
- Results are within a few percent compared to Monte Carlo simulations of the full theory
- Higher order interaction terms affect the results very little

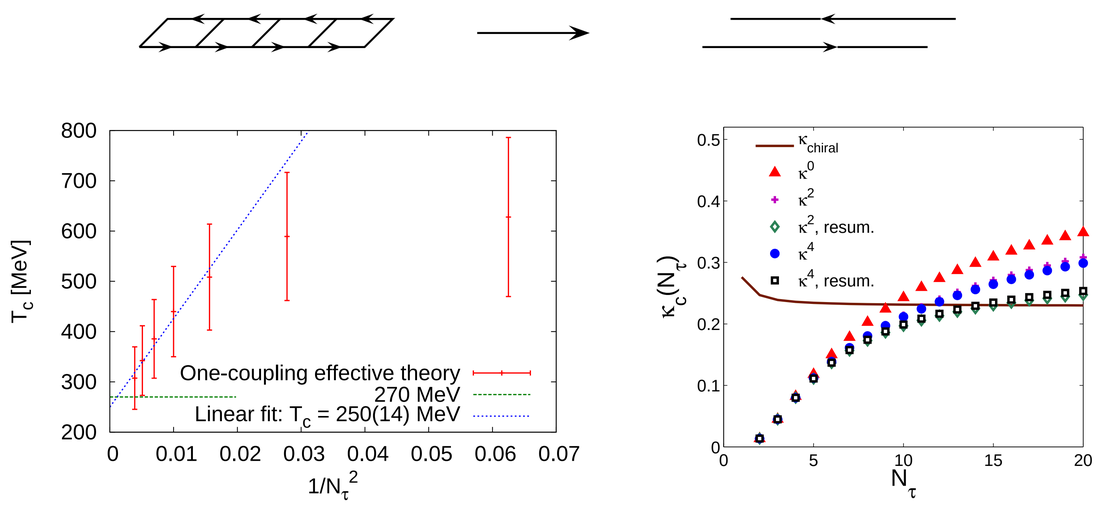

Figure 1: Left: Critical couplings $\beta_c(N_\tau)$ converted to temperatures. Right: Critical $\kappa_c(N_\tau)$. [1, 2].

Including static fermions

- Fermions can be integrated out exactly $\to$ Quark determinant: breaks $Z(3)$-symmetry explicitely
- Then: same steps as in pure gauge case in a combined strong coupling and hopping expansion

$$\int [dU_i] \exp\left[-S_g(\beta)\right] \det\left[Q(\kappa_f, \mu_f)\right] = \exp\left\{\sum_{i=1}^{\infty} \lambda_i S_i^s - 2\sum_f \sum_{i=1}^{\infty}\left[h_i S_i^a + \bar{h}_i S_i^{a,\dagger}\right]\right\} ,$$

$$\kappa_f = \frac{1}{2am_f + 8} , \qquad Q(\kappa_f, 0) = \delta_{yx} - \kappa_f \sum_\mu (1 + \gamma_\mu) U_\mu(\tau, \vec{x}) \delta_{y, x+\hat{\mu}} .$$

- Chemical potential introduced via a modification of the temporal quark hops

$$\kappa_f (1 \pm \gamma_0) U_{\pm 0}(\tau, \vec{x}) \delta_{y, x \pm \hat{0}} \qquad \longrightarrow \qquad \kappa_f e^{\pm a\mu} (1 \pm \gamma_0) U_{\pm 0}(\tau, \vec{x}) \delta_{y, x \pm \hat{0}}$$

- Modifies the effective coupling corresponding to quark loops with some power of

$$C = (2\kappa_f)^{N_\tau} \exp\left[a\mu_f N_\tau\right] = \exp\left[-\frac{m_f^{\mathrm{stat}} - \mu_f}{T}\right] , \qquad am_f^{\mathrm{stat}} = -\ln\left(2\kappa_f\right)$$

- In a κ_f-expansion: Leading contribution comes from Polyakov loops $\hat{=}$ Static approximation

$$\exp\left[-2h_1 S_1^a\right] = \det{}_{s,c}\left[Q_{\mathrm{stat}}\right] = \prod_{f,i} \det{}_c\left[1 + h_1 W_i\right]^2 \det{}_c\left[1 + \bar{h}_1 W_i^\dagger\right]^2 , \qquad h_1 = C\left[1 + \ldots\right]$$

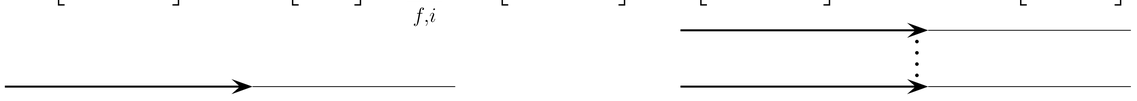

Figure 2: Left: Single Polyakov loop. Right: Generalized Polyakov loop, which winds n times before being traced.

- Allows to quantitatively explore the upper right corner of the Columbia plot, cf. fig. 4

Including dynamical fermions

Figure 3: Left: First correction graph, leading to an interaction between equally oriented loops. Right: Summation over winding numbers in order to fulfill the Pauli principle.

- Corrections also have spatial hops, cf. fig. 3. Summed contribution (neglecting antiquarks) is

$$\exp\left[-2h_2 S_2^a\right] = \prod_{<ij>}\left[1 - h_2 N_\tau \mathrm{Tr}\frac{W_i}{1 + CW_i}\mathrm{Tr}\frac{W_j}{1 + CW_j}\right] , \qquad h_2 = C^2\frac{\kappa^2}{N_c}\left[1 + \ldots\right] .$$

- For finite lattice spacing, baryon density should eventually (i.e. with increasing μ) saturate (Pauli principle); need to sum over all multiple windings of the loops to get this correct

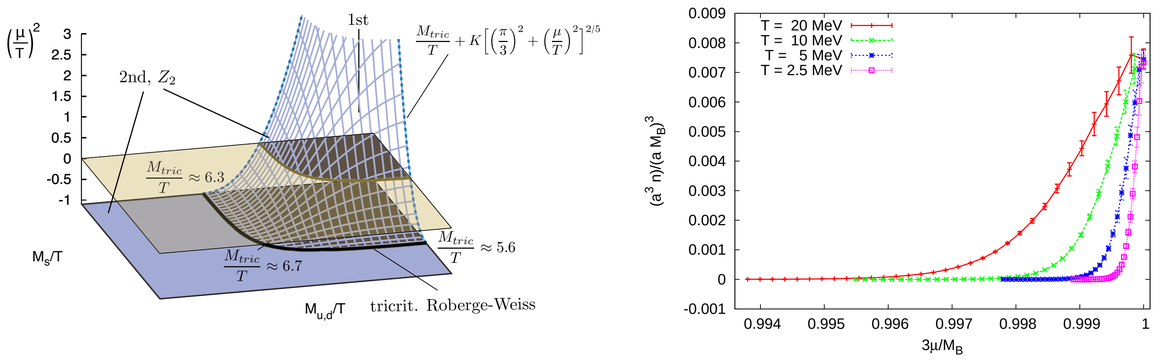

Figure 4: Left: Upper right corner of the Columbia plot for imaginary as well as real chemical potential. Right: First results compatible with the Silver Blaze phenomenon of QCD [3].

References

[1] J. Langelage, S. Lottini and O. Philipsen, JHEP **1102** (2011) 057 [Erratum-ibid. **1107** (2011) 014] [arXiv:1010.0951 [hep-lat]].

[2] M. Fromm, J. Langelage, S. Lottini and O. Philipsen, JHEP **1201** (2012) 042 [arXiv:1111.4953 [hep-lat]].

[3] T. D .Cohen, Phys. Rev. Lett. **91** (2003) 222001 [hep-ph/0307089].