Deconfinement transitions of large N QCD with chemical potential at weak and strong coupling

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Abstract

We calculate the deconfinement line of transitions for large N_c QCD at finite temperature and chemical potential in two different regimes: weak coupling in the continuum, and strong coupling on the lattice, working in the limit where N_f is of order N_c. In the first regime we extend previous weak-coupling results from one-loop perturbation theory on $S^1 \times S^3$ to higher temperatures and obtain the line of transitions that extends from the temperature-axis to the chemical potential-axis, where the theory reduces to a matrix model, analogous to that of Gross, Witten, and Wadia. In the second regime we use the same matrix model to obtain the deconfinement line of transitions as a function of the coupling strength and μ/T to leading order in a strong coupling expansion of lattice QCD with heavy quarks, extending previous $U(N_c)$ results to $SU(N_c)$. We show that in the case of zero chemical potential the result obtained for the Polyakov line from $S^1 \times S^3$ at weak coupling, reproduces the known results from the strong coupling expansion, under a simple change of parameters, which is valid for sufficiently low temperatures and chemical potentials.

1-loop QCD on $S^1 \times S^3$ vs. lattice strong coupling expansion

The action of large N_c QCD, with $\frac{N_f}{N_c}$ fixed, to leading order in the lattice strong coupling expansion and the hopping (heavy quark) expansion is given by [1]

$$\begin{aligned} S_{lat} - S_{Vdm} = &- JD \sum_x \left[\langle W\rangle W^\dagger(x) + \langle W^\dagger\rangle W(x) - \langle W\rangle\langle W^\dagger\rangle\right] \\ &- hN_c \sum_x \left[e^{\mu\beta} W(x) + e^{-\mu\beta} W^\dagger(x)\right] , \end{aligned} \tag{1}$$

- S_{Vdm} is the Vandermonde contribution,
- $J \equiv 2\left(\frac{\beta_{lat}}{2N_c^2}\right)^{N_\tau}$ with $\beta_{lat} = \frac{2N_c}{g^2}$, for N_τ time slices,
- $h \equiv 2\frac{N_f}{N_c}\kappa^{N_\tau}$ with $\kappa \equiv \frac{1}{am+1+D}$, for D spatial dimensions, and lattice spacing a,
- $W(x) = \mathrm{Tr}\prod_{t=0}^{N_\tau-1} U_{t,i}$ is the Polyakov line.

On $S^1 \times S^3$ the QCD action from one loop perturbation theory takes the form [2, 3]

$$\begin{aligned} S_{S^1\times S^3} - S_{Vdm} = &- N_c^2 \sum_{n=1}^\infty \frac{1}{n} z_v(n\beta/R)\rho_n\rho_{-n} \\ &+ N_f N_c \sum_{n=1}^\infty \frac{(-1)^n}{n} z_f(n\beta/R, mR)\left(e^{n\beta\mu}\rho_n + e^{-n\beta\mu}\rho_{-n}\right) , \end{aligned} \tag{2}$$

- $\beta = \frac{1}{T}$ is the length of the S^1, R is the radius of the S^3 which satisfies $R \ll \Lambda_{QCD}^{-1}$,

$$z_v(n\beta/R) = 2\sum_{l=1}^\infty l(l+2)e^{-n\beta(l+1)/R} = \frac{2e^{-2n\beta/R}(3 - e^{-n\beta/R})}{(1 - e^{-n\beta/R})^3} ,$$

$$z_f(n\beta/R, mR) = 2\sum_{l=1}^\infty l(l+1)e^{-n\frac{\beta}{R}\sqrt{(l+\frac{1}{2})^2 + m^2R^2}} ,$$

and the Polyakov lines are defined by $\rho_n = \frac{1}{N_c}\mathrm{Tr}\mathscr{P}e^{n\int_0^\beta \mathrm{d}t A_0(x)} = \frac{1}{N_c}e^{n\beta\alpha} = \frac{1}{N_c}\sum_{i=1}^{N_c} e^{in\theta_i}$.

Transform observables with a simple change of parameters

Due to the absence of terms with correlations between different sites in the action (1) an observable of the form $\langle F(W, W^\dagger)\rangle$ can be obtained as

$$\begin{aligned} \langle F(W, W^\dagger)\rangle &= \frac{1}{N_x Z}\int \prod_x \mathrm{d}W(x) e^{-S[W(x), W^\dagger(x)]} \sum_{x'} F[W(x'), W^\dagger(x')] , \\ &= \frac{\int \mathrm{d}W e^{-S(W,W^\dagger)} F(W, W^\dagger)}{\int \mathrm{d}W e^{-S(W,W^\dagger)}} . \end{aligned}$$

Therefore, when it is possible to approximate the sums over n in (2) by the $n = 1$ contribution, then it is possible to calculate observables of the form $\langle F(\rho_1, \rho_{-1})\rangle$ in weakly coupled QCD on $S^1 \times S^3$ and use the transformations

$$\begin{gathered} \rho_1 \leftrightarrow \frac{1}{N_c}\langle W\rangle , \qquad \rho_{-1} \leftrightarrow \frac{1}{N_c}\langle W^\dagger\rangle \\ z_v(\beta/R) \leftrightarrow JD \\ z_f(\beta/R, mR)\frac{N_f}{N_c} \leftrightarrow h \end{gathered} \tag{3}$$

to obtain the results for strongly coupled lattice QCD with heavy quarks, or vice versa.

Large N_c analysis

Add a term to the action to impose the $SU(N_c)$ constraint with Lagrange multiplier $\mathcal{N}$,

$$S \to S + i\mathcal{N}N_c \sum_{i=1}^{N_c} \theta_i .$$

Consider a contour $\mathcal{C}$ with the eigenvalues $z_j = e^{i\theta_j}$ of the Polyakov line distributed along it with density $\varrho(z)$, where $N_c \to \infty$ allows for

$$\frac{1}{N_c}\sum_{i=1}^{N_c} \to \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i z}\varrho(z) .$$

Use this to simplify the equation of motion from $\frac{\partial S}{\partial \theta_i} = 0$ to the form

$$\mathfrak{P}\int_{\mathcal{C}} \frac{\mathrm{d}z'}{2\pi i}\varrho(z')\frac{z'+z}{z'-z} = \sum_{n=1}^\infty \left(\alpha_{-n}z^n - \alpha_n z^{-n}\right) - \mathcal{N} , \tag{4}$$

where $\mathfrak{P}$ indicates that the principal value is taken with z left out of the range of integration and $\alpha_{\pm n} \equiv z_v(n\beta/R)\rho_{\pm n} + \frac{N_f}{N_c}z_f(n\beta/R)e^{\mp n\mu\beta}$.

Confined (ungapped) phase

Consider the case where $\mathcal{C}$ is closed. Solve the equation of motion (4) with $\varrho(z) = \sum_{n=1}^\infty \rho_n z^{-n-1}$ using Cauchy's theorem subject to the identity constraint

$$\frac{1}{N_c}\sum_{i=1}^{N_c} \to \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i}\varrho(z) = 1 , \tag{5}$$

to obtain the Polyakov lines

$$\rho_{\pm n} = \frac{N_f}{N_c}\frac{(-1)^{n+1}z_f(n\beta/R, mR)e^{\mp n\mu\beta}}{1 - z_v(n\beta/R)} . \tag{6}$$

Applying the change of parameters (3) to $\rho_{\pm 1}$ gives the strong coupling results in [4]

$$\frac{1}{N_c}\langle W\rangle = \frac{he^{-\mu\beta}}{1 - JD} , \qquad \frac{1}{N_c}\langle W^\dagger\rangle = \frac{he^{\mu\beta}}{1 - JD} . \tag{7}$$

Deconfined (gapped) phase

Consider the case where $\mathcal{C}$ lies on an arc which opens on the negative real-axis. Define a resolvent

$$\phi(z) = \int_{\mathcal{C}} \frac{\mathrm{d}z'}{2\pi i}\varrho(z')\frac{z'+z}{z'-z} . \tag{8}$$

Solve for it using singular integral techniques for open contours,

$$\begin{aligned} \phi(z) = &- \mathcal{N} + \sum_{n=1}^\infty \left(\alpha_{-n}z^n - \alpha_n z^{-n}\right) + \sqrt{(z - \tilde{z})(z - \tilde{z}^*)} \\ &\times \sum_{l=1}^\infty \sum_{k=0}^\infty P_k(\cos\psi)\left(\alpha_{l+k}r^{-k-1}z^{-l} + \alpha_{-l-k}r^k z^{l-1}\right) . \end{aligned} \tag{9}$$

where $P_k(x)$ are the Legendre Polynomials, the endpoints $\tilde{z}$, $\tilde{z}^*$ of $\mathcal{C}$ occur at radius r and angles $\pm\psi$, and $x \equiv \cos\psi$. Calculate the Polyakov lines using

$$\rho_n = \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i}\varrho(z)z^n = \oint_\Gamma \frac{\mathrm{d}z}{4\pi i z}\phi(z)z^n , \tag{10}$$

where Γ is a closed contour peeled off the arc. Require the resolvent (9) to satisfy the identity constraint (5), the $SU(N_c)$ constraint $\sum_{i=1}^{N_c}\theta_i = 0$, which takes the form

$$\int_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i}\varrho(z)\log(z) = 0 \to \oint_\Gamma \frac{\mathrm{d}z}{2\pi i z}\phi(z)\log(z) = 0 , \tag{11}$$

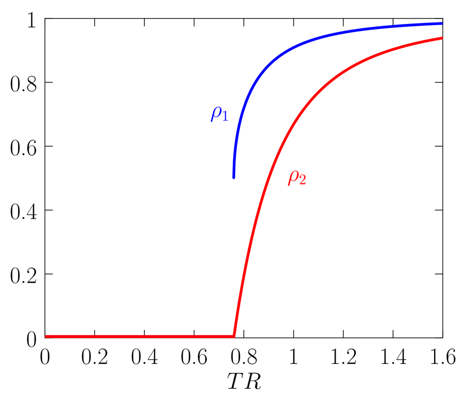
and the boundary conditions

$$\lim_{z\to 0}\phi(z) = 1 + 2\sum_{n=1}^\infty z^n\rho_{-n} ,$$

$$\lim_{z\to\infty}\phi(z) = -1 - 2\sum_{n=1}^\infty \frac{1}{z^n}\rho_n , \tag{12}$$

which result from (8) using (5) and (10).

Yang-Mills theory



In the confined phase (6) gives $\rho_n = 0$. In the deconfined phase with $z_v(1)$, $z_v(2) \neq 0$, (10) gives

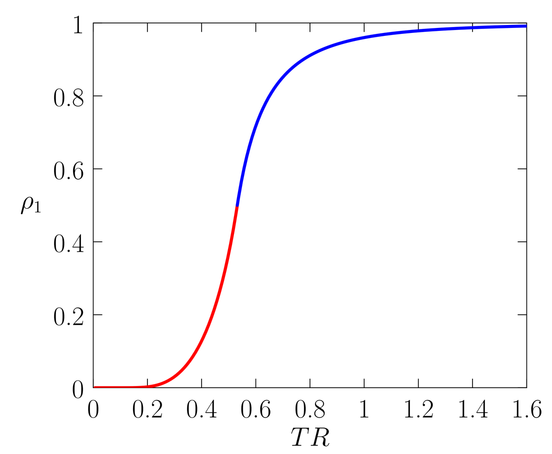
$$\rho_1 = \frac{4(1+x)^2}{4(1+3x) + z_v(1)(1-x)^3} ,$$

$$\rho_2 = \frac{8 - 2z_v(1)(1-x)(3+x)}{(1-x)z_v(2)\left[4(1+3x) + z_v(1)(1-x)^3\right]} ,$$

with $x \equiv \cos\psi$ the solution of (5), evaluated as

$$1 = \frac{8(1-x)^2(1+x)^4 z_v(1)z_v(2)}{\left[4 - z_v(1)(1-x)(3+x)\right]\left[16 + z_v(2)(1-x)\left[7 + x\left(3 + x(13+9x)\right)\right]\right]} .$$

QCD with $\mu = 0$



$\frac{N_f}{N_c} = 1$, $mR = 0$.

In the confining region ρ_1 is given by (6) with $\mu = 0$. The transition occurs when $\varrho(z) = \sum_{n=1}^\infty \rho_n z^{-n-1} = 0$ which corresponds to

$$1 - z_v - 2z_f\frac{N_f}{N_c} = 0 ,$$

with $z_v(1) \equiv z_v$, $z_f(1) \equiv z_f$, and $z_v(n) = z_f(n) = 0$ for $n > 1$. In the deconfined region

$$\rho_1 = \frac{z_v - z_f\frac{N_f}{N_c} + \sqrt{z_v^2 - z_v + 2z_v z_f\frac{N_f}{N_c} + z_f^2\frac{N_f^2}{N_c^2}}}{2z_v} .$$

The lattice strong coupling results in [1] are reproduced using the change of parameters (3). In the confining region $\langle W\rangle$ is given by (7) with $\mu = 0$. The transition occurs when

$$1 - JD - 2h = 0 .$$

In the deconfined region

$$\frac{1}{N_c}\langle W\rangle = \frac{1}{2}\left(1 - \frac{h}{JD}\right) + \frac{1}{2}\sqrt{\left(1 - \frac{h}{JD}\right)^2 - \frac{1}{JD}(1 - 4h)} .$$

QCD with $\mu \neq 0$: Polyakov lines I

The Polyakov lines in the confining region are given by (6) and (7) using the approximation $z_v(n) = z_f(n) = 0$ for $n > 1$. The Polyakov lines in the deconfined regions are obtained from (10) using (5) to get

$$\rho_{\pm 1} = \frac{z_f\frac{N_f}{N_c}(1-x)e^{\mp\mu\beta}\left[8 + 4e^{\pm 2\mu\beta}r^{\pm 2}(1+x) - z_v(1-x)^2(3+x)\right]}{16 - z_v(1-x)\left[16 - z_v(1-x)^2(3+x)\right]} , \tag{13}$$

where

$$r = \frac{4 - z_v(1-x)(3+x) + \sqrt{\left[4 - z_v(1-x)(3+x)\right]^2 - 16(1-x)^2 z_f^2\frac{N_f^2}{N_c^2}}}{4(1-x)e^{\mu\beta}z_f\frac{N_f}{N_c}} ,$$

and x is obtained by equating

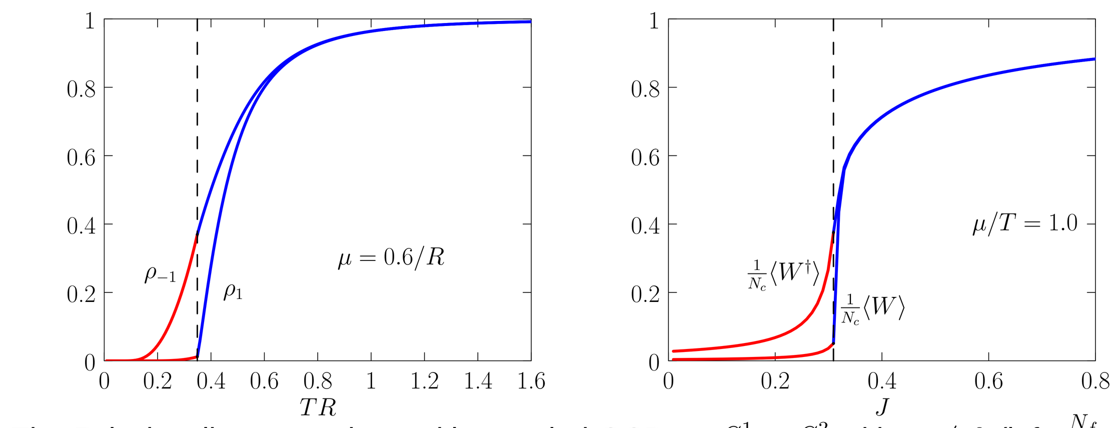
$$\mathcal{N} = (1+x)\left[-\frac{1}{1-x} + \frac{4z_f\frac{N_f}{N_c}r^{-1}e^{-\mu\beta}\left[z_v - z_v x^2 + 2r^2e^{2\mu\beta}(2 - z_v(1-x))\right]}{16 - z_v(1-x)\left[16 - z_v(1-x)^2(3+x)\right]}\right] , \tag{14}$$

obtained from the boundary conditions (12) with

$$\mathcal{N} = \frac{(1+x)\log r}{(1-x) - (1+x)\log\left(\frac{2}{1+x}\right)} , \tag{15}$$

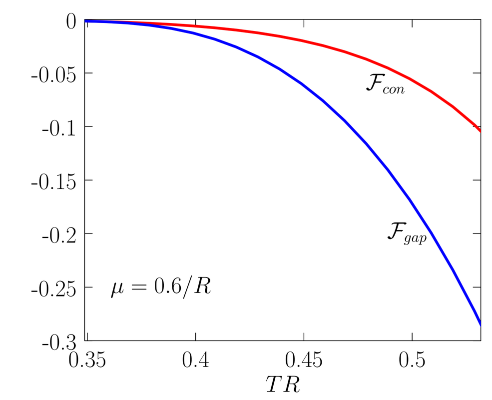
from the $SU(N_c)$ constraint (11). These quantities translate into their lattice strong coupling analogues by applying the transformations in (3).

QCD with $\mu \neq 0$: Polyakov lines II



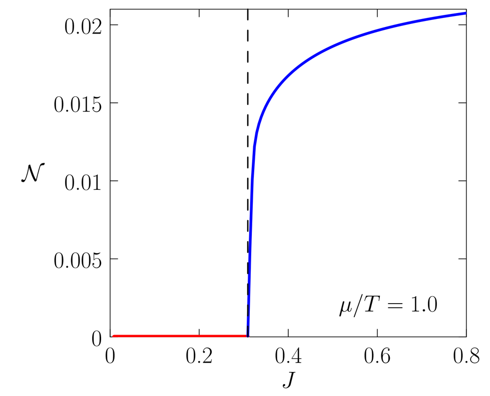
The Polyakov lines, $\rho_{\pm 1}$, in weakly coupled QCD on $S^1 \times S^3$ with $\mu \neq 0$ (left, $\frac{N_f}{N_c} = 1$, $mR = 0$) are obtained directly from (6), (13). In the lattice strong coupling theory $\frac{1}{N_c}\langle W\rangle$, $\frac{1}{N_c}\langle W^\dagger\rangle$ (right, $h = 0.01$) are obtained from (7), and (13) using the transformations (3).

QCD with $\mu \neq 0$: Free energy and quark number



In weakly-coupled QCD on $S^1 \times S^3$ solutions from the confined and deconfined phases are both possible for a range of T. A comparison of the free energies in this region indicates that the solution from the deconfined phase is favored as soon as it exists. The effective free energy $\mathcal{F} \equiv \frac{1}{N_c^2}F = \frac{1}{N_c^2}TS$ is plotted (left, $\frac{N_f}{N_c} = 1$, $mR = 0$) for the gapped and continuous distributions where

$$\mathcal{F} \simeq T\left[(1 - z_v(1))\rho_1\rho_{-1} - \frac{N_f}{N_c}z_f(1)\left(\rho_1 e^{\mu\beta} + \rho_{-1}e^{-\mu\beta}\right)\right] .$$

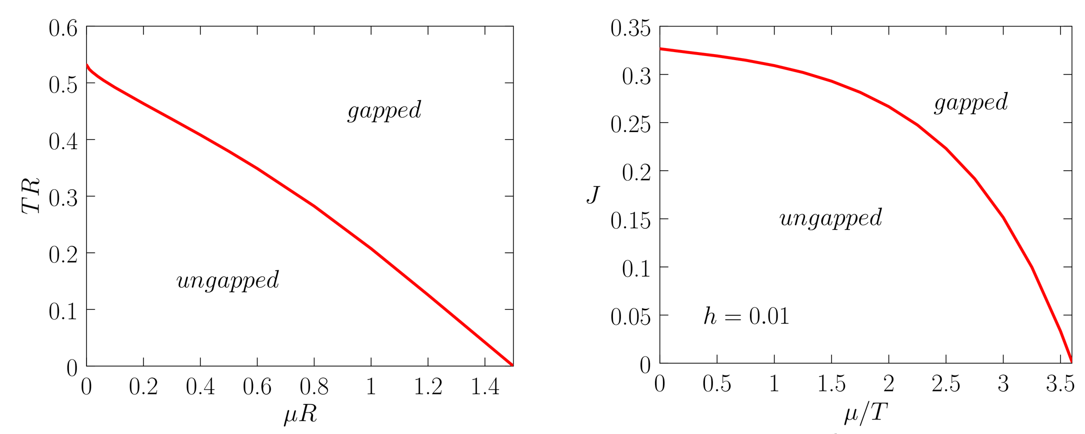


The Lagrange multiplier represents the effective quark number, $\mathcal{N} = \frac{1}{N_c^2}N_q$, since from (4)

$$\begin{aligned} \mathcal{N} &= \frac{1}{N_c}\sum_{i=1}^{N_c}\sum_{n=1}^\infty \left(\alpha_{-n}z_i^n - \alpha_n z_i^{-n}\right) \\ &\xrightarrow[N_c\to\infty]{} \frac{1}{N_c^2}\frac{1}{\beta}\frac{\partial \log Z}{\partial \mu} . \end{aligned}$$

$\mathcal{N}$ in the strongly coupled lattice theory is obtained from (14) and (15) using the transformations in (3) and is plotted (left, $h = 0.01$).

Phase diagrams



The line of transitions is given by the value of TR at which $\mathcal{N}$ becomes nonzero for weakly-coupled QCD on $S^1 \times S^3$ (left, $\frac{N_f}{N_c} = 1$, $mR = 0$), or the J value at which this happens for the lattice strong coupling theory (right, $h = 0.01$).

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