Functional Renormalization Group analysis of relativistic Bose-Einstein Condensation

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Introduction and motivation

The role of interactions in the phenomenon of Bose-Einstein Condensation (BEC) is a nontrivial fundamental question. In the nonrelativistic case, it was shown [1] that nonperturbative effects associated with interactions with nonzero momentum exchange affect significantly the critical parameters for condensation, with implications for cold atom physics. Here, we consider a relativistic context, with the application to pion BEC in isospin-dense QCD [2] media in mind. Besides its phenomenological motivation, the problem of relativistic BEC in isospin dense QCD represents a particularly interesting framework to develop efficient nonperturbative methods for dense systems: being Sign-Problem free, a robust reference from lattice simulations at finite isospin chemical potential is in principle available.

As a second-order phase transition, dominated thus by zero modes, the phenomenon of BEC is particularly suited for the implementation of a Functional Renormalization Group (FRG) analysis [3]. To isolate the physical problem of BEC, we consider a toy model of a complex scalar field theory with U(1) symmetry at finite density. We concentrate then on analyzing in detail how the FRG and the different possible approximations that must complement it perform in this case.

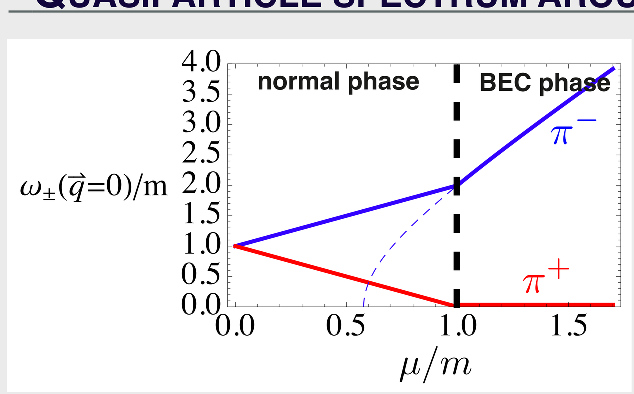
Effective Theory

We consider a complex scalar field $\pi=(\pi^1+i\pi^2)/\sqrt{2}$ with real mass ($m^2>0$) and U(1) symmetry at finite temperature and density. The euclidean action S$_E$, with the partition function given by $Z=\mathrm{Tr}\exp(-S_E)$, including the U(1) charge conservation constraint is:

$$S_E=\int d^4x\left\{-(\partial_0+i\mu)\pi^*(\partial_0-i\mu)\pi+(\nabla\pi^*)\cdot(\nabla\pi)+m^2\pi^*\pi+\lambda^2(\pi^*\pi)^2\right\},$$
$$=\int d^4x\Big\{\frac{1}{2}\partial_\nu\pi^1\partial_\nu\pi^1+\frac{1}{2}\partial_\nu\pi^2\partial_\nu\pi^2+\mu\left[\pi^1\partial_0\pi^2-\pi^2\partial_0\pi^1\right]+$$
$$+\frac{(m^2-\mu^2)}{2}[(\pi^1)^2+(\pi^2)^2]+\frac{\lambda^2}{4}[(\pi^1)^2+(\pi^2)^2]^2\Big\}$$

Comparing the latter form with the problem of the linear sigma model at finite isospin density, one can see that this is a simplified version of it in which we consider only the charged pion directions and $-\lambda^2v^2\mapsto m^2$, with v^2 being the minimum of the chiral potential.

- Quasiparticle spectrum around the condensate:



Indeed, the toy model result for the spectrum of fluctuations around the condensate resembles that of an isospin dense chiral model, featuring a BEC of π^+ for $\mu>m$.

Functional RG [3]

The FRG formalism provides a systematic framework for the inclusion of nonperturbative effects in field theory calculations. This is implemented via the construction of a set of effective actions $\Gamma_\kappa[\pi]$ which interpolate between the classical action $S[\pi]$ and the full effective action $\Gamma[\pi]$, i.e. a flow in the effective action space parameterized by κ.

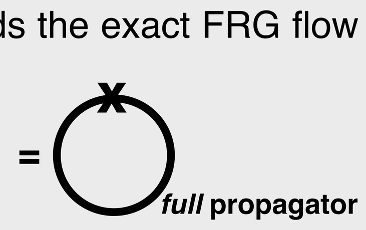
One interesting and convenient definition of an exact flow is that of the original theory with a modified propagator which suppresses IR modes with $q\lesssim\kappa$, satisfying thus:

$\kappa=0$	all quantum fluctuations are included ($\Gamma_{\kappa=0}=\Gamma$)
Intermediate κ	IR modes ($q^2\lesssim\kappa^2$) suppressed; UV fluctuations ($q^2>\kappa^2$) included
$\kappa=\Lambda\to\infty$	all fluctuations are suppressed; physics is classical ($\Gamma_{\kappa=\Lambda}=S$)

- Exact RG flow:

A standard procedure of deforming the theory yields the exact FRG flow equation for the κ-dependent effective action:

$$\partial_\kappa\Gamma_\kappa[\pi^i]=\frac{1}{2}\mathrm{Tr}\left[\partial_\kappa R_\kappa\left[\mathcal{G}^{-1}+R_\kappa\right]^{-1}\right]\quad=$$



where R_κ is the regulator and the *full* 2-point function appears:

$$\left[\mathcal{G}^{-1}(Q)\right]^{ij}=\frac{\delta^2\Gamma_\kappa}{\delta\pi^i(Q)\delta\pi^j(-Q)}$$

This flow equation actually encodes an infinite hierarchy of coupled exact RG equations involving the n–point functions.

- Truncations:

An adequate truncation or approximation is needed to render the exact FRG flow a practical nonperturbative method. Common choices are the derivative expansion and the disconsideration of higher order n-point functions, which are often combined, as we shall do in what follows.

Our aim here is to discuss the performance of different choices in the context of relativistic BEC in hot and dense systems.

Results in LPA and discussion

The Local Potential Approximation (LPA) is the lowest order of a derivative expansion for the effective action. Our LPA ansatz has the form of the classical euclidean action, but with κ-dependent parameters:

$$\Gamma_\kappa=\int d^4x\Big\{\frac{1}{2}(\partial_\nu\pi^i)(\partial_\nu\pi^i)+\mu_\kappa\left[\pi^1\partial_0\pi^2-\pi^2\partial_0\pi^1\right]+V_\kappa(\alpha)\Big\}$$

$$V_\kappa(\alpha)=\mathcal{V}_\kappa+\frac{(m_\kappa^2-\mu_\kappa^2)}{2}\alpha+\frac{\lambda_\kappa^2}{4}\alpha^2\qquad\qquad\alpha=(\pi^1)^2+(\pi^2)^2$$

which is also truncated beyond the 4-point function.

Inserting this ansatz in the flow equation and solving integrals and Matsubara sums, we obtain (for Litim regulator: $R_\kappa(\vec{q}^2)=(\kappa^2-\vec{q}^2)\theta(\kappa^2-\vec{q}^2)$)

$$\boxed{\partial_\kappa V_\kappa=\frac{\kappa^4}{12\pi^2}\sum_\pm\left\{\left(\pm\frac{\mu_\kappa^2}{w_\kappa}+1\right)\frac{1}{\sqrt{E_\pm^2}}\left[1+2\,n_b\left(\sqrt{E_\pm^2}\right)\right]\right\}}$$

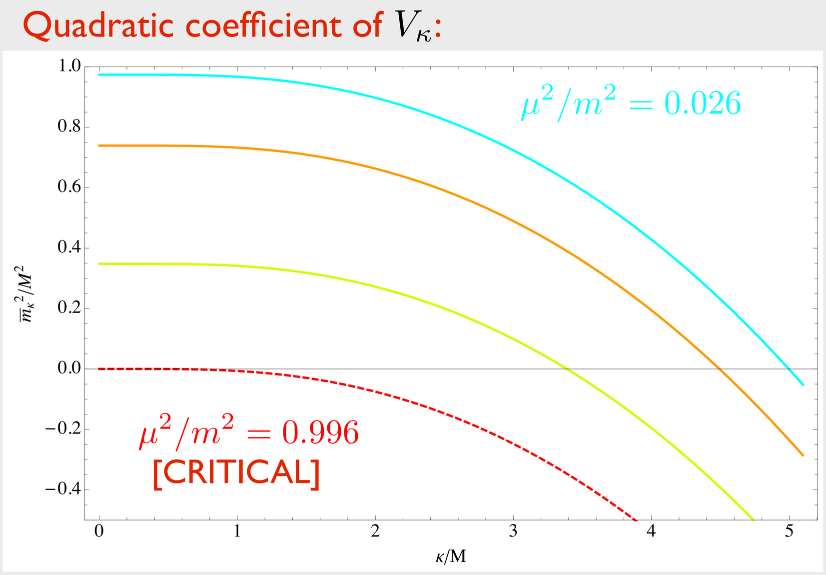
with n_b being the Bose distribution and: $\begin{cases}w_\kappa=\sqrt{\mu_\kappa^2(\kappa^2+2V_\kappa')+(\mu_\kappa^2+\alpha V_\kappa'')^2}\\ E_\pm^2=\kappa^2+2\alpha V_\kappa''+2V_\kappa'+2\mu_\kappa^2\pm2w_\kappa\end{cases}$

In this ansatz, we cannot access the flow of the chemical potential independently of that of the mass, so we set it constant and end up with 3 coupled differential equations.

- Cold and dense flow:

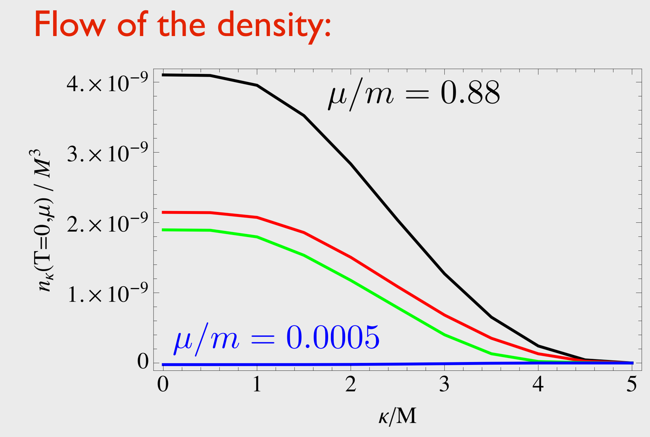
The second-order BEC transition occurs when the quadratic coefficient of the effective potential flows to zero at the physical theory (k=0).

Quadratic coefficient of V_κ:

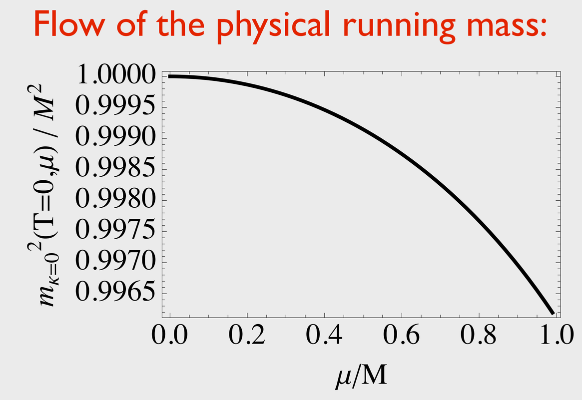


The critical chem. potential is decreased by fluctuations wrt mean-field result. This is due to running mass effects, triggered by interactions (cf. below). Condensation occurs at the *physical* running mass.

Flow of the density:

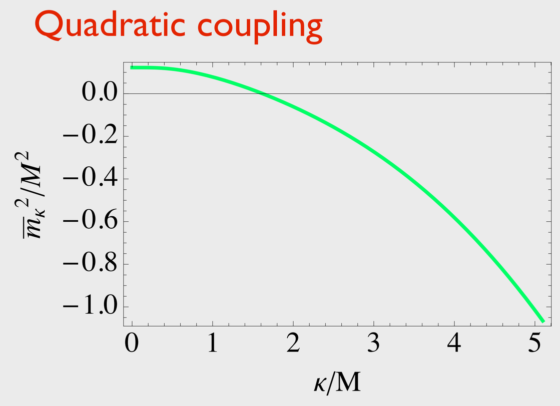


Flow of the physical running mass:

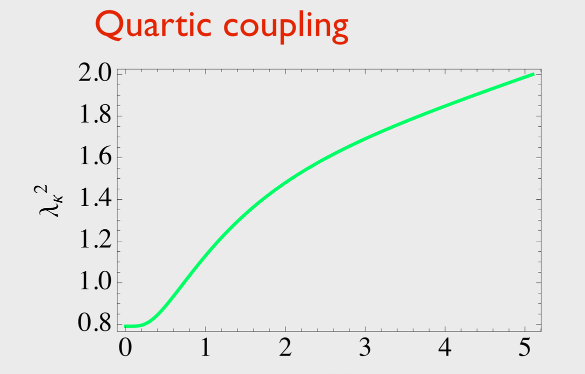


- Thermal flow:

Quadratic coupling

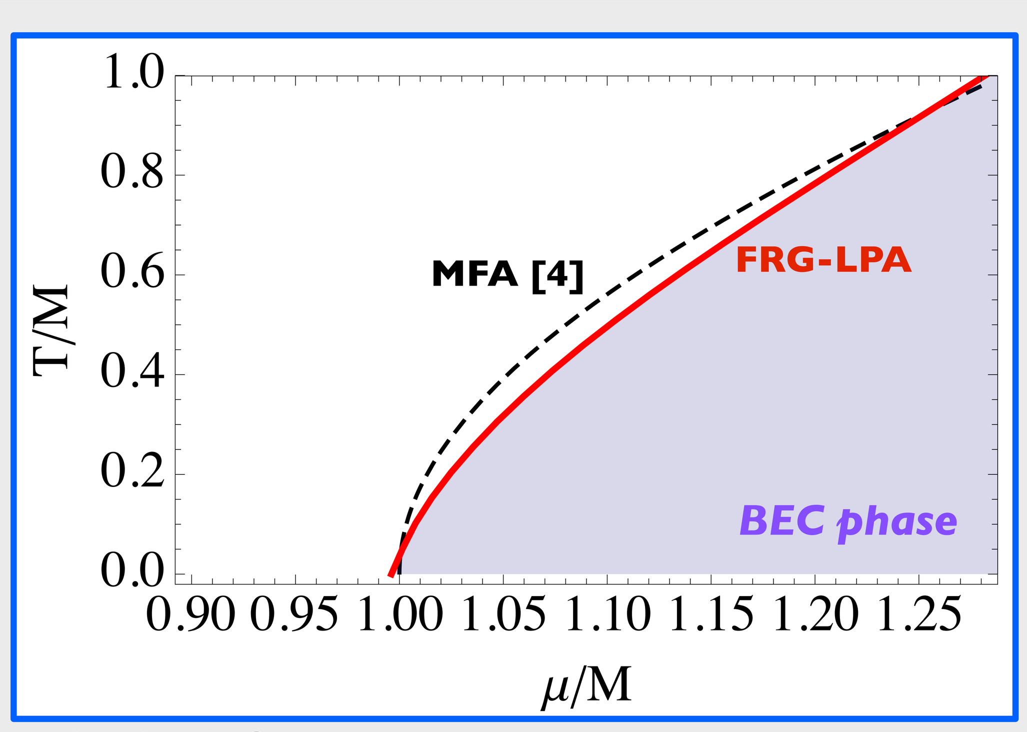


Quartic coupling

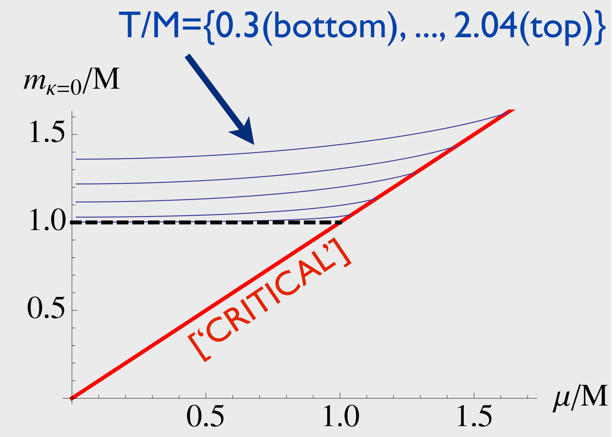


Flow at $T/M\approx0.51$ and $\mu^2/M^2\approx1.04$: $\bar{m}_\kappa^2$ versus κ (left) and λ_κ^2 versus κ (right).

- Phase diagram for BEC: mean-field *versus* FRG-LPA



- The second-order BEC transition point is shifted due to mass modifications in the medium (these are present even at 1-loop level).
- Possible LPA artefacts at low and high densities which we intend to investigate beyond LPA.
- Same features observed in chiral model [5].



Towards LPA': momentum exchange effects

In order to incorporate the effect from momentum-dependent interactions in our computation of nonperturbative corrections, we go one step further in the derivative expansion for the effective action, including the κ–dependent wavefunction renormalization Z$_\kappa$ in our ansatz. This improved truncation scheme is the so-called LPA' [3,6].

Our LPA' ansatz for the effective action includes different wavefunction renormalizations in the spatial and temporal directions and may feature a running chemical potential:

$$\Gamma_\kappa=\int d^4x\Big\{\frac{Z_{0\kappa}}{2}\partial_0\pi^a\partial_0\pi^a+\frac{Z_\kappa}{2}\partial_i\pi^a\partial_i\pi^a+\mu_\kappa\left[\pi^1\partial_0\pi^2-\pi^2\partial_0\pi^1\right]+V_\kappa(\alpha)\Big\}$$

The flow of the wavefunction renormalizations are then obtained via the adequate projection of the momentum dependence of the full 2-point function, e.g.:

$$Z_\kappa=\frac{1}{\beta^2}\lim_{Q^2\to0}\frac{\partial}{\partial\vec{q}^2}\left.\frac{\delta^2\Gamma_\kappa}{\delta\Delta^1(-Q)\delta\Delta^1(Q)}\right|_{\Delta=0}$$

Here, Δ is a small inhomogeneous fluctuation around the π condensate.

The solution of the set of coupled flow equations obtained in this way is currently work in progress.

Final remarks

We have investigated the role of interactions on the BEC transition at finite temperature and densities in a complex scalar field theory with U(1) symmetry using the nonperturbative framework of the FRG.

Our results [7] within the Local Potential Approximation are consistent with what was obtained previously in a chiral model and we believe that the relativistic BEC physics is fully present (and clean) in our toy model. We show that the shifts of the critical parameters can be directly related to medium-modification of the physical mass, present also at zero temperature (and even in perturbative results).

It is clear how one may render, through sensible approximations, the FRG formalism a powerful tool to address in a nonperturbative fashion the phase structure of in-medium field theories. The adequacy of approximations and truncations is, however, a subtle system-dependent issue. Our formal aim in the on-going part of this work is to scrutinize the results provided by FRG and its approximations within a sufficiently simple theory containing a physically motivating phenomenon (BEC), gaining understanding of how this nonperturbative flow implements nontrivial contributions and what are the limitations of the approximations used. In the near future, results beyond the leading order in the derivative expansion [7] should eventually answer whether the puzzling features found in the description of the T – μ phase-diagram are physical or rather artefacts of the LPA ansatz.

The generalization of the results to a QCD chiral model for pion condensation at finite isospin density should in principle be straightforward.

Acknowledgements

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