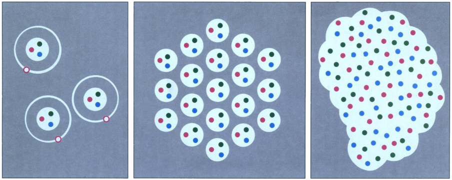
2 Flavor Formulation for Strong Coupling LQCD

Wolfgang Unger, ETH Zürich
with Philippe de Forcrand, ETH Zürich/CERN
SEWM 2012, Swansea

10.07.2012

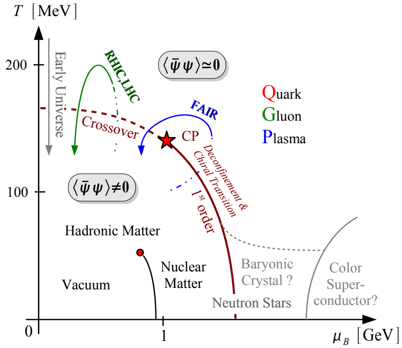


Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

1. Motivation for Strong Coupling LQCD in Continuous Time
   - Continuous Time Limit and $a/a_t = f(\gamma)$
   - Continuous Time Partition Function $Z(\beta)$
2. Hamiltonian Formulation
   - Spin Representation
   - Stochastic Series Expansion
3. Application: 2 flavor SC-LQCD
   - Generalization of Spin Representation
   - Preliminary Results

Why Strong Coupling Lattice QCD?

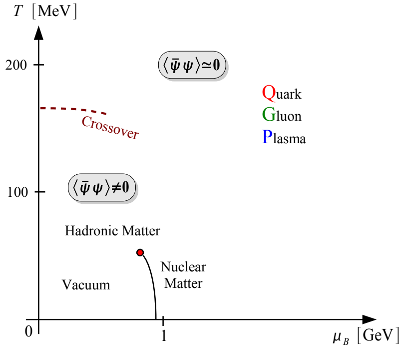
The QCD (μ, T) **phase diagram**:



- rich phase structure conjectured
- chiral and deconfinement transition
- QGP at high temperatures
- exotic matter at high density

Why Strong Coupling Lattice QCD?

The QCD (μ, T) **phase diagram**:



- because of the sign problem: very little is known
- so far: agreement on crossover temperature T_c at zero density

QCD has a severe **sign problem** for non-zero chemical potential $\mu = \frac{1}{3}\mu_B$:

- fermions anti-commute:

$$\gamma_5(i\not{p}+m+\mu\gamma_0)\gamma_5 = (i\not{p}+m+\mu\gamma_0)^\dagger$$

- the **fermion determinant** $\det M(\mu)$ becomes **complex!**

$$e^{-S_f} = \det M(\mu) = \overline{\det M(-\bar{\mu})}$$

- little hope that it can be circumvented:
  - Taylor expansion,
  - imaginary μ with analytic continuation,
  - reweighting method

  are all **limited to small** $\mu/T \lesssim 1$

Why Strong Coupling Lattice QCD?

Look at Lattice QCD in a regime where the **sign problem** can be made mild:

Strong Coupling Limit: $\beta = \frac{2N_c}{g^2} \rightarrow 0$

- allows to integrate out the gauge fields completely, as **link integration factorizes**
  $\Rightarrow$ no fermion determinant
- drawback: strong coupling limit is converse to asymptotic freedom, lattice is maximally coarse

Strong coupling LQCD shares important features with QCD:

- exhibits **confinement**, i.e. only color singlet degrees of freedom survive:
  - **mesons** (represented by monomers and dimers)
  - **baryons** (represented by oriented self-avoiding loops)
- and **spontaneous chiral symmetry breaking/restoration:** (restored at T_c)
  $\Rightarrow$ SC-LQCD is a great laboratory to study the full (μ, T) **phase diagram**

SC-LQCD is a useful toymodel for **nuclear matter**

Why Strong Coupling Lattice QCD?

Look at Lattice QCD in a regime where the **sign problem** can be made mild:
Strong Coupling Limit: $\beta = \frac{2N_c}{g^2} \to 0$

- allows to integrate out the gauge fields completely, as **link integration factorizes**
  $\Rightarrow$ no fermion determinant
- drawback: strong coupling limit is converse to asymptotic freedom, lattice is maximally coarse

Strong coupling LQCD shares important features with QCD:

- exhibits **confinement**, i.e. only color singlet degrees of freedom survive:
  - **mesons** (represented by monomers and dimers)
  - **baryons** (represented by oriented self-avoiding loops)
- and **spontaneous chiral symmetry breaking/restoration:** (restored at T_c)
  $\Rightarrow$ SC-LQCD is a great laboratory to study the full (μ, T) **phase diagram**

SC-LQCD is a useful toymodel for **nuclear matter**

SC-LQCD is a **1-parameter deformation of QCD**

SC-LQCD at finite temperature

How to vary the temperature?

- $aT = 1/N_\tau$ is discrete with N_τ even
- $aT_c \simeq 1.5$, i.e. ${N_\tau}^c < 2$ $\Rightarrow$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings:

$$\mathcal{Z}(m_q,\mu,\gamma,N_\tau) = \sum_{\{k,n,\ell\}} \prod_{b=(x,\mu)} \frac{(3-k_b)!}{3!k_b!}\gamma^{2k_b\delta_{\mu 0}} \prod_x \frac{3!}{n_x!}(2am_q)^{n_x} \prod_\ell w(\ell,\mu)$$

$$k_b \in \{0,\ldots N_c\},\; n_x \in \{0,\ldots N_c\}$$

Should we expect $a/a_\tau = \gamma$, as suggested at weak coupling?

- **No**: meanfield predicts $a/a_\tau = \gamma^2$, since $\gamma_c^2 = N_\tau \frac{(d-1)(N_c+1)(N_c+2)}{6(N_c+3)}$

  $\Rightarrow$ sensible, N_τ-independent definition of the temperature: $aT \simeq \frac{\gamma^2}{N_\tau}$

- Moreover, SC-LQCD partition function is a function of γ^2

However: **precise correspondence between a/a_τ and γ^2 not known**

SC-LQCD at finite Temperature and Continuous Time:

Strategy for **unambiguous** answer: the **continuous Euclidean time limit** (CT-limit):

$$N_\tau \to \infty, \qquad \gamma \to \infty, \qquad \gamma^2/N_\tau \equiv aT \quad \text{fixed}$$

- same as in analytic studies: $a_\tau = 0$, $aT = \beta^{-1} \in \mathbb{R}$

SC-LQCD at finite Temperature and Continuous Time:

Strategy for **unambiguous** answer: the **continuous Euclidean time limit** (CT-limit):

$$N_\tau \to \infty, \qquad \gamma \to \infty, \qquad \gamma^2/N_\tau \equiv aT \quad \text{fixed}$$

- same as in analytic studies: $a_\tau = 0$, $aT = \beta^{-1} \in \mathbb{R}$

Several **advantages** of continuous Euclidean time approach:

- ambiguities arising from the functional dependence of observables on the anisotropy parameter will be circumvented, **only one parameter** setting the temperature
- no need to perform the continuum extrapolation $N_\tau \to \infty$
- allows to estimate critical temperatures more precisely, with a faster algorithm (about 10 times faster than $N_t = 16$ at T_c)
- baryons become static in the CT-limit, the **sign problem is completely absent!**

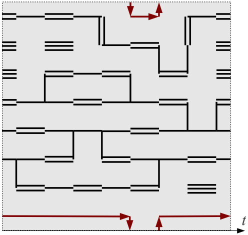
Continuous Time Partition Function, 1 flavor

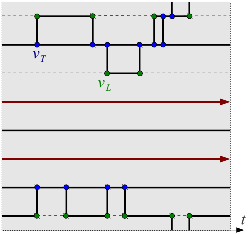
Partition function in inverse temperature $\beta = 1/aT$ and in the **chiral limit**:

$$\mathcal{Z}(\beta,\mu) = \sum_{\kappa\in 2\mathbb{N}} \frac{(\beta/2)^\kappa}{\kappa!} \sum_{\mathcal{C}\in\Gamma_\kappa} v_L^{n_L(\mathcal{C})} v_T^{n_T(\mathcal{C})} e^{3\beta\mu B(\mathcal{C})}, \quad v_L = 1,\; v_T = 2/\sqrt{3}$$

Typical (2-dimensional) configurations in discrete and continuous time at the same temperature:

- multiple spatial dimer become resolved into **single spatial dimers** as $a_t \to 0$
- **baryons** become **static** in continuous time!





- $\kappa = \frac{1}{2}\sum_x n_L(x) + n_T(x)$ is the **number of spatial dimers**, B is baryon number
- weight of configuration given by number of spatial dimers and **vertices** v_L, v_T **regardless of time coordinates**

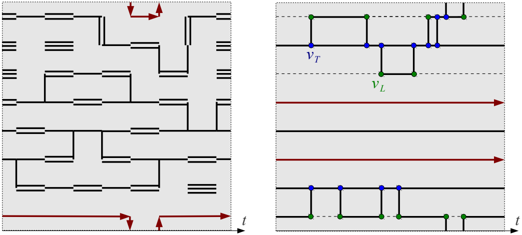
Continuous Time Partition Function, 1 flavor

Partition function in inverse temperature $\beta = 1/aT$ and in the **chiral limit**:

$$\mathcal{Z}(\beta,\mu) = \sum_{\kappa \in 2\mathbb{N}} \frac{(\beta/2)^\kappa}{\kappa!} \sum_{\mathcal{C}\in\Gamma_\kappa} v_L^{n_L(\mathcal{C})} v_T^{n_T(\mathcal{C})} e^{3\beta\mu B(\mathcal{C})}, \quad v_L = 1,\ v_T = 2/\sqrt{3}$$

Typical (2-dimensional) configurations in discrete and continuous time at the same temperature:

- multiple spatial dimer become resolved into **single spatial dimers** as $a_t \to 0$
- **baryons** become **static** in continuous time!



- sum over all spatial dimer time coordinates $\sim N_\tau/2 \Rightarrow$ **expansion in** $\beta = N_\tau/\gamma^2$
- Γ_κ is the set of equivalence classes of configurations with κ spatial dimers, time coordinates of spatial dimers irrelevant

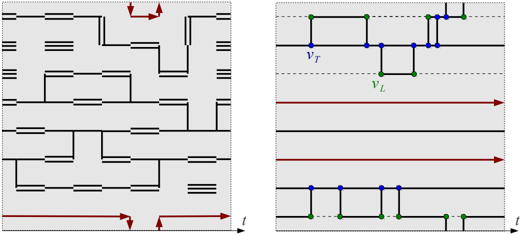
Continuous Time Partition Function, 1 flavor

Partition function in inverse temperature $\beta = 1/aT$ and in the **chiral limit**:

$$\mathcal{Z}(\beta,\mu) = \sum_{\kappa \in 2\mathbb{N}} \frac{(\beta/2)^\kappa}{\kappa!} \sum_{\mathcal{C}\in\Gamma_\kappa} v_L^{n_L(\mathcal{C})} v_T^{n_T(\mathcal{C})} e^{3\beta\mu B(\mathcal{C})}, \quad v_L = 1,\ v_T = 2/\sqrt{3}$$

Typical (2-dimensional) configurations in discrete and continuous time at the same temperature:

- multiple spatial dimer become resolved into **single spatial dimers** as $a_t \to 0$
- **baryons** become **static** in continuous time!

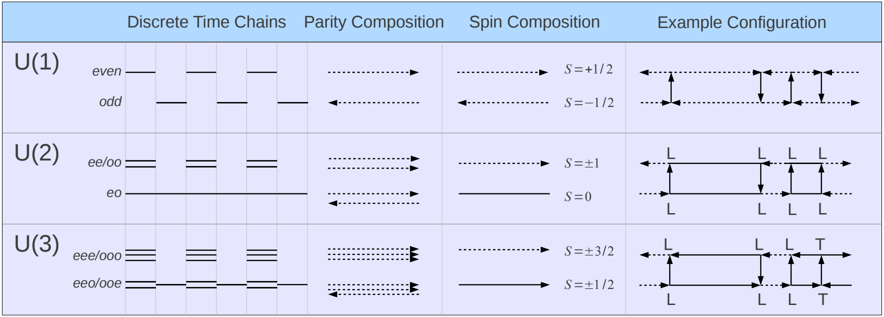


- each term Γ_κ is represented by a **world line configuration**
- allows to apply QMC techniques: **continuous time worm** (Beard & Wiese), loop cluster algorithm (Evertz et al.), **stochastic series expansion** (Sandvik)

Mapping of 1-flavor SU(N_c) to a spin system

Continuous time methods can be applied to any gauge group SU(N_c):

- baryons become static for $N_c \geq 3$
- mesonic discrete time chains **classified by parity:**



⇒ mesonic CT line types **classified by "spin"**: $S = -N_c/2 \ldots N_c/2$
(remnant of staggered even/odd ordering), $\Delta S = \pm 1$ (absorption/emission)

- generalizes to arbitrary $U(N_c)$

Stochastic Series Expansion

Idea: rewrite partition function, based on decomposition in diagonal and non-diagonal elements $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, truncation L:

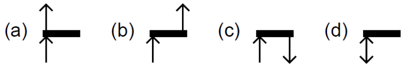
$$\mathcal{Z}(\beta) = \mathrm{Tr}\left\{e^{-\beta\mathcal{H}}\right\} = \sum_{\chi}\sum_{S_L} \frac{\beta^\kappa (L-\kappa)!}{L!}\left\langle \chi \left| \prod_{i=1}^{L} \mathcal{H}_{a_i,b_i} \right| \chi \right\rangle, \qquad \begin{array}{l} \mathcal{H}_{1,b} = \varepsilon\mathbb{1},\ \varepsilon \geq 0 \\ \mathcal{H}_{2,b} = \frac{1}{2} S_x^+ S_y^- \end{array}$$

with S_L a **time-ordered sequence** of operator-indices: $S_L = [a_1, b_1], [a_2, b_2], \ldots [a_L, b_L]$

- $a_i = 0$: identity, $a_i = 1, 2$: diagonal/non-diagonal matrix element
- $b_i = \langle x, y \rangle \in Vd$ denotes a bond

Two kinds of **updates:**

1. **changing order in β,** $\kappa \mapsto \kappa \pm 1$:
   $P([1,b]_p \mapsto [0,0]_p) = \frac{L-\kappa+1}{Vd\beta\langle\chi|\mathcal{H}_{1,b}|\chi\rangle}, \quad P([0,0]_p \mapsto [1,b]_p) = \frac{Vd\beta\langle\chi|\mathcal{H}_{1,b}|\chi\rangle}{L-\kappa}$
2. **operator loop:** visit bonds b_i successively from an input leg, determine output leg with heatbath probability $\langle \chi_x \chi_y | \mathcal{H}_{a_i b_i} | \chi'_x \chi'_y \rangle$



L is set larger than $\kappa_{\max}$ $\Rightarrow$ **SSE is approximation free** (like CT-Worm)

SSE applied to SC-LQCD

Strong Coupling $U(1)$ is identical to **XY Model** in zero field!
Extension to $U(N_c)$ for SC-LQCD straightforward:

$$\mathcal{H} = \frac{1}{2} \sum_{\langle x,y \rangle} J_x^+ J_y^-$$

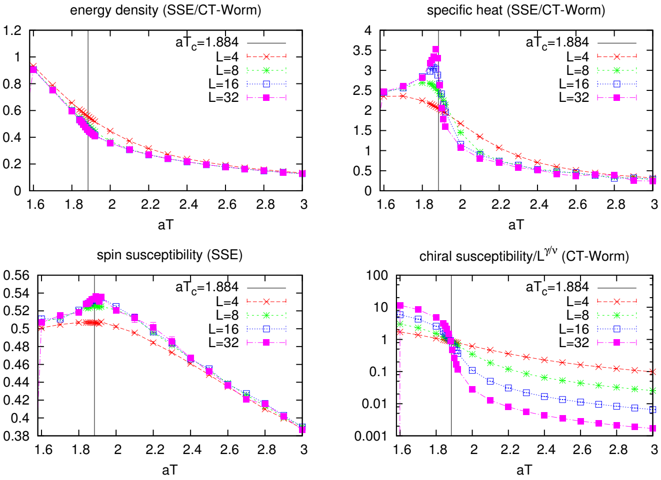
with $J^+ = \begin{pmatrix} 0 & & & & \\ v_1 & 0 & & & \\ & v_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & v_{N_c} & 0 \end{pmatrix}, \quad v_k = \sqrt{\frac{k(1+N_c-k)}{N_c}}$

and $J^- = (J^+)^T$ for absorption/emission

- $N_c = 3: \;\; v_L \equiv v_1 = v_3 = 1, \;\; v_T \equiv v_2 = 2/\sqrt{3}$
- state vector characterizing time slice:
  $$|S^z\rangle(t) \in \left\{ \bigotimes_{\vec{x}\in V} S^z_{\vec{x}} | S^z_{\vec{x}} \in \{-N_c/2, \dots N_c/2\} \right\}$$
- **oriented spatial dimers** act at time t_i on $|S_x\rangle$ by raising/lowering spin at absorption/emission site
- lowest/highest weight: $\quad J^+|N_c/2\rangle = 0, \quad J^-|-N_c/2\rangle = 0$
- S^z counts net number of (odd-even) time like meson sites at each site
- $\frac{N_c}{2}[J^+, J^-] = J^z = \mathrm{diag}(-N_c/2, \dots, N_c/2)$ fulfilled, $\quad J^z|S^z\rangle = S^z|S^z\rangle$
- new observable: **spin susceptibility** $\chi_S = \beta \left\langle (\sum_i S^z_i)^2 \right\rangle / N$

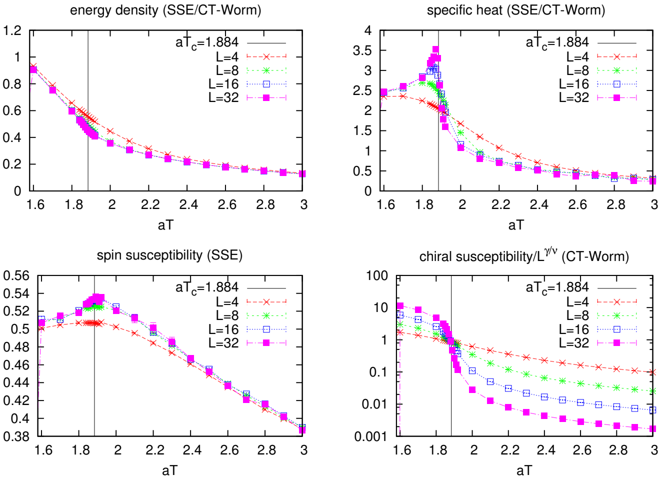
Observables for the Chiral Transition

- CT-Worm: energy density/specific heat, chiral susceptibility
- SSE: energy density/specific heat, "spin" susceptibility



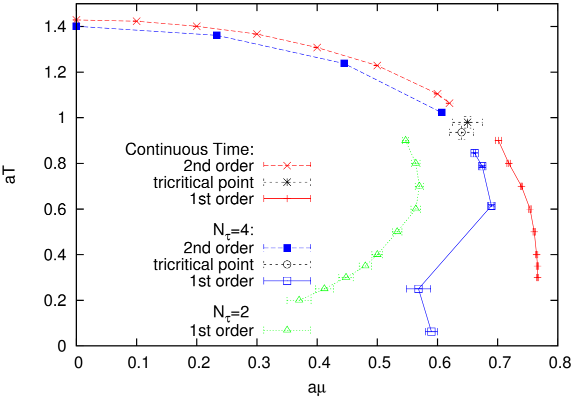
Observables for the Chiral Transition

- CT-Worm: energy density/specific heat, chiral susceptibility
- SSE: energy density/specific heat, **"spin" susceptibility feels chiral transition**



1-flavor SC-QCD Phase Diagram

Comparison of phase diagram in continuous time with $N_\tau = 4$ data (M. Fromm, 2010) via identification $a\mu = \gamma^2 a_\tau \mu$, studied via Worm algorithm [hep-lat/1111.1434]:

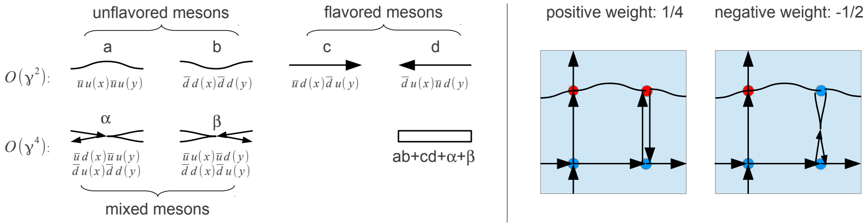


- behavior at low μ agrees well, location of TCP agrees within errors
- no re-entrance seen at small aT (also confirmed by Ohnishi *et. al*, LAT2012)

Application: Generalization of SC-LQCD to 2 chiral flavors!

Aim: obtain phase diagram for 2-flavor SC-LQCD, where **pion exchange** may play a crucial role for nuclear transition, but:

- at present, no 2-flavor formulation for staggered SC-LQCD suitable for MC
- already the mesonic sector has a severe **sign problem** (worse than for finite μ HMC)
- 2 new types of mixed dimers give negative sign in mesonic loops already for U(2):



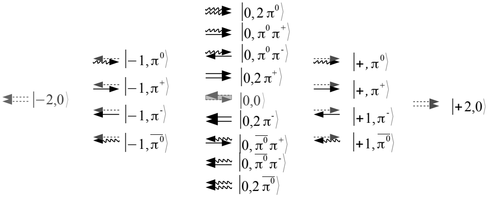
Observation in **continuous time** formulation:

- **static lines** for 2 staggered flavors have all **postive weight!**
- again: only single spatial dimers (no α, β spatial dimers)
- Hamiltonian formulation feasible

Continuous Time Transition Rules for $N_{\mathrm{f}} = 2$

Flavored static lines:

- new classification in terms of **quantum numbers** $|S^z, Q_{\pi^0}, Q_{\pi^+}\rangle$
- $N_{\mathrm{c}} = 2$: in total 19 types of lines, 18 have weight 1/4 per a_t, “vacuum state“ $|0,0,0\rangle$ has weight $1/\sqrt{8}$



- state multiplicities $= N_{\mathrm{c}} + 1 + \frac{2}{3} N_{\mathrm{c}}(N_{\mathrm{c}}+1)(N_{\mathrm{c}}+2)$, 44 states for $N_{\mathrm{c}} = 3$
- “spin” S^z counts number of emission/absorption events (remnant of even/odd decomposition of lattice for staggered fermions) $S^z = -\frac{1}{2} N_{\mathrm{c}} N_{\mathrm{f}}, \ldots, +\frac{1}{2} N_{\mathrm{c}} N_{\mathrm{f}}$
- two “charges” $Q_i = -N_{\mathrm{c}}, \ldots, +N_{\mathrm{c}}$ denote the flavor content
- **spin/charge conservation:** transitions at spatial dimers, raising charges at one site, lowering at a neighboring site:

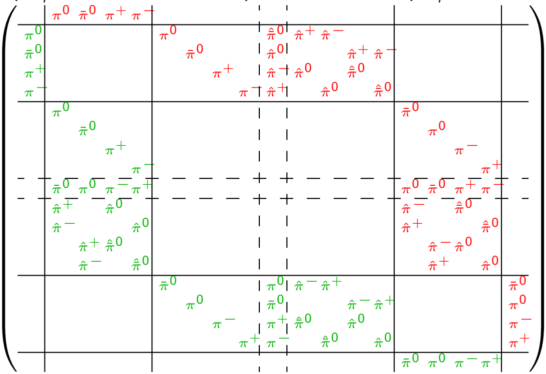
$$|\Delta S^z| = 1, \quad |\Delta Q_{\pi^0}| + |\Delta Q_{\pi^+}| = 1$$

Hamiltonian for $N_\mathrm{f} = 2$

$$\mathcal{H} = \frac{1}{2}\sum_{\langle x,y\rangle}\left(J^+_{\pi^0(x)}J^-_{\pi^0(y)} + J^+_{\hat\pi^0(x)}J^-_{\hat\pi^0(y)} + J^+_{\pi^+(x)}J^-_{\pi^+(y)} + J^+_{\pi^+(x)}J^-_{\pi^-(y)}\right)$$

Absorption ($J^+_{\pi_i}$, lower left triangle) and Emission ($J^-_{\pi_i}$, upper triangle), state vector:

$J^{+/-}_{\pi_i} =$



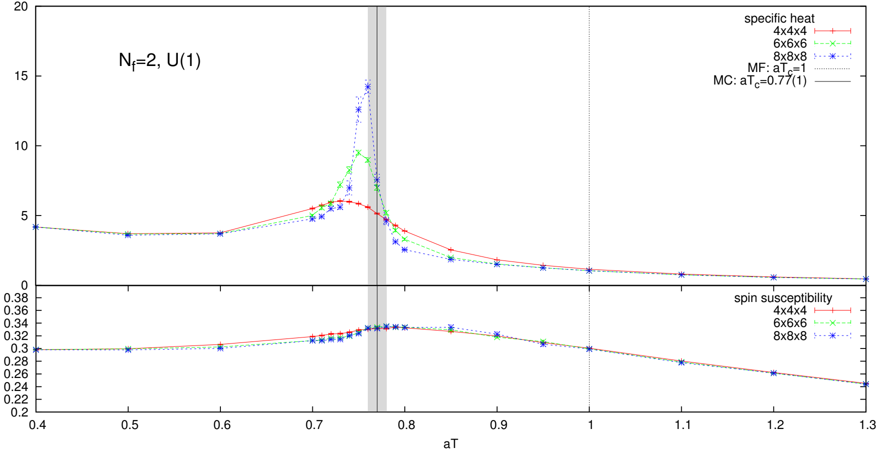
$$\chi = \begin{pmatrix} -2,\,0 \\ -1,\,\pi^0 \\ -1,\,\hat\pi^0 \\ -1,\,\pi^+ \\ -1,\,\pi^- \\ 0,\,2\pi^0 \\ 0,\,2\hat\pi^0 \\ 0,\,2\pi^+ \\ 0,\,2\pi^- \\ 0,\,0 \\ 0,\,\pi^0\pi^+ \\ 0,\,\pi^0\pi^- \\ 0,\,\hat\pi^0\pi^+ \\ 0,\,\hat\pi^0\pi^- \\ +1,\,\pi^0 \\ +1,\,\hat\pi^0 \\ +1,\,\pi^+ \\ +1,\,\pi^- \\ +2,\,0 \end{pmatrix}$$

- vertex weights are $v_{\pi_i} = 1$ for vertices not mixing the two charges Q_i and $v_{\hat\pi_i} = \frac{1}{\sqrt{2}}$ for vertices mixing the charges

Preliminary Result for 2-flavor SC-LQCD

Comparison of aT_c from MC data with mean field:

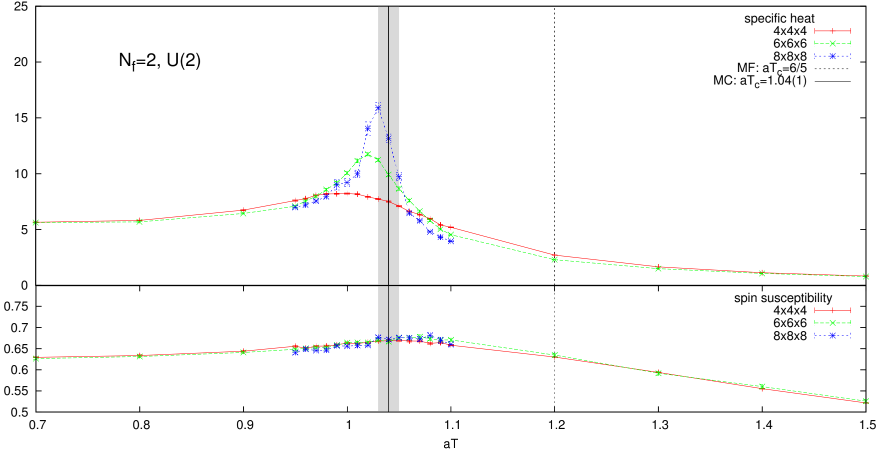
$N_{\rm f}$	$N_{\rm c}=1$	$N_{\rm c}=2$	$N_{\rm c}=3$
1	3/2 [1.102(1)]	4/2 [1.467(1)]	5/2 [1.884(1)]
2	**5/5 [0.77(1)]**	6/5 [1.04(1)]	7/5



Preliminary Result for 2-flavor SC-LQCD

Comparison of aT_c from MC data with mean field:

N_f	$N_\mathrm{c} = 1$	$N_\mathrm{c} = 2$	$N_\mathrm{c} = 3$
1	3/2 [1.102(1)]	4/2 [1.467(1)]	5/2 [1.884(1)]
2	5/5 [0.77(1)]	**6/5 [1.04(1)]**	7/5



Summary

Achievements:

- CT partition function: new formulation as a **quantum spin system!**
- "spin" formulation and Hamiltonian follow from conservation laws for even/odd chains of time-like dimers and flavors - this **generalizes to arbitrary N_c, N_f**
- new observable: spin susceptibility, sensitive to chiral transition
- quantum Monte Carlo applicable: e.g. continuous time worm or **stochastic series expansion** (most convenient)
- now also applied to U(2) with two flavors (incorporates pion exchange)
- extension to SU(3) with finite baryon chemical potential straightforward (Hamiltonian worked out, but no simulations yet)

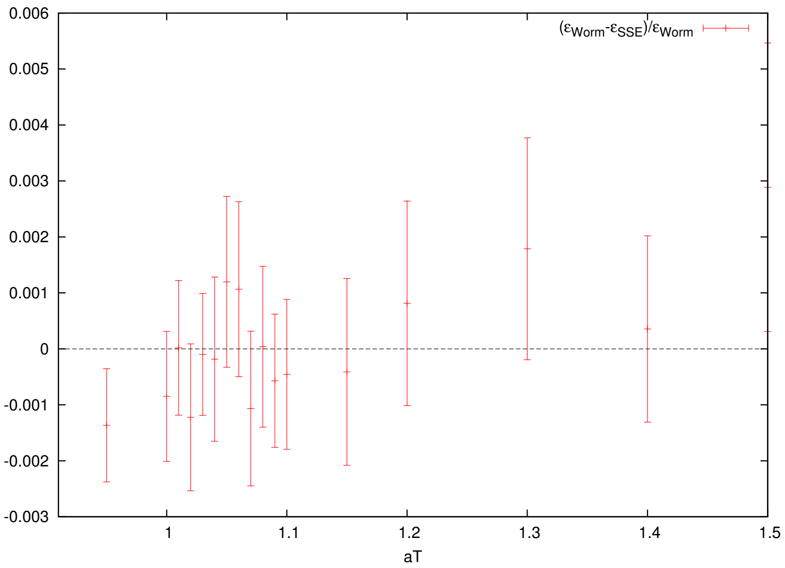
Summary

Achievements:

- CT partition function: new formulation as a **quantum spin system!**
- "spin" formulation and Hamiltonian follow from conservation laws for even/odd chains of time-like dimers and flavors - this **generalizes to arbitrary N_c, N_f**
- new observable: spin susceptibility, sensitive to chiral transition
- quantum Monte Carlo applicable: e.g. continuous time worm or **stochastic series expansion** (most convenient)
- now also applied to U(2) with two flavors (incorporates pion exchange)
- extension to SU(3) with finite baryon chemical potential straightforward (Hamiltonian worked out, but no simulations yet)

Soon: obtain full 2-flavor SC-LQCD phase diagram in (μ, T)-plane!

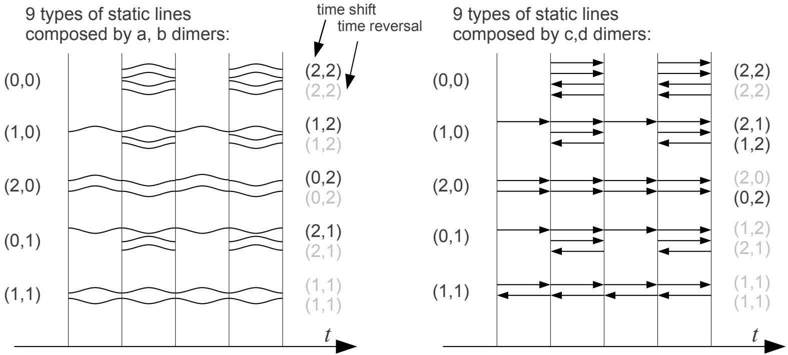
Comparison of SSE and CT-Worm



Static Line Rules

Combine temporal dimers of alternating orders in γ^2 (here for $N_c = 2$):

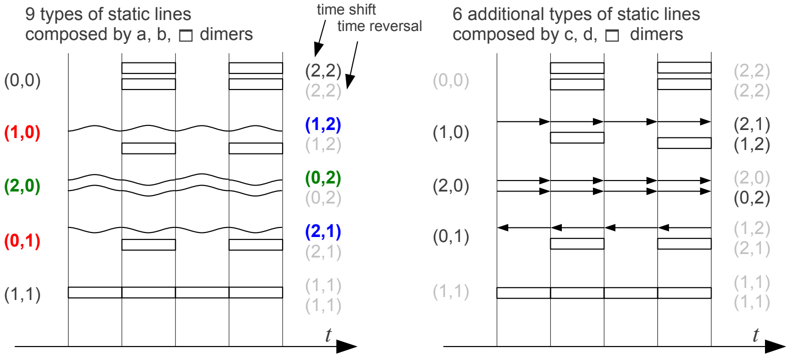
- first: consider (a, b) and (c, d) dimers sepaterely
- then: resum them to obtain flux representation



Static Line Rules

Combine temporal dimers of alternating orders in γ^2 (here for $N_c = 2$):

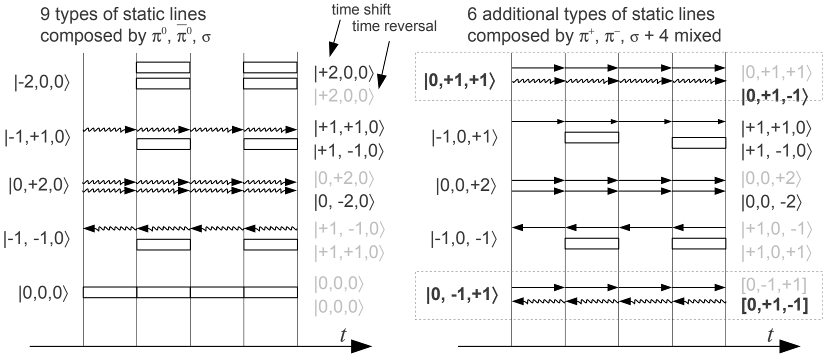
- resummation of $a - b - a$ and $b - a - b$ chains
- resummation of $ab + cd + \alpha + \beta$ into $\square$ dimers



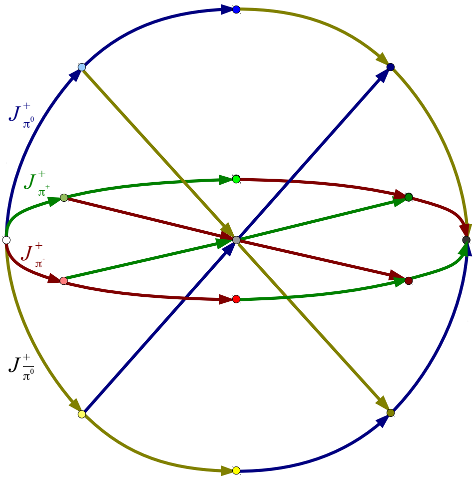
Static Line Rules

Combine temporal dimers of alternating orders in γ^2 (here for $N_c = 2$):

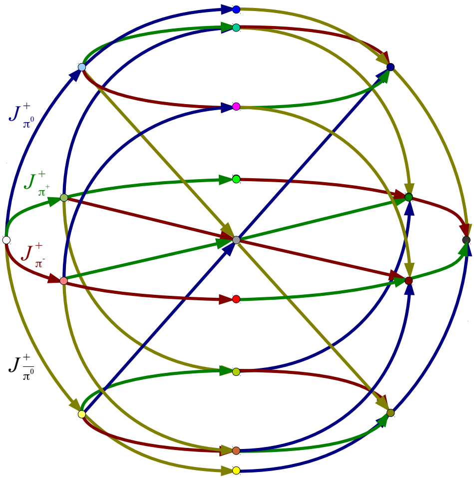
- new classification in terms of **quantum numbers** $|S^z, Q_{\pi^0}, Q_{\pi^+}\rangle$
- in total: 19 types of lines, 18 have weight $1/16$ per $2a$, $|0,0,0\rangle$ has weight $1/8$



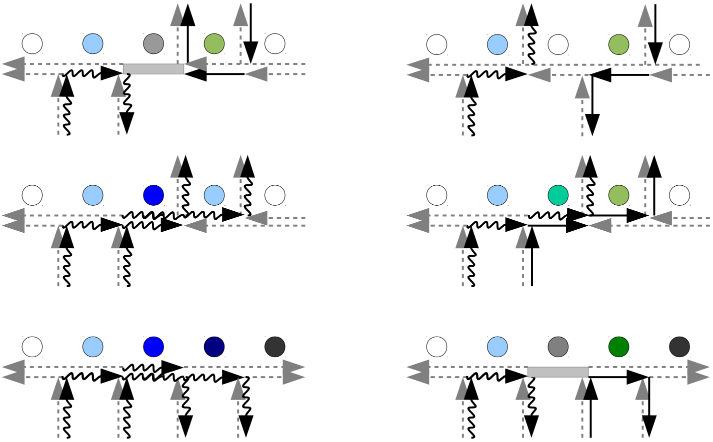
The Transition Rules Encoded in $J^{\pm}$



The Transition Rules Encoded in $J^{\pm}$



The Transition Rules Encoded in $J^{\pm}$



Why Study Strong Coupling QCD on the Lattice?

Two possible scenarios for the relation between
SC-LQCD (back) and the (L)QCD phase diagram for four flavors (front):