Exact results in supersymmetric field theories and holography

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Based on arXiv:1503.05537 with Assel, Cassani, Di Pietro, Komargodski, Lorenzen†

and previous work by subsets of authors

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Outline

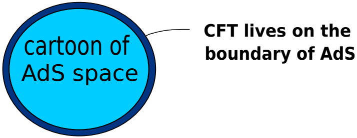
1. Introduction and motivations
2. Rigid supersymmetry
3. Localization of path integrals in four dimensions
4. Supersymmetric index and Casimir energy
5. Supersymmetric Casimir energy from quantum mechanics

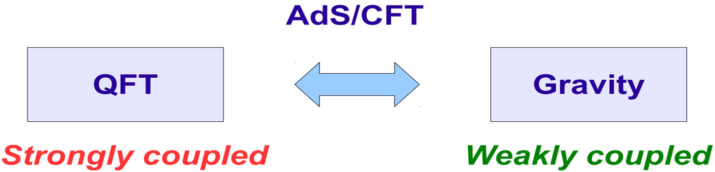
Motivations

- We are interested in exact (and preferably analytic) non-perturbative results in (strongly coupled) quantum field theories
- In dimension higher than 2 the most powerful tools are probably
  - Holography
  - Localization
- When both can be applied, it is instructive to compare them

Gauge/gravity duality

Equivalence between (quantum) **gravity** in bulk space-times and **quantum field theories** on their boundaries





Localization

[See S. Murthy's talk]

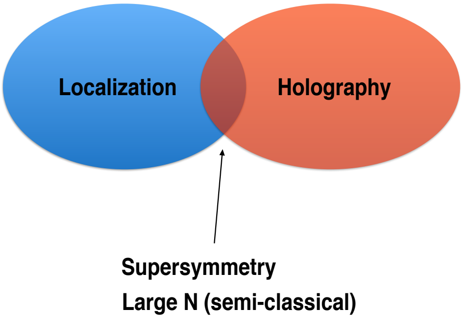
- For certain supersymmetric field theories defined on (compact) curved Riemannian manifolds the path integral may be computed exactly
- Localization: functional integral over all fields of a theory $\rightarrow$ integral/sum over a reduced set of field configurations
- Saddle point around a supersymmetric locus gives the exact answer
- A priori the path integral ("partition function" $\mathbf{Z}$) depends on the parameters of the theory and of the background geometry

Uses of localization

- Partition functions on $\mathbf{S^2}$ and $\mathbf{S^4}$ compute exact Kähler potential on the space of marginal deformations of certain supersymmetric CFTs
- When the manifold $\mathbf{M_d}$ is of the form $\mathbf{S^1} \times \mathbf{M_{d-1}}$ the path integral may be interpreted as an index Tr $\mathbf{(-1)^F\, e^{-(operator)}}$, "counting" states in the field theory (Hamiltonian formalism). In this case the name "partition function" is more appropriate: think of $\mathbf{S^1}$ as compactified time
- Indices and other partition functions may be used to test conjectured non-perturbative Seiberg(-like) dualities
- Quantum entropy of black holes [See S. Murthy's talk]

... many more ...

Localization vs holography



The simplest "observable" to be compared is the partition function, through

$$e^{-S_{\text{supergravity}}[M_{d+1}]} = Z[M_d = \partial M_{d+1}]$$

where the supergravity action is evaluated on a solution $\mathbf{M_{d+1}}$ with conformal boundary $\mathbf{M_d}$, on which the supersymmetric QFT is defined.

Four dimensional $\mathcal{N} = 1$ supersymmetric field theories

- Here we focus on $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{1}$ supersymmetric gauge theories with matter
- Vector multiplet: gauge field $\mathcal{A}$; Weyl spinor λ; auxiliary scalar $\mathbf{D}$, all transforming in the adjoint representation of a group $\mathbf{G}$
- Chiral multiplet: complex scalar ϕ; Weyl spinor ψ; auxiliary scalar $\mathbf{F}$, all transforming in a representation $\mathcal{R}$ of the group $\mathbf{G}$
- In flat space with Lorentzian signature, supersymmetric Lagrangians containing these fields are textbook material
- A first caveat in Euclidean space is that degrees of freedom in multiplets are *a priori* doubled: $\lambda^\dagger \to \tilde{\lambda}$, $\phi^\dagger \to \tilde{\phi}$, etcetera, where tilded fields are regarded as independent

Supersymmetry and Lagrangians (flat space)

- For example, the supersymmetry transformations of the fields in the vector multiplet are

$$\delta \mathcal{A}_\mu = i\zeta\sigma_\mu\widetilde{\lambda} \qquad \delta D = -\zeta\sigma^\mu D_\mu\widetilde{\lambda}$$

$$\delta\lambda = \mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}\zeta + iD\zeta \qquad \delta\widetilde{\lambda} = 0$$

  where ζ is a constant spinor parameter, $D_\mu = \partial_\mu - i\mathcal{A}_\mu\cdot$, and $\mathcal{F}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$

- The supersymmetric Yang-Mills Lagrangian reads

$$\mathcal{L}_{\text{vector}} = \text{tr}\left[\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}D^2 + i\widetilde{\lambda}\,\widetilde{\sigma}^\mu D_\mu\lambda\right]$$

- Similarly, there are supersymmetry transformations and supersymmetric Lagrangians for the fields in the chiral multiplet

Rigid supersymmetry on curved manifolds

- One can try to define supersymmetric field theories on curved manifolds: clearly $\partial_\mu \to \nabla_\mu$, but this is not enough
- The supersymmetry transformations and Lagrangians must be modified. [Witten]: "twist" $\mathcal{N} = 2$ SYM $\to$ supersymmetric on arbitrary Riemannian manifod
- Local supersymmetry studied since long time ago $\to$ supergravity
- [Festuccia-Seiberg]: take supergravity with some gauge and matter fields and appropriately throw away gravity $\to$ "rigid limit"
- Important: in the process of throwing away gravity, some extra fields of the supergravity multiplet remain, but are non-dynamical $\to$ background fields

Rigid new minimal supersymmetry

- For $\mathbf{d = 4}$ field theories with an $\mathbf{R}$ symmetry, one can use (Euclidean) new minimal supergravity [Sohnius-West]. Gravitini variations:

$$\delta\psi_\mu \sim (\nabla_\mu - iA_\mu)\,\zeta + iV_\mu\zeta + iV^\nu\sigma_{\mu\nu}\zeta = 0$$

$$\delta\tilde{\psi}_\mu \sim (\nabla_\mu + iA_\mu)\,\tilde{\zeta} - iV_\mu\tilde{\zeta} - iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta} = 0$$

- A_μ, V_μ are background fields and ζ, $\tilde{\zeta}$ are supersymmetry parameters

- Existence of ζ or $\tilde{\zeta}$ is equivalent to Hermitian metric [Klare-Tomasiello-Zaffaroni], [Dumitrescu-Festuccia-Seiberg]

- The supersymmetry transformations of the vector multiplet are

$$\delta\mathcal{A}_\mu = i\zeta\sigma_\mu\widetilde{\lambda} + i\widetilde{\zeta}\,\widetilde{\sigma}_\mu\lambda$$

$$\delta\lambda = \mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}\zeta + iD\zeta \qquad \delta\widetilde{\lambda} = \mathcal{F}_{\mu\nu}\,\widetilde{\sigma}^{\mu\nu}\widetilde{\zeta} - iD\widetilde{\zeta}$$

$$\delta D = -\zeta\sigma^\mu\left(D_\mu\widetilde{\lambda} - \tfrac{3i}{2}V_\mu\widetilde{\lambda}\right) + \widetilde{\zeta}\,\widetilde{\sigma}^\mu\left(D_\mu\lambda + \tfrac{3i}{2}V_\mu\lambda\right)$$

where $D_\mu = \nabla_\mu - i\mathcal{A}_\mu\cdot - iq_R A_\mu$

Localization on four-manifolds: strategy outline

[Assel-Cassani-DM]

- Work in Euclidean signature and start with generic background fields $\mathbf{A}_\mu$, $\mathbf{V}_\mu$ associated to a Hermitian manifold
- Construct δ-exact Lagrangians for the vector and chiral multiplets $\rightarrow$ set-up localization on a general Hermitian manifold
- Focus on manifolds with topology $\mathbf{M_4} \simeq \mathbf{S^1} \times \mathbf{S^3}$ (these admit a second spinor $\tilde{\zeta}$ with opposite R-charge)
- Prove that the localization locus is given by gauge field $\mathcal{A}_\tau =$ constant, with all other fields $(\lambda, \mathbf{D}; \phi, \psi, \mathbf{F})$ vanishing
- Partition function reduces to a matrix integral over $\mathcal{A}_\tau$ $\rightarrow$ integrand is infinite product of 3d super-determinants

Localizing Lagrangians and saddle point equations

- The bosonic parts of the localizing terms constructed with ζ only are

$$\mathcal{L}^{(+)}_{\text{vector}} = \operatorname{tr}\left(\frac{1}{4}\mathcal{F}^{(+)}_{\mu\nu}\mathcal{F}^{(+)\,\mu\nu} + \frac{1}{4}D^2\right)$$

$$\mathcal{L}_{\text{chiral}} = \left(g^{\mu\nu} - iJ^{\mu\nu}\right) D_\mu\widetilde{\phi}D_\nu\phi + \widetilde{F}F$$

- With the obvious reality conditions on the fields, $\mathcal{A}, D$ Hermitian, $\widetilde{\phi} = \phi^\dagger$, $\widetilde{F} = F^\dagger$, we obtain the saddle point equations

$$\begin{aligned} \textbf{vector :} \qquad & \mathcal{F}^{(+)}_{\mu\nu} = 0\,, \qquad D = 0 \\ \textbf{chiral :} \qquad & J^\mu{}_\nu D^\nu\widetilde{\phi} = iD^\mu\widetilde{\phi}\,, \quad F = 0 \end{aligned}$$

Hopf surfaces

- A Hopf surface is essentially a four-dimensional complex manifold with the topology of $S^1 \times S^3$. It can be described as a quotient of $\mathbb{C}^2 - (0,0)$, with coordinates z_1, z_2 identified as

$$(z_1, z_2) \sim (pz_1, qz_2)$$

  where p, q are in general two complex parameters

- We show that on a Hopf surface we can take a very general metric

$$ds^2 = \Omega^2 d\tau^2 + f^2 d\rho^2 + m_{IJ} d\varphi_I d\varphi_J \qquad I, J = 1, 2$$

  while preserving two spinors ζ and $\widetilde{\zeta}$

- τ is a coordinate on S^1, while the 3d part has coordinates $\rho, \varphi_1, \varphi_2$, describing S^3 as a T^2 fibration over an interval

The matrix model

- The localizing locus simplifies drastically, e.g. $\to \mathcal{F}^{(+)} = \mathcal{F}^{(-)} = \mathbf{0} \to$ full contribution comes from zero-instanton sector! Flat connections $\mathcal{A}_\tau =$ constant, and all other fields vanishing

- The localized path integral is reduced an infinite products of $\mathbf{d = 3}$ super-determinants, that may be computed explicitly using the method of pairing of eigenvalues [Hama et al], [Alday et al]

- Using a "natural" regularisation prescription for infinite products, formulas for elliptic gamma functions, we obtain (more on the regularisation prescription later!)

$$\mathbf{Z}^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho \in \Delta_{\mathcal{R}}} \prod_{\mathbf{n} \in \mathbb{Z}} \mathbf{Z}^{\text{chiral}}_{\text{1-loop (3d)}} \left[\sigma_0^{(\mathbf{n},\rho)}\right]$$

$$\to \; e^{i\pi \Psi^{(0)}_{\text{chi}}} e^{i\pi \Psi^{(1)}_{\text{chi}}} \prod_{\rho \in \Delta_{\mathcal{R}}} \Gamma_{\mathbf{e}} \left(e^{2\pi i \rho_{\mathcal{A}_0}} (\mathbf{pq})^{\frac{\mathbf{r}}{2}}, \mathbf{p}, \mathbf{q} \right)$$

Supersymmetric index

- Adding the contribution of the vector multiplet, everything combines nicely into the following formula

$$\mathbf{Z}[\mathcal{H}_{\mathbf{p},\mathbf{q}}] \;=\; \mathrm{e}^{-\mathcal{F}(\mathbf{p},\mathbf{q})}\,\mathcal{I}(\mathbf{p},\mathbf{q})$$

where $\mathcal{I}(\mathbf{p},\mathbf{q})$ is the supersymmetric index with $\mathbf{p},\mathbf{q}$ fugacities

$$\mathcal{I}(\mathrm{p},\mathrm{q}) \;=\; \frac{(\mathrm{p};\mathrm{p})^{\mathrm{r_G}}(\mathrm{q};\mathrm{q})^{\mathrm{r_G}}}{|\mathcal{W}|}\int_{\mathrm{T^{r_G}}}\frac{\mathrm{dz}}{2\pi\mathrm{iz}}\prod_{\alpha\in\Delta_+}\theta\left(\mathrm{z}^{\alpha},\mathrm{p}\right)\theta\left(\mathrm{z}^{-\alpha},\mathrm{q}\right)\prod_{\mathrm{J}}\prod_{\rho\in\Delta_{\mathrm{J}}}\Gamma_{\mathrm{e}}\left(\mathrm{z}^{\rho}(\mathrm{pq})^{\frac{\mathrm{r_J}}{2}},\mathrm{p},\mathrm{q}\right)$$

which may be defined as a sum over states as

$$\mathcal{I}(\mathbf{p},\mathbf{q}) \;=\; \mathrm{Tr}[(-1)^{\mathsf{F}}\mathbf{p}^{\mathsf{J}+\mathsf{J}'-\frac{\mathsf{R}}{2}}\mathbf{q}^{\mathsf{J}-\mathsf{J}'-\frac{\mathsf{R}}{2}}]$$

- The fact that the index is computed by the localized path integral on a Hopf surface was anticipated by [Closset-Dumitrescu-Festuccia-Komargodski]

A curious pre-factor

- Localised path integral + "natural" regularisation produced a pre-factor $\mathcal{F}(\mathbf{p},\mathbf{q})$ explicitly given by ($\mathbf{p} \equiv \mathbf{e}^{-2\pi|\mathbf{b}_1|}$, $\mathbf{q} \equiv \mathbf{e}^{-2\pi|\mathbf{b}_2|}$)

$$\mathcal{F}(\mathbf{p},\mathbf{q}) = \frac{4\pi}{3}\left(|\mathbf{b}_1| + |\mathbf{b}_2| - \frac{|\mathbf{b}_1| + |\mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}\right)(\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27}\frac{(|\mathbf{b}_1| + |\mathbf{b}_2|)^3}{|\mathbf{b}_1||\mathbf{b}_2|}(3\,\mathbf{c} - 2\,\mathbf{a})$$

$$\mathbf{a} = \frac{3}{32}\left(3\,\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}\right), \qquad \mathbf{c} = \frac{1}{32}\left(9\,\mathrm{tr}\mathbf{R}^3 - 5\,\mathrm{tr}\mathbf{R}\right)$$

- Invariant depending only on complex structure of the manifold and anomaly coefficients of the QFT $\mathbf{a}, \mathbf{c}$ → expect to be a physical observable

"Who ordered that?"

Supersymmetric Casimir energy

- For simplicity, from now we focus on the case $\mathbf{p} = \mathbf{q} = \mathbf{e}^{-\beta}$
- In analogy with the ordinary ("zero temperature") Casimir energy, i.e. energy of the vacuum, we can define

$$\mathbf{E}_{\text{susy}} \equiv - \lim_{\beta \to \infty} \frac{\mathrm{d}}{\mathrm{d}\beta} \log \mathbf{Z}(\beta)$$

- Since $\mathbf{Z}(\beta) = \mathrm{e}^{-\mathcal{F}(\beta)} \mathcal{I}(\beta)$ and in the limit $\beta \to \infty$ the index $\mathcal{I}(\beta) \to$ constant, we see that only $\mathcal{F}(\beta)$ contributes
- $\mathcal{F}(\beta)$ captures a supersymmetric version of the Casimir energy
- However, this is very sensitive to the regularisation used! The regularisation in [ACM] yields the result

$$\mathbf{E}_{\text{susy}} = \frac{4}{27}(\mathbf{a} + 3\mathbf{c})$$

Is the supersymmetric Casimir energy unambiguous?

From its path integral definition

$$\mathbf{E}_{\mathbf{susy}} \equiv -\lim_{\beta\to\infty} \frac{\mathrm{d}}{\mathrm{d}\beta} \log \mathbf{Z}(\beta)$$

one may be worried that this is an ambiguous quantity

1. There could be finite counterterms, i.e. integrals of local densities that can be added to **log Z** shifting arbitrarily its value
2. The result can depend on the details of the regularisation prescription

The key is supersymmetry

- From a systematic analysis of counterterms in new minimal supergravity we proved that all finite supersymmetric counterterms on $\mathbf{S}^1 \times \mathbf{M}_3$ vanish [Assel-Cassani-DM.2]
- Make sure that the regularisation respects the relevant Ward identities [Assel,Cassani,Di Pietro,Lorenzen,Komargodski,DM]

Hamiltonian formalism

- An alternative, but equivalent, point of view is to consider the decompactified theory on $\mathbf{R} \times \mathbf{S}^3$ and perform canonical quantization

- The supersymmetric Casimir energy should be recovered as the vacuum expectation value of the (supersymmetric) Hamiltonian $\mathbf{H}_{\mathbf{susy}}$ that appears in the definition of the index [Kim-Kim]

$$\mathcal{I}(\beta) = \mathbf{Tr}[(-1)^{\mathbf{F}} \mathbf{e}^{-\beta \mathbf{H}_{\mathbf{susy}}}]$$

  namely

$$\mathbf{E}_{\mathbf{susy}} = \langle \mathbf{H}_{\mathbf{susy}} \rangle$$

- $\mathbf{H}_{\mathbf{susy}}$ is supersymmetric because $[\mathbf{H}_{\mathbf{susy}}, \mathbf{Q}] = \mathbf{0}$, where $\mathbf{Q}$ is the supercharge

- In [Lorenzen,DM] we showed that using a natural regularisation prescription

$$\langle \mathbf{H}_{\mathbf{susy}} \rangle = \frac{4}{27}(\mathbf{a} + 3\mathbf{c})$$

The Casimir energy is a subtle quantity

- Regularising infinite sums is tricky: slightly different prescriptions (e.g. different order in which operations are done) lead to different results!
- This is the case for the regularisation of determinants in [Assel,Cassani,DM], as well as of the sum giving $\langle \mathbf{H}_{\mathbf{susy}} \rangle$ in [Lorenzen,DM]
- More generally, this issue arises in any computation of 1-loop determinants, as those appeared in the several papers using localization
- It would be useful to understand what is the "correct" method to regularise in general $\longrightarrow$ the key is supersymmetry

Casimir energy in two dimensions

- For a 2d CFT defined on the cylinder $\mathbb{R} \times \mathbf{S^1}$ the (ordinary) Casimir energy is defined as

$$\mathbf{E_0 = \int_{S^1} \langle T_{tt} \rangle}$$

- It measures degrees of freedom, indeed one finds that is proportional to the central charge **c** [Cardy,...] [See also talks of J. David and A. O'Bannon]

$$\mathbf{E_0 = -\frac{c}{12 r_1}}$$

- This follows simply from starting with a theory on the plane, where $\langle \mathbf{T}_{\mu\nu} \rangle = \mathbf{0}$, and performing a conformal transformation

- In $\mathbf{d = 2}$ the only dimensionless counterterm that can be written is

$$\int \mathbf{d^2 x \sqrt{g} R}$$

  where **R** denotes the Ricci scalar. This vanishes on the cylinder and thus does not shift the vacuum energy → this is a physical quantity

Casimir energy in higher dimensional CFT's

- In higher dimensional conformal field theories, defined on the "cylinder" $\mathbb{R} \times \mathbf{S}^{2n+1}$, with $\mathbf{n} \geq \mathbf{1}$, one can similarly consider the vacuum energy

$$\mathbf{E_0} = \int_{\mathbf{S}^{2n+1}} \langle \mathbf{T_{tt}} \rangle$$

- A generalisation of the above rescaling trick allows to write an expression for the energy momentum tensor [Brown et al,...]
- In $\mathbf{d} = \mathbf{4}$ one finds $\mathbf{E_0} = \frac{3}{4r_3}\left(\mathbf{a} - \frac{\mathbf{b}}{2}\right)$, where $\mathbf{a}$ is the anomaly coefficient and $\mathbf{b}$ is a coefficient in $\langle \mathbf{T}^{\mu}_{\mu} \rangle = \frac{1}{(4\pi)^2}\left(\mathbf{aE_{(4)}} - \mathbf{cW^2} + \mathbf{b}\Box\mathbf{R}\right)$
- These depend ambiguously on the local counterterm $-\frac{b}{12(4\pi)^2}\int d^4x\sqrt{g}R^2$
- The Casimir energy in dimension $\mathbf{d} = \mathbf{2n} \geq \mathbf{4}$ is ambigous and therefore is not a physical observable of CFTs!

Back to the supersymmetric Casimir energy

- Now back to the supersymmetric the Casimir energy: which strategy?
- In the presence of background fields, $\mathbf{T}_{\mu\nu}$ is not conserved, so $\mathbf{E_0}$ is not even a conserved quantity in general
- Supersymmetry on $\mathbf{S^1} \times \mathbf{S^3}$ requires to have a fixed $\mathbf{A_t = 1/r_3}$: "no large gauge transformations allowed"
- This flat **A** changes dramatically the vacuum, i.e. it is not a "small deformation" of flat space: the rescaling trick fails
- Reduce to supersymmetric Quantum Mechanics [Assel,Cassani,Di Pietro,Lorenzen,Komargodski,DM]
- Supersymmetric Ward identity selects the regularisation prescription

Implications of the supersymmetry algebra

- Out of the four preserved supercharges we focus on two ($\mathbf{Q}$ and $\mathbf{Q}^\dagger$), whose algebra takes the form

$$\frac{1}{2}\{Q, Q^\dagger\} = H_{\rm susy} - \frac{1}{r_3}(R + 2J_3) \,, \qquad [H_{\rm susy}, Q] = [R + 2J_3, Q] \; = \; 0$$

- From this we deduce that on the vacuum we have the Ward identity

$$r_3\langle H_{\rm susy}\rangle = \langle R + 2J_3\rangle$$

- If we also use (we don't really have to) the other two supercharges it's immediate to see that $\langle J_3\rangle = 0$, and hence our Ward identity is simply the (vacuum!) "energy=charge" relation

$$r_3\langle H_{\rm susy}\rangle = \langle R\rangle$$

Reduction to SUSY QM

- As we are interested in the limit of $\beta \to \infty$, it's natural to do a KK reduction on the $\mathbf{S}^3$
- Supersymmetry implies that we get supersymmetric quantum mechanics for infinitely many degrees of freedom, organised in 1d supermultiplets
- The Ward identity implies that the term in the effective action that computes $\langle \mathbf{H}_{\mathbf{susy}} \rangle$ is

$$\mathcal{W} \sim \int \mathbf{dt} \left(\frac{1}{r_3} \sqrt{|g_{tt}|} + A_t^R \right) \quad (*)$$

  where $\mathbf{A}_t^R$ is the background gauge field associated to the $\mathbf{R}$ symmetry

- In QM (*) is a local term $\to$ looks like $\langle \mathbf{H}_{\mathbf{susy}} \rangle$ and of $\langle \mathbf{R} \rangle$ are ambiguous
- However, the quantum-mechanical term (*) cannot descend from a higher-dimensional counterterm and thus it is scheme independent

1d multiplets: long and short

- Since $\langle \mathbf{R} \rangle$ is computed by the CS coefficient of $\int \mathbf{dtA_t}$, the susy Casimir energy cannot depend on continuous coupling constants (and hence on the RG scale)

- It is sufficient to calculate the susy Casimir energy starting from the free field theory limit in 4d (we assume a Lagrangian exists)

- Focus on a free chiral multiplet in 4d $(\Phi, \Psi, \mathbf{F})$ and KK reduce to 1d $\rightarrow$ two types of multiplets: *chiral multiplet* (ϕ, ψ) and a *Fermi multiplet* $(\lambda, \mathbf{f})$

$$\begin{aligned} \textbf{chiral} \;:\quad & \delta\phi = \sqrt{2}\zeta\psi \,, & \delta\psi = -\sqrt{2}i\zeta^\dagger D_t\phi \\ \textbf{Fermi} \;:\quad & \delta\lambda = \sqrt{2}\zeta f + p\sqrt{2}\zeta^\dagger\phi \,, & \delta f = -\sqrt{2}i\zeta^\dagger D_t\lambda - p\sqrt{2}\zeta^\dagger\psi \end{aligned}$$

- When $\mathbf{p} = \mathbf{0}$ the chiral and Fermi multiplets are independent $\rightarrow$ short multiplets. When $\mathbf{p} \neq \mathbf{0}$ the two multiplets form one reducible but indecomposable representation of supersymmetry $\rightarrow$ long multiplets

Shortening conditions

The supersymmetric Lagrangian of a long multiplet takes the form

$$\begin{aligned} \mathbf{L} &= |\mathbf{D_t}\phi|^2 - \mathbf{i}\mu(\phi \mathbf{D_t}\phi^\dagger - \phi^\dagger \mathbf{D_t}\phi) + \mathbf{i}\psi^\dagger \mathbf{D_t}\psi - 2\mu\psi\psi^\dagger \\ &+ \mathbf{i}\lambda^\dagger \mathbf{D_t}\lambda + |\mathbf{f}|^2 - \mathbf{p}^2|\phi|^2 - \mathbf{p}(\lambda\psi^\dagger + \psi\lambda^\dagger) \end{aligned}$$

- Starting from 4d Lagrangian, we reduce to 1d expanding in harmonics. E.g.

$$\Phi = \sum_{\ell,\mathbf{m},\mathbf{n}} \phi_{\ell,\mathbf{m},\mathbf{n}} \mathbf{Y}_\ell^{\mathbf{m},\mathbf{n}}$$

- The parameter governing the shortening of the multiplets depends only on the quantum numbers $\ell, \mathbf{m}$ and reads

$$\mathbf{r}_3^2\mathbf{p}^2 = (\ell - 2\mathbf{m})(2 + \ell + 2\mathbf{m})$$

- When $\mathbf{p}^2 = \mathbf{0}$ the long multiplet becomes short and reduces to

- a 1d chiral multiplet for $\mathbf{m} = \ell/2$

- a 1d Fermi multiplet for $\mathbf{m} = -1 - \ell/2$

Canonical quantization

- The "oscillator" form of $\mathbf{H}_{\mathbf{susy}}$, **R**, **Q** are obtained straightforwardly
- Calculating the spectrum is a cute exercise in QM
- For the long multiplets we find

$$\langle \mathbf{H}_{\mathbf{long}} \rangle \;=\; \langle \mathbf{R}_{\mathbf{long}} \rangle \;=\; \mathbf{0}$$

- For the short multiplets we find

$$\begin{aligned}
&\textbf{chiral} \quad \left(\mathbf{m} = \frac{\ell}{2}\right): && \langle \mathbf{H}_{\mathbf{chiral}} \rangle \;=\; \frac{1}{2\mathbf{r}_3}(\ell + \mathbf{r}) \\
&\textbf{Fermi} \quad \left(\mathbf{m} = -\frac{\ell}{2} - 1\right): && \langle \mathbf{H}_{\mathbf{Fermi}} \rangle \;=\; -\frac{1}{2\mathbf{r}_3}(\ell + 2 - \mathbf{r})
\end{aligned}$$

A supersymmetric regularisation

The expectation value of the Hamiltonian is obtained by adding up the contributions of all chiral and Fermi multiplets

$$\mathbf{H}_{\mathbf{susy}} = \sum_{\ell,m,n} \mathbf{H}^{\mathbf{long}}_{\ell,m,n} + \sum_{\ell,m,n} \mathbf{H}^{\mathbf{chiral}}_{\ell,m,n} + \sum_{\ell,m,n} \mathbf{H}^{\mathbf{fermi}}_{\ell,m,n}$$

$$\Rightarrow \langle \mathbf{H}_{\mathbf{susy}} \rangle = \sum_{\ell \geq 0} \frac{1}{2r_3}(\ell+1)(\ell+r) - \sum_{\ell \geq 0} \frac{1}{2r_3}(\ell+1)(\ell+2-r)$$

- In order to preserve supersymmetry in every multiplet, the contributions of two types of short multiplets are regularised separately, e.g.

$$\rightarrow \lim_{s\to 0} \sum_{\ell \geq 0} \frac{1}{2r_3}(\ell+1)(\ell+r)^{-s+1} - \sum_{\ell \geq 0} \frac{1}{2r_3}(\ell+1)(\ell+2-r)^{-s+1}$$

Eventually:
$$\langle \mathbf{H}_{\mathbf{susy}} \rangle = \mathbf{E}_{\mathbf{susy}} = \frac{4}{27r_3}(a+3c)$$

Summary

The final result for the supersymmetric partition function is

$$\mathbf{Z}^{\text{susy}}_{\mathbf{S}^3\times \mathbf{S}^1_\beta} = \mathbf{e}^{-\frac{4\beta}{27\mathbf{r}_3}(\mathbf{a}+3\mathbf{c})}\mathcal{I}_{\mathbf{S}^3\times \mathbf{S}^1_\beta}$$

where $\mathcal{I}_{\mathbf{S}^3\times \mathbf{S}^1_\beta}$ is the usual supersymmetric index

The Casimir pre-factor should be important for modular-like properties, generalising the 2d case [Cardy]

There is a similar result also for the partition function with fugacities $\mathbf{p},\mathbf{q}$, with Casimir energy pre-factor being ($\mathfrak{b} \sim \log \mathbf{p}/\mathbf{q}$)

$$\mathbf{E}_{\text{susy}} = \frac{2}{3\mathbf{r}_3}\left(|\mathfrak{b}| + |\mathfrak{b}|^{-1}\right)(\mathbf{a}-\mathbf{c}) + \frac{2}{27\mathbf{r}_3}(|\mathfrak{b}| + |\mathfrak{b}|^{-1})^3(3\,\mathbf{c} - 2\,\mathbf{a})$$

Finally, we now have to go back to holography!

Comments on holography

- The standard holographically renormalised on shell action in AdS_5 does not agree with the supersymmetric Casimir energy
- It agrees with the non-supersymmetric computation in the round cylinder, as well as its infinitesimal deformations
- Adding an appropriate flat $\mathbf{A}$ necessary for supersymmetry on $\mathbf{S^1} \times \mathbf{S^3}$ does not affect the on shell action
- The problem is open: there are at least two logical alternatives
  1. There are new missing holographic boundary terms, that depend on $\mathbf{A}$, that would change the result when included
  2. There exist a different solution, which is asymptotically AdS_5 + appropriate flat $\mathbf{A}$, that has smaller on shell action, and hence dominates the semiclassical path integral

  ... or both .... or something else ...

Outlook

- Push the localization technique: how many more path integrals can we compute exactly and explicitly, and what can we learn from them? In dimensions $\mathbf{d} \geq \mathbf{4}$ rigid supersymmetry allows for large classes of geometries
- The supersymmetric Casimir energy may be defined for theories on $\mathbf{S^1} \times \mathbf{M}$ and it is a physical observable of a theory, generalising well-known results in $\mathbf{d} = \mathbf{2}$ [Cardy et el]. This also leads to revisit carefully the regularisation prescription on $\mathbf{S^1} \times \mathbf{M}$, and a priori more general localization calculations
- Supersymmetric localization yields very precise predictions for the gauge/gravity duality, allowing to perform detailed tests. Supergravity solutions should reproduce exactly numbers and functions. E.g. the supersymmetric Casimir energy should be reproduced!!
- This is forcing us to refine the holographic dictionary and to think about "why" computations on the two sides match $\rightarrow$ progress towards "proving" the gauge/gravity duality in large sectors

Extra slides

Conserved charges

- In the presence of background vector fields $\mathbf{A}^{\mathbf{I}}_{\mu}$ the energy-momentum tensor, defined as

$$\mathbf{T}_{\mu\nu} = \frac{-2}{\sqrt{-\mathbf{g}}} \frac{\delta \mathbf{S}}{\delta \mathbf{g}^{\mu\nu}}$$

is not conserved, but instead obeys the Ward identity

$$\nabla^{\mu} \mathbf{T}_{\mu\nu} = \sum_{\mathbf{I}} \left(\mathbf{F}^{\mathbf{I}}_{\mu\nu} \mathbf{J}^{\mu}_{\mathbf{I}} - \mathbf{A}^{\mathbf{I}}_{\nu} \nabla_{\mu} \mathbf{J}^{\mu}_{\mathbf{I}} \right)$$

where the currents $\mathbf{J}^{\mu}_{\mathbf{I}} = \frac{1}{\sqrt{-\mathbf{g}}} \frac{\delta \mathbf{S}}{\delta \mathbf{A}^{\mathbf{I}}_{\mu}}$ are not necessarily conserved

- However for any Killing symmetry ξ of the background one can define a conserved current

$$\mathbf{Y}^{\mu}_{\xi} \;=\; \xi_{\nu} \left(\mathbf{T}^{\mu\nu} + \sum_{\mathbf{I}} \mathbf{J}^{\mu}_{\mathbf{I}} \mathbf{A}^{\nu}_{\mathbf{I}} \right)$$

Supersymmetry algebra

- In particular, the canonical Hamiltonian takes the form

$$\mathbf{H} = \int_{S^3} d^3x \left(T^{tt} + J_R^t A^t - \frac{3}{2} J_{FZ}^t V^t \right)$$

- It receives contributions from the R-charge operator

$$\mathbf{R} = \int_{S^3} d^3x \, J_R^t$$

  as well as the Ferrara-Zumino charge (not conserved)

- In terms of the supercharge **Q**, one can compute explicitly the super-algebra

$$[H, Q] = -\mathfrak{q} Q \,, \qquad [R, Q] = Q \,, \qquad [J_3, Q] = -\frac{1}{2} Q \,,$$

$$\{Q, Q^\dagger\} = H + (1 + \mathfrak{q}) R + 2 J_3$$

$$\Rightarrow H = H_{susy} \text{ if and only if } \mathfrak{q} = 0$$