On Classical Solutions of
$4d$ Supersymmetric Higher Spin Theory

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Holography, Strings and Higher Spins

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Introduction and motivation

- In this talk I will study classical solutions of higher spin theories.
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- Black holes with all their properties: Horizon, temperature, entropy…
- They may be black branes, so similar to black holes, but of different dimensionality.
- If there are extra charges, the solutions can be extremal. And with SUSY they can be BPS. They can indicate the existence of solitonic objects in the theory, like D-branes.

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Holographically

- They can describe a thermal state.
- They can describe a pure state ("bubbling solutions").
- Should be able to perform calculations in those backgrounds instead of the usual AdS vacuum.

Higher spin theory

- We will work with the case of a 4d bulk theory.
- Rather complicated interacting theory of fields of arbitrary integer spin (brief review to come). [Vasiliev,...]
- May also include half integer spins.
- Can be consistently defined in asymptotically AdS_4 space (but not flat space).

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- It is known how to calculate correlation functions in the AdS_4 vacuum and match to field theory correlation functions. [Giombi,Yin][...]
- How would one describe thermal states. Or other ensambles?
- Is the theory complete, or are there more degrees of freedom? Extended objects?
- ...

Outline

- Introduction and motivation
- Lightning review of Vasiliev theory
- The Didenko-Vasiliev solution
- Embedding in SUSY theories
- Preserved SUSYs
  - Bulk
  - Boundary
- Summary

Review of (SUSY) Vasiliev theory

- The theory is defined by the equations of motion of the three master fields

W: The higher-spin connection, which is a space-time one-form. It containing the massless higher-spin gauge fields of spin $s \geq 2$, as well as auxiliary fields.

B: A space-time zero-form, which contains the curvature of the fields, such as the Weyl tensor and its higher-spin generalisation, as well as the massive scalar, massless fermion and Maxwell field.

S: is an auxiliary field introduced to turn on interactions. It is a space-time zero-from, but a one-form in Z-space

$$S = S_\alpha dz^\alpha + \bar{S}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}}$$

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- All the master-fields depend on the following variables

x: the space-time coordinates.

$y_\alpha, \bar{y}_{\dot{\alpha}}$: Bosonic spinors. Expanding in them gives the higher spin fields. Collectively denoted Y.

$z_\alpha, \bar{z}_{\dot{\alpha}}$: Introduced to turn on interactions in an explicitly gauge-invariant way. Collectively denoted Z.

ϑ^i: n SUSY parameters satisfying the Clifford algebra $\{\vartheta^i, \vartheta^j\} = 2\delta^{ij}$ turning every field into a 2^n component superfield ($n = 0$ is the bosonic theory). [Chang,Minwalla,Sharma,Yin]

If we combine $\mathcal{A} = W + S$, then the Vasiliev equations can be written in the compact form

$$\mathcal{F} \equiv d\mathcal{A} - \mathcal{A} \wedge_\star \mathcal{A} = -f_\star \left(B \star v\right) dz^2 - \bar{f}_\star \left(B \star \bar{v}\Gamma\right) d\bar{z}^2 \,,$$
$$dB - \mathcal{A} \star B + B \star \pi(\mathcal{A}) = 0 \,.$$

This requires the following definitions

- Multiplication is performed using the star product

$$\Phi(Y,Z) \star \Theta(Y,Z) = \Phi(Y,Z) \exp\left[-\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\right)\left(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\right) + \text{c.c.}\right] \Theta(Y,Z) \,.$$

- $f(X) = 1 + X e^{i\theta(X)}$ controls the interactions. I'll assume θ is a constant.

- The Kleiniens $v = e^{z_\alpha y^\alpha}$, $\bar{v} = e^{\bar{z}_{\dot\alpha} \bar{y}^{\dot\alpha}}$ which satisfy

$$v \star v = 1 \,, \qquad v \star \Phi(Y,Z) \star v = \Phi(-\gamma Y, -\gamma Z) \,,$$
$$\bar{v} \star \bar{v} = 1 \,, \qquad \bar{v} \star \Phi(Y,Z) \star \bar{v} = \Phi(\gamma Y, \gamma Z) \,,$$
$$\gamma(\circ_\alpha) = \circ_\alpha \,, \qquad \gamma(\bar{\circ}_{\dot\alpha}) = -\bar{\circ}_{\dot\alpha}$$

- The generalized twist operators π and $\bar{\pi}$ acting by

$$\pi\left(\Phi(Y,Z,dZ)\right) = \Phi(-\gamma Y, -\gamma Z, -\gamma dZ) \,,$$
$$\bar{\pi}\left(\Phi(Y,Z,dZ)\right) = \Phi(\gamma Y, \gamma Z, \gamma dZ)$$

- The chirality operator $\Gamma = i^{\frac{n(n-1)}{2}} \vartheta^1 \vartheta^2 \ldots \vartheta^n$.

- The equations of motion are invariant under the gauge transformations

$$\delta\mathcal{A} = d\epsilon - [\mathcal{A}, \epsilon]_\star \, , \qquad \delta B = \epsilon \star B - B \star \pi(\epsilon) \, .$$

where $\epsilon(Y, Z|x, \vartheta)$ is a zero-form which satisfies the same reality conditions and truncations as W.

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- In components the e.o.m. are

$$\begin{aligned}
&dW - W\wedge_\star W = 0\,,\\
&dB - W\star B + B\star\pi(W) = 0\,,\\
&dS_\alpha - [W,S_\alpha]_\star = 0\,, \qquad d\bar S_{\dot\alpha} - \left[W,\bar S_{\dot\alpha}\right]_\star = 0\,,\\
&B\star\pi(S_\alpha) + S_\alpha\star B = 0\,, \qquad B\star\pi(\bar S_{\dot\alpha}) - \bar S_{\dot\alpha}\star B = 0\,,\\
&S_\alpha\star S^\alpha = 2f_\star(B\star v)\,, \qquad \bar S_{\dot\alpha}\star\bar S^{\dot\alpha} = 2\bar f_\star(B\star\bar v\Gamma)\,, \qquad [S_\alpha,\bar S_{\dot\alpha}]_\star = 0\,,
\end{aligned}$$

  and the gauge transformations take the form

$$\delta W = d\epsilon - [W,\epsilon]_\star\,, \quad \delta B = \epsilon\star B - B\star\pi(\epsilon)\,, \quad \delta S_\alpha = [\epsilon,S_\alpha]_\star\,, \quad \delta\bar S_{\dot\alpha} = \left[\epsilon,\bar S_{\dot\alpha}\right]_\star\,.$$

Spin statistics

- We project onto half of the components of all the fields by

$$
\begin{aligned}
W(Y,Z|x,\vartheta) &= \Gamma W(-Y,-Z|x,\vartheta)\Gamma\,,\\
B(Y,Z|x,\vartheta) &= \Gamma B(-Y,-Z|x,\vartheta)\Gamma\,,\\
S_\alpha(Y,Z|x,\vartheta) &= -\Gamma S_\alpha(-Y,-Z|x,\vartheta)\Gamma\,,\\
\bar S_{\dot\alpha}(Y,Z|x,\vartheta) &= -\Gamma \bar S_{\dot\alpha}(-Y,-Z|x,\vartheta)\Gamma\,,\\
\epsilon(Y,Z|x,\vartheta) &= \Gamma \epsilon(-Y,-Z|x,\vartheta)\Gamma\,.
\end{aligned}
$$

- Using the properties of the kleinians this is

$$
[v\bar v\Gamma, W]_\star = [v\bar v\Gamma, B]_\star = [v\bar v\Gamma, \epsilon]_\star = \{v\bar v\Gamma, S\}_\star = 0\,.
$$

- Thus even functions of ϑ^i (bosons) are even functions in Y and Z (even spin). Odd functions of the ϑ^i (fermions) are odd functions in Y, Z (odd spin).

Reality conditions

- First we define the complex conjugation of the variables

$$(y_\alpha)^\dagger = \bar{y}_{\dot\alpha}\,, \qquad (z_\alpha)^\dagger = \bar{z}_{\dot\alpha}\,, \qquad (dz_\alpha)^\dagger = d\bar{z}_{\dot\alpha}\,, \qquad (\vartheta^i)^\dagger = \vartheta^i$$

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- One then defines a second operator τ, which reverses the order of the ϑ^i and otherwise acts by

$$\tau\left[\Phi(Y, Z, dZ|x, \vartheta)\right] = \Phi(iY, -iZ, -idZ|x, \tau[\vartheta])\,.$$

- It also satisfies

$$\tau\left(\Phi \star \Theta\right) = \tau(\Theta) \star \tau(\Phi)\,,$$

and

$$\tau\left(\Gamma\right)^\dagger = \Gamma^{-1} = \Gamma\,.$$

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  We will use the non-minimal reality conditions

$$\tau(W)^\dagger = -W\,, \quad \tau(S)^\dagger = -S\,, \quad \tau(B)^\dagger = \bar{v} \star B \star \bar{v}\Gamma = \Gamma v \star B \star v.$$

The AdS_4 vacuum solution

- We use global coordinates for AdS_4 (of unit radius)

$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu \equiv (1 + \lambda^{-2} r^2) dt^2 - \frac{1}{1 + \lambda^{-2} r^2} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \, .$$

- Setting $\lambda = 1$, we take for the vielbeins

$$h^0 = \sqrt{1 + r^2} \, dt \, , \qquad h^1 = \frac{1}{\sqrt{1 + r^2}} \, dr \, , \qquad h^2 = r \, d\theta \, , \qquad h^3 = r \sin\theta \, d\varphi \, ,$$

- The connection one-forms are

$$\omega_{01} = r \, dt \, , \qquad \omega_{12} = \sqrt{1 + r^2} \, d\theta \, , \qquad \omega_{13} = \sqrt{1 + r^2} \sin\theta \, d\varphi \, , \qquad \omega_{23} = \cos\theta \, d\varphi \, ,$$

with all others zero.

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  with all others zero.

- The AdS_4 vacuum solution to the interacting theory is then

$$W_0 = -\frac{1}{4} \left(\omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\dot\alpha\dot\beta} \bar{y}^{\dot\alpha} \bar{y}^{\dot\beta} - \sqrt{2} h_{\alpha\dot\beta} y^\alpha \bar{y}^{\dot\beta} \right) ,$$
$$B_0 = 0 \, , \qquad S_0 = z_\alpha dz^\alpha + \bar{z}_{\dot\alpha} d\bar{z}^{\dot\alpha} .$$

Black holes in AdS_4

- The simplest black holes in AdS_4 can be written in the Kerr-Schild form

$$g_{\mu\nu} = g^0_{\mu\nu} - \frac{2M}{r} k_\mu k_\nu \,, \quad g^{\mu\nu} = g^{0\mu\nu} + \frac{2M}{r} k^\mu k^\nu \,, \quad k_\mu dx^\mu = dt - \frac{dr}{1+r^2}.$$

- One can also construct traceless completely symmetric tensors

$$\phi_{\mu_1\ldots\mu_s} = \frac{2M}{r} k_{\mu_1} \ldots k_{\mu_s} \,.$$

- They satisfy the equations of motion of a massless spin-s field in a AdS background

$$D^\mu D_\mu \phi_{\nu(s)} - s D_\mu D_\nu \phi^\mu_{\nu(s-1)} = -2(s-1)(s+1)\phi_{\nu(s)}.$$

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Lessons:

- A black-hole solution in AdS_4 can be written as a one loop perturbation.
- The perturbation is constructed from a vector k.
- k generates an infinite tower of massless higher-spin fields.
- k can be expressed in terms of the killing vector $V = \sqrt{2}\partial/\partial t$ and the associated Killing two-form $\kappa_{\alpha\beta}, \kappa_{\dot\alpha\dot\beta}$ through

$$k_{\alpha\dot\alpha} = \frac{1}{1+r^2}\left(v_{\alpha\dot\alpha} - \frac{\kappa_\alpha{}^\beta v_{\beta\dot\alpha}}{r}\right) .$$

The Didenko-Vasiliev solution

[Didenko,Vasiliev]

- From V (or k) we construct the Killing matrix

$$K_{AB} = \begin{pmatrix} \sqrt{2}\kappa_{\alpha\beta} & v_{\alpha\dot{\beta}} \\ v_{\alpha\dot{\beta}} & \sqrt{2}\kappa_{\dot{\alpha}\dot{\beta}} \end{pmatrix} ,$$

- It satisfies the covariantly constant condition

$$D_0 \left(K_{AB} Y^A Y^B \right) = 0 \, .$$

- We normalize it such that

$$K_A{}^B K_B{}^C = -\delta_A^C \, ,$$

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- For the solution we take

$$B = bF_K \star \delta(y)\,, \qquad F_K \equiv 4\exp\left(\frac{i}{2}K_{AB}Y^AY^B\right).$$

- This solution has very interesting properties

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$$F_K \star \delta(y) = F_K \star \delta(\bar y) \,, \qquad F_K \star F_K = F_K .$$

- By virtue of the covariant constancy, it solves the equations of motion

$$dB - W_0 \star B + B \star \pi(W_0) = 0 \,,$$

- By performing the star-product explicitly we obtain

$$B = \frac{4b}{r} \exp \left[\frac{i}{2\kappa^2} \left(\kappa_{\alpha\beta} y^\alpha y^\beta + \kappa_{\dot\alpha\dot\beta} \bar{y}^{\dot\alpha} \bar{y}^{\dot\beta} + 2i\kappa_{\alpha\gamma} v^\gamma{}_{\dot\beta} y^\alpha \bar{y}^{\dot\beta} \right) \right].$$

- We find the components of the generalised higher-spin Weyl tensors

$$C_{\alpha(2s)} = \frac{b}{s! 2^{s-2} r} \left(\frac{i\kappa_{\alpha\alpha}}{\kappa^2} \right)^s, \qquad \bar{C}_{\dot\alpha(2s)} = \frac{b}{s! 2^{s-2} r} \left(\frac{i\kappa_{\dot\alpha\dot\alpha}}{\kappa^2} \right)^s.$$

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- The solution for S is more complicated.

- First we define a reduces set of oscillators

$$F_K \star Z_A \equiv F_K\, A_A\,, \qquad A_A \equiv (a_\alpha, \bar{a}_{\dot\alpha}) \equiv Z_A + iK_A{}^B Y_B\,, \qquad [A_A, A_B] = 4\epsilon_{AB}\,,$$

such that Z will always appear in this combination with Y.

- The star product of F_K with any function then gives

$$F_K \star \phi(Z|x) = F_K\, \phi(A|x)\,.$$

- Functions of this form define a subalgebra

$$(F_K\phi_1) \star (F_K\phi_2) = F_K\,(\phi_1 * \phi_2)\,,$$

- with the $*$-product

$$(\phi_1 * \phi_2)(A) = \int d^4u\, \phi_1(A + 2U_+)\phi_2(A - 2U_-)e^{2U_{+A}U_-^A}.$$

- Here $U_{\pm A} = \Pi_{\pm A}{}^B U_B$ are defined in terms of the projectors

$$\Pi_{\pm AB} = \frac{1}{2}\,(\epsilon_{AB} \pm iK_{AB})\,, \qquad \Pi_{\pm A}{}^B \Pi_{\pm B}{}^C = \Pi_{\pm A}{}^C\,, \qquad \Pi_{\pm A}{}^B \Pi_{\mp B}{}^C = 0.$$

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- In particular

$$Y_{-A} \star F_K = F_K \star Y_{+A} = 0\,,$$

- and

$$(F_K\, \phi) \star Y_{+A} = (F_K\, \phi) \star F_K \star Y_{+A} = 0.$$

- For any function $\phi(a)$ holomorphic in a

$$[a_\alpha, \phi(a)]_* = 2\partial_{a^\alpha}\phi(a)\,, \qquad \{a_\alpha, \phi(a)\}_* = 2\left(a_\alpha + i\kappa_\alpha{}^\beta \partial_{a^\beta}\right)\phi(a)\,,$$

- The $*$-product possesses Kleinien operators $\mathcal{K}$, $\bar{\mathcal{K}}$

$$F_K \star \delta(z) = F_K \mathcal{K}\,, \qquad \mathcal{K} = \frac{1}{r}\exp\left[\frac{i\kappa_{\alpha\beta}}{2\kappa^2}a^\alpha a^\beta\right],$$

$$F_K \star \delta(\bar{z}) = F_K \bar{\mathcal{K}}\,, \qquad \bar{\mathcal{K}} = \frac{1}{r}\exp\left[\frac{i\kappa_{\dot{\alpha}\dot{\beta}}}{2\kappa^2}\bar{a}^{\dot{\alpha}}\bar{a}^{\dot{\beta}}\right].$$

- They satisfy

$$\mathcal{K} * \mathcal{K} = \bar{\mathcal{K}} * \bar{\mathcal{K}} = 1\,, \qquad \{\mathcal{K}, a_\alpha\}_* = \left\{\bar{\mathcal{K}}, \bar{a}_{\dot{\alpha}}\right\}_* = 0\,,$$

$$\left[\mathcal{K}, \bar{\mathcal{K}}\right]_* = [\mathcal{K}, \bar{a}_{\dot{\alpha}}]_* = \left[\bar{\mathcal{K}}, a_\alpha\right]_* = 0.$$

- We take the ansatz

$$W = W_0 + F_K \left[\Omega(a|x) + \bar{\Omega}(\bar{a}|x)\right].$$

$$B = bF_K \star \delta(y)\,,$$

$$S_\alpha = z_\alpha + F_K \sigma_\alpha(a|x)\,, \qquad \bar{S}_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} + F_K \bar{\sigma}_{\dot{\alpha}}(\bar{a}|x)\,,$$

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- Then in terms of

$$\varsigma_\alpha = a_\alpha + \sigma_\alpha(a|x)\,, \qquad \bar{\varsigma}_{\dot\alpha} = \bar{a}_{\dot\alpha} + \bar{\sigma}_{\dot\alpha}(\bar{a}|x)\,, \qquad \mathcal{Q} = \left(\hat{d} - \frac{i}{2} d\kappa^{\alpha\beta} \partial_{a^\alpha} \partial_{a^\beta}\right)\,,$$

with $\hat{d}$ the standard exterior derivative except that $\hat{d}a = \hat{d}\bar{a} = 0$

- We find the equations

$$[\varsigma_\alpha(a|x), \varsigma_\beta(a|x)]_* = 2\epsilon_{\alpha\beta}\left(1 + e^{i\theta} b\mathcal{K}\right)\,, \qquad [\varsigma_\alpha(a|x), \bar{\varsigma}_{\dot\alpha}(\bar{a}|x)]_* = 0\,,$$

$$\{\mathcal{K}, \varsigma_\alpha(a|x)\}_* = 0\,, \qquad \left\{\varsigma_\alpha(a|x), \bar{\mathcal{K}}\right\}_* = 0\,,$$

$$\mathcal{Q}\Omega - \Omega \wedge_* \Omega = 0\,, \qquad \mathcal{Q}\varsigma_\alpha - [\Omega, \varsigma_\alpha]_* = 0\,,$$

- We take the ansatz

$$W = W_0 + F_K \left[\Omega(a|x) + \bar{\Omega}(\bar{a}|x)\right].$$

$$B = bF_K \star \delta(y)\,,$$

$$S_\alpha = z_\alpha + F_K \sigma_\alpha(a|x)\,, \qquad \bar{S}_{\dot\alpha} = \bar{z}_{\dot\alpha} + F_K \bar{\sigma}_{\dot\alpha}(\bar{a}|x)\,,$$

- Then in terms of

$$\varsigma_\alpha = a_\alpha + \sigma_\alpha(a|x)\,, \qquad \bar{\varsigma}_{\dot\alpha} = \bar{a}_{\dot\alpha} + \bar{\sigma}_{\dot\alpha}(\bar{a}|x)\,, \qquad \mathcal{Q} = \left(\hat{d} - \frac{i}{2} d\kappa^{\alpha\beta}\partial_{a^\alpha}\partial_{a^\beta}\right)\,,$$

with $\hat{d}$ the standard exterior derivative except that $\hat{d}a = \hat{d}\bar{a} = 0$

- We find the equations

$$[\varsigma_\alpha(a|x), \varsigma_\beta(a|x)]_* = 2\epsilon_{\alpha\beta}\left(1 + e^{i\theta}b\mathcal{K}\right)\,, \qquad [\varsigma_\alpha(a|x), \bar{\varsigma}_{\dot\alpha}(\bar{a}|x)]_* = 0\,,$$

$$\{\mathcal{K}, \varsigma_\alpha(a|x)\}_* = 0\,, \qquad \left\{\varsigma_\alpha(a|x), \bar{\mathcal{K}}\right\}_* = 0\,,$$

$$\mathcal{Q}\Omega - \Omega \wedge_* \Omega = 0\,, \qquad \mathcal{Q}\varsigma_\alpha - [\Omega, \varsigma_\alpha]_* = 0\,,$$

- To linear order in b one finds

$$\sigma_\alpha(a|x) = \frac{be^{i\theta}}{r}\pi_\alpha^{+\beta}a_\beta \int_0^1 dt \exp\left(\frac{it}{2}\frac{\kappa_{\alpha\beta}}{\kappa^2}a^\alpha a^\beta\right)\,,$$

with a new set of projectors

$$\pi^\pm_{\alpha\beta} = \frac{1}{2}\left(\epsilon_{\alpha\beta} \pm \frac{\kappa_{\alpha\beta}}{\sqrt{-\kappa^2}}\right).$$

- In fact, this solves the equations to all orders in b with $\Omega = 0$.
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W = W_0 &= -\frac{1}{4}\left(\omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta} - \sqrt{2} h_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta}\right) \\
B &= \frac{4b}{r}\exp\left[-\frac{i}{2r^2}\left(\kappa_{\alpha\beta} y^\alpha y^\beta + \kappa_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta} + 2i\kappa_{\alpha\gamma}v^\gamma{}_{\dot\beta} y^\alpha \bar y^{\dot\beta}\right)\right] \\
S_\alpha &= z_\alpha + F_K \frac{e^{i\theta}b}{r}\pi^{+\beta}_\alpha a_\beta \int_0^1 dt\,\exp\left(-\frac{it\kappa_{\alpha\beta}}{2r^2}a^\alpha a^\beta\right) \\
\bar S_{\dot\alpha} &= \bar z_{\dot\alpha} + F_K \frac{e^{-i\theta}b}{r}\pi^{+\dot\beta}_{\dot\alpha} \bar a_{\dot\beta} \int_0^1 dt\,\exp\left(-\frac{it\kappa_{\dot\alpha\dot\beta}}{2r^2}\bar a^{\dot\alpha} \bar a^{\dot\beta}\right)
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\end{aligned}
$$

- This solution is valid for the theory with arbitrary interaction term given by the angle θ_0.

Embeddings in the SUSY theory

- We can decompose the fields with the projectors

$$W = \Gamma^+ W_+ + \Gamma^- W_- \,,$$
$$B = \Gamma^+ B_+ + i\Gamma^- B_- \,, \qquad \Gamma^\pm = \frac{1 \pm \Gamma}{2} \,,$$
$$S = \Gamma^+ S_+ + \Gamma^- S_- \,,$$

- Each of the $\Phi_\pm$ will have even and odd components (in ϑ).
- For the classical solutions the odd components vanish.
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- Projecting the equation for W we find identical equations for the two blocks

$$\Gamma^\pm(dW-W\wedge_\star W)=\Gamma^\pm W_\pm-\Gamma^\pm W_\pm\wedge_\star\Gamma^\pm W_\pm=\Gamma^\pm\left(dW_\pm-W_\pm\wedge_\star W_\pm\right).$$

- Likewise for the flatness equation for B.
- The equations for S has an explicit Γ, which complicates things

$$S_{+\alpha}\star S_+^\alpha=2f_\star\left(B_+\star v\right)\,,\qquad \bar S_{+\dot\alpha}\star\bar S_+^{\dot\alpha}=2\bar f_\star(B_+\star\bar v)\,,\qquad [S_{+\alpha},\bar S_{+\dot\alpha}]_\star=0\,,$$
$$S_{-\alpha}\star S_-^\alpha=2f_\star\left(iB_-\star v\right)\,,\qquad \bar S_{-\dot\alpha}\star\bar S_-^{\dot\alpha}=2\bar f_\star(-iB_-\star\bar v)\,,\qquad [S_{-\alpha},\bar S_{-\dot\alpha}]_\star=0\,.$$

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$$S_{+\alpha} \star S_+^\alpha = 2 f_\star \left(B_+ \star v\right) \,, \qquad \bar{S}_{+\dot\alpha} \star \bar{S}_+^{\dot\alpha} = 2\bar{f}_\star(B_+ \star \bar{v}) \,, \qquad [S_{+\alpha}, \bar{S}_{+\dot\alpha}]_\star = 0 \,,$$
$$S_{-\alpha} \star S_-^\alpha = 2 f_\star \left(iB_- \star v\right) \,, \qquad \bar{S}_{-\dot\alpha} \star \bar{S}_-^{\dot\alpha} = 2\bar{f}_\star(-iB_- \star \bar{v}) \,, \qquad [S_{-\alpha}, \bar{S}_{-\dot\alpha}]_\star = 0 \,.$$

- The equation for $S_-, \bar{S}_-$ is the same as for the theory with $\theta \to \theta + \pi/2$.

- We take the following embeddings of the DV solution

$$W(Y,Z|x) = W_0\,,$$

$$B(Y,Z|x) = b\left[\eta_p F_K + \eta_m F_{-K}\right] \star \delta(y)\,,$$

$$S_\alpha(Y,Z|x) = z_\alpha + s\big[\eta_p F_K \sigma_\alpha(a,K|x) + \eta_m F_{-K}\sigma_\alpha(a,-K|x)\big]\,,$$

$$\bar{S}_{\dot\alpha}(Y,Z|x) = \bar{z}_{\dot\alpha} + \bar{s}\big[\eta_p F_K \bar\sigma_{\dot\alpha}(\bar{a},K|x) + \eta_m F_{-K}\bar\sigma_{\dot\alpha}(\bar{a},-K|x)\big]\,,$$

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- We allowed two different solutions with the Killing matrix K replaced by $-K$.
- η_p and η_m are orthogonal projectors, such that these two solutions don't interact.
- In a diagonal basis

$$b \equiv b_+ + i b_-\,,$$
$$s \equiv e^{i\theta_0} s_+ + e^{i(\theta_0+\pi/2)} s_-\,,$$
$$\bar s \equiv e^{-i\theta_0} \bar s_+ + e^{-i(\theta_0+\pi/2)} \bar s_-\,,$$

- The equations of motion then impose

$$s_\pm = b_\pm\,, \qquad bs = sb\,, \qquad s\bar s = \bar s s\,,$$

- The reality conditions are

$$b_\pm^\dagger = b_\pm\,, \qquad s^\dagger = \bar s\,.$$

SUSY invariance, bulk

- Symmetries are given by trivial gauge transformations

$$d\epsilon - [W_0, \epsilon]_\star = 0.$$

- Taking an odd gauge parameter

$$\epsilon(Y|x, \vartheta) = \Xi_\alpha(x, \vartheta) y^\alpha + i\bar{\Xi}_{\dot{\alpha}}(x, \vartheta)\bar{y}^{\dot{\alpha}}\,,$$

- The equation for the connection reduces to the Killing spinor equation

$$\tilde{\nabla}\begin{pmatrix}\Xi\\ \bar{\Xi}\end{pmatrix} \equiv \left(d - \frac{i}{2}\omega_{ab}\gamma^{ab} + \frac{i}{\sqrt{2}}h_a\gamma^a\right)\begin{pmatrix}\Xi\\ \bar{\Xi}\end{pmatrix} = 0\,,$$

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- The general solution is given by the four Killing spinors of AdS_4: $\psi^I = \psi^I_\alpha y^\alpha + i\bar\chi^I_{\dot\alpha}\bar{y}^{\dot\alpha}$

$$\epsilon(Y|x,\vartheta) = \psi^I(Y|x)\xi^I(\vartheta)\,, \qquad I = 1, 2, \bar{1}, \bar{2}\,.$$

- The reality condition imposes (with $i = 1, 2$)

$$\left(\xi^i\right)^\dagger = \xi^{\bar{i}}\,,$$

The parameters ξ are also $2^{n-1} \times 2^{n-1}$ constant matrices.

- The gauge invariance of B and S imposes

$$\psi^I \star B\,\xi^I b - B \star \pi(\psi^I)\, b\xi^I = 0\,,$$
$$\psi^I \star S\,\xi^I s - S \star \psi^I\, s\xi^I = 0\,.$$

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- Both B and S are proportional to F_K, which acts as a projector on Y

$$(\Pi_{-A}{}^B Y_B) \star F_K = F_K \star (\Pi_{+A}{}^B Y_B) = 0\,, \qquad \Pi_{\pm AB} = \frac{1}{2}\,(\epsilon_{AB} \pm iK_{AB})$$

- Since K_{AB} is a bilinear in the AdS_4 Killing spinors, it projects the four Killing spinors onto a two-dimensional subspace

$$\psi^i \star F_K = F_K \star \psi^{\bar{i}} = 0\,, \qquad F_K \star \psi^i = 2F_K\psi^i\,, \qquad \psi^{\bar{i}} \star F_K = 2\psi^{\bar{i}} F_K\,.$$

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- With our ansatz for the matrix structure of B and S we get the equations for b, η_p and η_m

$$\psi^i F_{-K}\xi^i \eta_m b + \psi^{\bar{i}} F_K \xi^{\bar{i}} \eta_p b - F_K \psi^i \eta_p b \xi^i - F_{-K} \psi^{\bar{i}} \eta_m b \xi^{\bar{i}} = 0\,.$$

- This equation is satisfied if

$$\xi^i \eta_m b = \eta_p b \xi^i = 0\,.$$

- This is a simple matrix equation.

- Let me consider the case of the theory with $n = 4$.

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- The simplest choice of η's is

$$\eta_p = \mathrm{diag}(1,1,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,1,1)\,,$$

- It preserves the supserymmetries

$$\xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix},$$

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- These are half of the SUSY generators (since ξ is odd).
- Another choice is

$$\eta_p = \mathrm{diag}(1,0,1,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

- This is $1/4$ BPS.

If b is not full rank, we can preserve more SUSY.

- $3/8$ BPS conficuration:

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0&0&0&0\\ 0&0&*&0\\ *&0&0&0\\ *&0&0&0 \end{pmatrix}.$$

- Another $1/2$ BPS cases:

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,0)\,, \qquad \xi^i = \begin{pmatrix} 0&0&0&0\\ 0&0&*&*\\ *&0&0&0\\ *&0&0&0 \end{pmatrix},$$

- A $5/8$-BPS case:

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0&0&0&0\\ 0&0&*&0\\ *&*&0&0\\ *&*&0&0 \end{pmatrix}.$$

- And the $3/4$ case:

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,0,0)\,, \qquad \xi^i = \begin{pmatrix} 0&0&0&0\\ 0&0&*&*\\ *&*&0&0\\ *&*&0&0 \end{pmatrix}.$$

SUSY invariance, boundary

- The theory has massless fields and we need to choose boundary conditions for the scalar, fermions and vector fields.
- For a scalar in AdS_{d+1}

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2}\,.$$

- The field in the bulk has fall-off

$$C(r,x) = \frac{a}{r^{\Delta_-}} + \frac{b}{r^{\Delta_+}} + \ldots$$

- The holographic dual depends of the choice of boundary conditions.

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- The holographic dual depends of the choice of boundary conditions.
- The general fall-off of B takes the form (for the scalar $m^2 = -2$)

$$B^{(0)} = \frac{1}{r}\left(\Gamma^+ \cos\gamma + i\Gamma^- \sin\gamma\right)\tilde{f}_1 + \frac{1}{r^2}\left(\Gamma^- \cos\gamma + i\Gamma^+ \sin\gamma\right)\tilde{f}_2 + O\Big(\frac{1}{r^3}\Big),$$

$$B^{(1)} = \frac{1}{r^2}\left[e^{i\beta}F_{\alpha\beta}y^\alpha y^\beta + \Gamma e^{-i\beta}\bar{F}_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}\right] + O\Big(\frac{1}{r^3}\Big),$$

- Different theories will have different conditions on $\tilde{f}_{1,2}$, F and $\bar{F}$.

- All our solutions are based on the DV solution with asymptotics

$$B = \frac{4}{r}\, b(\eta_p + \eta_m) - \frac{2i}{r^2}\, b(\eta_p - \eta_m) \left(\frac{\kappa_{\alpha\beta}}{r} y^\alpha y^\beta + \frac{\kappa_{\dot\alpha\dot\beta}}{r} \bar{y}^{\dot\alpha} \bar{y}^{\dot\beta} \right) + O(Y^4)\,.$$

- In particular there is no $1/r^2$, component, so $\tilde{f}_2 = 0$.
- We can invert the equation ($\tilde{f} \equiv \tilde{f}_1$)

$$4b(\eta_p + \eta_m) = \left(\Gamma^+ \cos\gamma + i\Gamma^- \sin\gamma \right) \tilde{f}$$

$\mathcal{N} = 2$ with $SU(2)$ flavor symmetry

- Let us consider a theory with $n = 4$ and the boundary conditions

$$\beta = \gamma = \theta_0 \,, \qquad \left[\vartheta^1, \tilde{f}\right] = 0 \,.$$

- The symmetry among ϑ^2, ϑ^3 and ϑ^4 will be an $SU(2)$ flavor symmetry of the boundary theory.
- Two supersymmetries generated by ϑ^1 and $\Gamma\vartheta^1$ are preserved.

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- We need to solve

$$\xi^i \eta_m b = \eta_p b \xi^i = 0 \,.$$

- If $\xi^i = \Gamma^+\vartheta^1$, then η_m is an eigenstate of Γ^- while η_p is an eigenstate of Γ^+.
- ξ^i can only be either of $\Gamma^\pm\vartheta^1$, but not a linear combination.

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- ξ^i can only be either of $\Gamma^\pm\vartheta^1$, but not a linear combination.
- One can check that $\Gamma^-\vartheta^1$ can be embedded in $\xi^i = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ *&*&0&0\\ *&*&0&0 \end{pmatrix}$, but none of the other examples, so $\eta_p = \mathrm{diag}(1,1,0,0)$ and $\eta_m = \mathrm{diag}(0,0,1,1)$.
- Lastly we see that b_+ and b_- are proportional to each other. Giving a two parameter family of $1/2$ BPS solutions.

$\mathcal{N} = 2$ with $U(1) \times U(1)$ flavor symmetry

- Another theory with $n = 4$ has boundary conditions that leave two supersymmetries and a $U(1) \times U(1)$ flavor symmetry.
- The boundary conditions are

$$\beta = \theta_0 \,, \qquad \gamma = \theta_0 P_{1,\vartheta^3\vartheta^4} \,, \qquad \tilde{f} \in \text{span}\left\{1, \vartheta^3\vartheta^4, \vartheta^3\vartheta^1, \vartheta^3\vartheta^2, \vartheta^4\vartheta^1, \vartheta^4\vartheta^2\right\} \,,$$

  where $P_{1,\vartheta^3\vartheta^4}$ projects onto the subspace spanned by $1, \vartheta^3\vartheta^4$.
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- These SUSYs are compatible with $\xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}$.
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- Again b_+ and b_- will have to be proportional to each other, and we find a two parameter family of $1/2$ BPS solutions.

Summary

- We simplified the DV solution: $W = W_0$.
- We showed how to embed the DV solution into a several examples of SUSY higher spin theories.
- Non-trivial conditions for preserving bulk SUSY and preserving those not broken by boundary conditions.
- This can be done in other examples as well, including the theory conjectured to be dual to a certain limit of ABJ theory.
- Also developed the formalism for embedding more general soluitons.
- This can be applied also to theories with Chan-Paton factors.
- Allows to implement and test ideas in higher spin holography in a SUSY setting.

Summary

- We simplified the DV solution: $W = W_0$.
- We showed how to embed the DV solution into a several examples of SUSY higher spin theories.
- Non-trivial conditions for preserving bulk SUSY and preserving those not broken by boundary conditions.
- This can be done in other examples as well, including the theory conjectured to be dual to a certain limit of ABJ theory.
- Also developed the formalism for embedding more general soluitons.
- This can be applied also to theories with Chan-Paton factors.
- Allows to implement and test ideas in higher spin holography in a SUSY setting.
- Still missing a proper understanding of these solutions…

The end