Large spin systematics in CFT

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Introduction

The problem

In this talk we will consider operators with higher spin:

$$Tr\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi, \qquad Tr\varphi\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}Tr\varphi\varphi$$

And study their dimension Δ for large values of the spin ℓ.

The motivation

- They form the leading twist sector of many theories.
- Important role in QCD analysis of deep inelastic scattering.
- Related to Wilson loops with cusps and UV divergences of scattering amplitudes.
- Fundamental role in applying integrability to AdS/CFT.

Introduction

The method

- The method relies only on symmetries and basic properties of CFT's (conformal symmetry, analiticity, unitarity, structure of the OPE,...)
- The results will hold for a vast family of theories!
- Similar in spirit to the conformal bootstrap program.

Leading twist operators

Generic four dimensional gauge theory:

Leading twist operators built from scalars and derivatives

$$\mathcal{O}_\ell = Tr\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi + \cdots$$

- At large values of the spin, they acquire a logarithmic anomalous dimension:

$$\Delta_\ell - \ell = f(g)\log\ell + \cdots$$

- This behavior is valid to all orders in perturbation theory!
- $f(g)$ appears in many computations!

What can we say about sub-leading corrections?

$$\Delta_\ell - \ell = f(g)\log\ell + f^{(0)}(g) + \frac{f^{(1)}(g,\log\ell)}{\ell} + \frac{f^{(2)}(g,\log\ell)}{\ell^2} + \cdots$$

Reciprocity

Reciprocity principle

- Odd powers of $1/\ell$ are fixed in terms of the even ones!

$$\Delta_\ell - \ell - 2 = \gamma_\ell \quad \rightarrow \quad \gamma_\ell \equiv f(\ell + \frac{1}{2}\gamma_\ell)$$

$$\Downarrow$$

$$f(\ell) = a_0(\log J_0) + \frac{a_2(\log J_0)}{J_0^2} + \frac{a_4(\log J_0)}{J_0^4} + \cdots$$

where $J_0^2 = \ell(\ell+1)$.

- First observed in QCD [Moch, Vermaseren, Mogt, Gribov, Lipatov, Drell, Levy, Yan, Dokshitzer, Marchesini, Salam...]
- Then checked for other theories, including MSYM [Basso, Korchemski, Beccaria, Forini, Tirziu, Tseytlin, ...]
- Led to great activity but a proof was still missing!

Reciprocity

Equivalent formulation

Define the full Casimir:

$$J^2 = (\ell + \gamma_\ell/2)(\ell + \gamma_\ell/2 + 1)$$

Reciprocity principle

$$\gamma_\ell = \alpha_0(\log J) + \frac{0}{J} + \frac{\alpha_2(\log J)}{J^2} + \frac{0}{J^3} + \frac{\alpha_4(\log J)}{J^4} + \cdots$$

Let us prove this for a generic four dimensional CFT!

CFT - Symmetries

Conformal algebra:

- Scale transformations → dilatation D
- Poincare Algebra: P_μ and $M_{\mu\nu}$
- Special conformal transformations: K_μ

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = \eta_{\mu\nu} D - i M_{\mu\nu}$$

We may also have flavor symmetries.

CFT - Ingredients

Main ingredient:

- Local conformal primary operators: $\mathcal{O}_{\Delta,\ell}(x)$, $\quad [K_\mu, \mathcal{O}_{\Delta,\ell}] = 0$

Dimension

Lorentz spin

In addition we have descendants $P_{\mu_k} ... P_{\mu_i} \mathcal{O}_{\Delta,\ell}$.

Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk} |x|^{\Delta_k - \Delta_i - \Delta_j} \mathcal{O}_k(0)$$

- The set Δ_i and C_{ijk} (for all operators) characterizes the CFT and is called the *CFT data*.

CFT - Observables

Main observable:

Correlation functions of local operators

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle$$

Conformal symmetry + OPE:

- CFT data → all correlation functions!

2 and 3pt-function

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_{ij}-\Delta_k}|x_{13}|^{\Delta_{ik}-\Delta_j}|x_{23}|^{\Delta_{jk}-\Delta_i}}$$

OPE factorization

ϕ: Scalar operator of dimension d on a generic CFT, $e.g\ Tr\varphi^2$.

Conformal symmetry

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2d}x_{34}^{2d}}, \qquad u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

Crossing relation: $v^d\mathcal{G}(u,v)=u^d\mathcal{G}(v,u)$

OPE

$$\phi(x_1)\phi(x_2)=x^{-2d}\left(1+\sum_k C_{\phi\phi k}|x_{12}|^{\Delta_k-\Delta_{ij}}\mathcal{O}_k(x_1)\right)$$

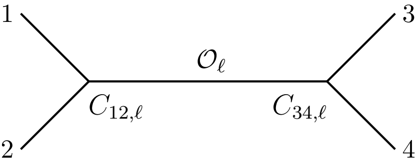
Identity operator

- Conformal primaries $[K_\mu,\mathcal{O}]=0$
- Descendants $P_{\mu_1}\dots P_{\mu_n}\mathcal{O}$

OPE factorization

OPE → 4-pt function factorizes!

$$\mathcal{G}(u,v) \quad = \sum_{\ell}$$



$$\mathcal{G}(u,v) = 1 + \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

Identity operator   Conformal primaries   Conformal blocks

- The conformal blocks are explicitly known.

Conformal blocks

Conformal blocks properties

- Small u behavior:

$$G_{\Delta,\ell}(u,v) \sim u^{\frac{1}{2}(\Delta-\ell)} k_{\Delta,\ell}(1-v)$$

$$k_{\Delta,\ell}(1-v) = (1-v)^{\ell}\ {}_2F_1(\tfrac{1}{2}(\Delta+\ell), \tfrac{1}{2}(\Delta+\ell), \Delta+\ell; 1-v)$$

The power of u is controlled by the twist $\tau = \Delta - \ell$.

- There exist a second order operator $\mathcal{D}$ such that

$$\mathcal{D}\left(u^{\frac{1}{2}(\Delta-\ell)} k_{\Delta,\ell}(v)\right) = J^2 u^{\frac{1}{2}(\Delta-\ell)} k_{\Delta,\ell}(v)$$

with $J^2 = \frac{1}{4}(\Delta+\ell-2)(\Delta+\ell)$.

Leading twist contribution

- We can focus in the contribution of leading twist operators by considering the small u limit:

$$\mathcal{G}(u,v) = 1 + u^{\tau_{lead}/2} h(v) + \ldots$$

- In perturbative CFT they are given by the tower of operators mentioned above!

$$\tau_\ell = \Delta_\ell - \ell = 2 + \gamma_\ell$$

$$\Downarrow$$

$$\mathcal{G}(u,v) = 1 + u\, h(\log u, v) + \ldots$$

$$\sum_\ell a_\ell u^{1+\gamma_\ell/2} k_{\ell+2+\gamma_\ell,\ell}(v) = u\, h(\log u, v)$$

Tree level

First consider the problem at tree level: $\gamma_\ell = 0$

$$\sum_{\ell=0,2,\ldots} a_\ell^{(0)} u k_{\ell+2,\ell}(1-v) = u\left(\frac{1}{v}+1\right)$$

- We can solve:

$$a_\ell^{(0)} = 2\frac{\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}$$

- Each term in the sum diverges logarithmically as $v \to 0$.
- We need an infinite number of terms to reproduce the divergence on the right hand side!
- The divergence comes from the region $\ell \gg 1$.
- Large ℓ behavior of $a_\ell^{(0)}$ is fixed by the divergence of the r.h.s.

Loop level

Let us turn on the coupling!

$$\tau_\ell = 2 + \gamma_\ell(g), \qquad a_\ell = a_\ell^{(0)} \hat{a}_\ell(g)$$

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + g\mathcal{G}^{(1)}(u,v) + ...$$

To any order in perturbation theory:

$$\frac{1}{v}+1 \to \frac{1}{v}h^{(0)}(\log u, \log v)+h^{(1)}(\log u, \log v)+vh^{(2)}(\log u, \log v)+\cdots$$

- Where only integer powers of v can appear due to analiticity

  [Dolan-Osborn]

$$\sum_\ell a_\ell u^{\gamma_\ell/2} k_{\ell+2+\gamma_\ell,\ell}(v) = \frac{1}{v}h^{(0)}(\log u, \log v)+h^{(1)}(\log u, \log v)+\cdots$$

Method

- Consider

$$\sum_{\ell} a_\ell u^{\gamma_\ell/2} k_{\ell+2+\gamma_\ell,\ell}(v) \quad (*)$$

- Assume a general large ℓ behavior:

$$\gamma_\ell = p_0(\log J) + \frac{p_1(\log J)}{J} + \frac{p_2(\log J)}{J^2} + \cdots$$
$$\hat{a}_\ell = q_0(\log J) + \frac{q_1(\log J)}{J} + \frac{q_2(\log J)}{J^2} + \cdots$$

  where $J^2 = (\ell + \gamma_\ell/2)(\ell + \gamma_\ell/2 + 1)$.

- Compute the divergent terms in $(*)$ as $v \to 0$.
- Since they come from the large ℓ region, they should be controlled by the above expansion!

Method

Method to compute the divergent contributions [L.F.A, Maldacena, Fitzpatrick et. al., Komargodski at. al.]

- Focus in the small v/large ℓ region:

$$v=\epsilon, \quad \ell=\frac{x}{\epsilon^{1/2}}, \quad \sum_{\ell} \rightarrow \frac{1}{2}\int_0^\infty dx$$

- Perform a change of variables $x \rightarrow j$:

$$\frac{j^2}{\epsilon}=(\frac{x}{\epsilon^{1/2}}+\gamma_\ell)(\frac{x}{\epsilon^{1/2}}+\gamma_\ell+1)$$

- Use integral representation for the hypergeometric function and use saddle point.

Method

$$(*) = \frac{1}{\epsilon} 4 \int_0^\infty q_0 u^{p_0/2} j K_0(2j) dj + \\ + \frac{1}{\epsilon^{1/2}} \int_0^\infty u^{p_0/2} (q_0 p_1 \log u + 2q_1 - q_0 p_0') K_0(2j) dj \\ + \text{finite}$$

Claim: all divergent terms (not only the leading one) are captured!

But $(*) = \frac{1}{\epsilon} h^{(0)}(\log u, \log \epsilon) + h^{(1)}(\log u, \log \epsilon) + \cdots$

$$\int_0^\infty u^{p_0/2} (q_0 p_1 \log u + 2q_1 - q_0 p_0') K_0(2j) dj = 0$$

Results

- At any order in perturbation theory p_i, q_i are polynomials in $\log j^2/\epsilon$.

$$\int_0^\infty P(\log j^2/\epsilon) K_0(2j) dj = 0 \to P(\log j^2/\epsilon) = 0$$

$$\Downarrow$$

$$p_1 = 0, \quad q_1 = \frac{1}{2} q_0 p_0'$$

- Acting with $\mathcal{D}$ multiplies the integrand by j^2/ϵ and increases the degree of divergence by one (preserving analiticity).
- We obtain a constraints, involving the higher order terms in the expansions!

Results

Results

1. $\gamma(\ell)$ expanded for large ℓ contains only even powers of $1/J$!
2. $\frac{\hat{a}(\ell)}{2+\gamma'(\ell)}$ expanded for large ℓ contains only even powers of $1/J$!

- The results extend to arbitrary dimensions and non-scalar operators!

Example: $\mathcal{N}=4$ SYM

$$\gamma(\ell) = 2h_1(\ell)\lambda + ...$$
$$\hat{a}(\ell) = -h_2(\ell)\lambda + ...$$

- Both statements can be explicitly checked up to three loops!

Generalizations

Non-perturbative results

- Given $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$ there are double trace operators $\mathcal{O}\partial^{\ell}\mathcal{O}$ of dimension [L.F.A., Maldacena, Fitzpatrick et. al; Komargodski et.al.]

$$\Delta_\ell - \ell = 2\Delta_{\mathcal{O}} + \gamma_\ell, \quad \gamma_\ell = -\frac{c}{\ell^{\tau_{\min}}} + \cdots$$

τ_{min}: Twist of the minimal twist operator in $\mathcal{O} \times \mathcal{O}$.

- Our method applies also to this case!

$$\gamma_\ell = -\frac{c}{J^{\tau_{\min}}}\left(1 + \frac{p_2}{J^2} + \ldots\right)$$

- Extensively checked for the critical $O(N)$ model and theories with holographic dual.

Generalizations

Non-conformal theories

- We can consider a theory with non-vanishing beta function.
- The anomalous dimensions will depend on the scheme. Using DREG with $D = 4 - 2\epsilon$

$$\beta_\epsilon(g) = -2\epsilon + \beta(g)$$

- The beta function vanishes at $\epsilon_{cr} = \beta(g)/2$.
- "Even" expansion in terms of the corrected Casimir

$$J_\beta^2 = (\ell + \gamma_\ell/2 - \beta/2)(\ell + \gamma_\ell/2 - \beta/2 + 1)$$

- We have checked this is the case for QCD and $\mathcal{N} = 0, 1, 2$ SYM theories!

Conclusions

- We have derived an infinite number of constraints for the large spin expansion of the anomalous dimension and structure constants of higher spin operators.
- The derivation relied solely in CFT arguments and symmetries, and applies to a large class of theories.
- Reciprocity was assumed to make progress in the computation of the dimension of twist-two operators in MSYM. Can use this to make progress in the computation of structure constants?
- Unexplored prediction for OPE coefficients, can we check it for QCD?
- It would be interesting to use the full power of crossing-symmetry.