# Quantum Entanglement of locally perturbed thermal states

Joan Simón

University of Edinburgh and Maxwell Institute of Mathematical Sciences

Holography, strings and higher spins
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Based on arXiv:1410.2287 and work in progress with P. Caputa, A. Štikonas, T. Takayanagi & K. Watanabe

# Motivation

Consider a critical physical system in 1+1 dimensions in some thermal state

$$\rho_\beta$$

Perturb the state by a local primary operator

$$\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)$$

Evolve the system unitarily

$$e^{-iHt}\,\left(\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)\right)\,e^{iHt}$$

Question : Is there any sense in which subsystems behave thermally after some time scale $t^\star_\omega$ ?

$$\Delta I_{A:B}(t^\star_\omega) = 0$$

# Holographic Motivation

1. Eternal BH $\simeq$ thermo field double (Maldacena)
   - Recently re-interpreted in terms of EPR=ER (Maldacena-Susskind)
   - Local Perturbation of this scenario : time evolution of the throat ?
   - Improvement in the holographic dictionary
2. BH physics suggest speed at which thermality is regained is faster than in diffusive systems (scrambling) (Susskind-Sekino)
   - No-cloning argument & causality bounds

$$t_\omega \sim \beta \log S$$

   - Small perturbations get blue shifted near horizon (Shenker-Stanford)

$$t^\star \sim \beta \log m_p \beta$$

Question : Any CFT evidence for any of these bulk effects ?

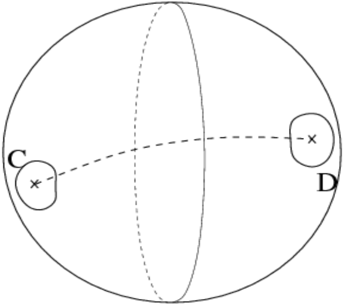
# Entanglement vs Correlations

Question : Is there any relation between quantum entanglement and correlation lengths ?
Consider as measure of entanglement, mutual information

$$I(C:D) = S(\rho_C) + S(\rho_D) - S(\rho_{CD})$$

Using Pinksler's inequality, one can show (Wolf, Verstraete, Hastings, Cirac)

$$I(C:D) \geq \frac{(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle)^2}{2\|\mathcal{O}_C\|^2 \|\mathcal{O}_D\|^2}$$



# Connected correlators as geodesics in AdS/CFT

The connected 2-pt correlation function of a heavy operator behaves like (Balasubramanian & Ross)

$$\langle \mathcal{O}_C(x_c)\mathcal{O}_D(x_D)\rangle - \langle \mathcal{O}_C(x_c)\rangle\langle \mathcal{O}_D(x_D)\rangle \sim e^{-mL_{\text{bulk}}(x_C,x_D)}$$

- $L_{\text{bulk}}(x_C, x_D)$ is the bulk geodesic distance between the boundary points $x_C$ and $x_D$
- $\Delta_{\mathcal{O}} = m\ell \gg 1$ (not scaling with N or c)
- Holographic dual correlation only depends mildly on the dual operator (through $\Delta_{\mathcal{O}}$)

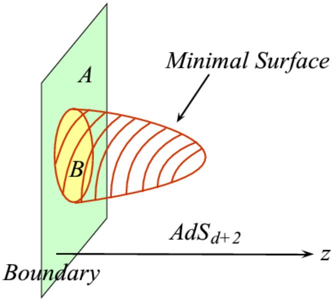
# Entanglement entropy in AdS/CFT

Entanglement entropy in an strongly coupled CFT vs bulk geometry.
Ryu & Takayanagi

$$S(\rho_B) \propto \text{Area}(\partial B) \propto \text{Area}(\Sigma_{\text{bulk}})$$

where $\Sigma_{\text{bulk}}$ is a bulk minimal surface anchored to $\partial B$

- Non-local diffeomorphism invariant observables
- Deep relation between the set of minimal surfaces and Einstein's equations



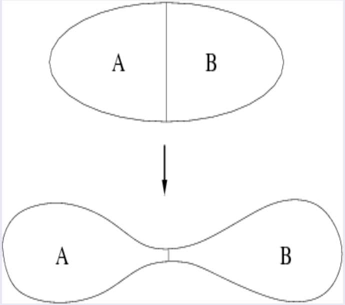
# Entanglement vs spacetime connectedness

Consider a full quantum system described by $A \cup B$
Study the limit of vanishing entanglement holographically

## Spacetime connectedness (van Raamsdonk)

Sending entanglement to zero, requires :

1. Proper bulk distance to infinity
2. Area of the common boundary to zero $\Rightarrow$ pinching



# Consequences

## Quantum mechanics

Consider a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with no interactions

1. Product states have vanishing connected correlators
2. Entangled states have non-vanishing correlators !!

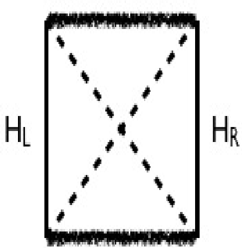
## AdS/CFT

Consider 2 decoupled CFTs

1. Product states having holographic duals correspond to disconnected asymptotically AdS spacetimes
   Example : $|\mathrm{vac}\rangle \otimes |\mathrm{vac}\rangle \Rightarrow$ 2 disconnected AdS spacetimes
2. Entangled states $\Rightarrow \exists$ correlations $\Rightarrow$ connected geometry !!
   Example : eternal AdS black hole

# Eternal AdS BH revisited

1. Classical maximal extension of the eternal AdS BH
2. Connectedness through BH event horizon




For certain observables and low energies, an observer in $\mathcal{H}_R$ measures a thermal state :

$$\rho_{\rm BH} = \frac{1}{Z(\beta)} \sum_i e^{-\beta E_i} |E_i\rangle\langle E_i| \,, \qquad |E_i\rangle \in \mathcal{H}_R$$

Can we interpret $\rho_{\rm BH}$ as a reduced density matrix ? (Maldacena)

$$\rho_{\rm BH} = {\rm tr}_{\mathcal{H}_L} |\Psi\rangle\langle\Psi| \;\; \text{with} \;\; |\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\beta E_i/2} |E_i\rangle \otimes |E_i\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$$

Quantum entanglement is responsible for the existence of correlations

# EPR = ER (Maldacena & Susskind)

## Eternal black hole re-interpreted

1. Non-vanishing correlators between $\mathcal{H}_L$ and $\mathcal{H}_R$ are due to quantum entanglement (EPR)
2. These correlations are holographically captured by the bulk geodesic distance between opposite boundaries $\Rightarrow$ length of the ER bridge
3. Entanglement entropy = black hole entropy $\Rightarrow$ maximal cross-section of the ER bridge

# Outline

- 2d CFT set-up
- Free scalar 2d CFT at $T = 0$ (warm-up)
- Large c 2d CFTs at finite $T$ & thermo field double
- Holographic remarks
- Final remarks

# Set-up (single CFT)

Consider an excited state in a 2d CFT

$$|\Psi_{\mathcal{O}}(t)\rangle = \sqrt{\mathcal{N}}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0,-\ell)|0\rangle$$

- $\mathcal{O}$ is inserted at $t = 0$ and $x = -\ell$ and dynamically evolved afterwards
- $\epsilon$ is a small parameter smearing the UV behaviour of the local operator

Density matrix :

$$\begin{aligned}\rho(t) &= \mathcal{N}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0,-\ell)|0\rangle\langle 0|\mathcal{O}^\dagger(0,-\ell)\, e^{iHt}\, e^{-\epsilon H} \\ &= \mathcal{N}\,\mathcal{O}(\omega_2,\bar{\omega}_2)|0\rangle\langle 0|\mathcal{O}^\dagger(\omega_1,\bar{\omega}_1)\end{aligned}$$

where $\omega_1 = -\ell + i(\epsilon - it)$, $\omega_2 = -\ell - i(\epsilon + it)$ $(\bar{\omega}_1 = -\ell - i(\epsilon - it))$

# Set-up (notation)

Our calculations will be done in euclidean signature :

$$\omega = x + i\tau\,, \qquad \bar{\omega} = x - i\tau$$

We use the euclidean continuation : $\tau = it$

- The normalization factor $\mathcal{N}$ is fixed by $\mathrm{Tr}(\rho(t)) = 1$
- The cut-off $\epsilon$ can be viewed as a separation in the insertion time appearing in $\rho(t)$

# Replica trick - I

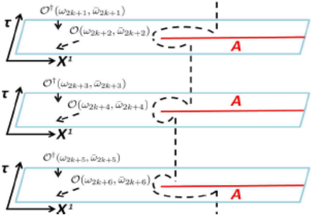
Following Cardy & Calabrese

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left(\frac{\mathrm{Tr}\rho_A^n}{\mathrm{Tr}\left(\rho_A^{(0)}\right)^n}\right)$$

$$= \frac{1}{1-n}\log\left[\frac{\langle\mathcal{O}(\omega_1,\bar{\omega}_1)\mathcal{O}^\dagger(\omega_2,\bar{\omega}_2)...\mathcal{O}^\dagger(\omega_{2n},\bar{\omega}_{2n})\rangle_{\Sigma_n}}{\left(\langle\mathcal{O}(\omega_1,\bar{\omega}_1)\mathcal{O}^\dagger(\omega_2,\bar{\omega}_2)\rangle_{\Sigma_1}\right)^n}\right]$$

Notice no twisted operators but CFT defined on a Riemann surface

- $\omega_{2k+1} = e^{2\pi i k}\omega_1$
- $\omega_{2k+2} = e^{2\pi i k}\omega_2$



# Replica trick - II

Following Cardy & Calabrese

$$\mathrm{Tr}\rho_A^n \sim \langle\psi|\sigma(\omega_1,\bar{\omega}_1)\tilde{\sigma}(\omega_2,\bar{\omega}_2)|\psi\rangle$$

$|\psi\rangle$ stands for whatever CFT state you want to consider (vacuum or excited state)

- Non-trivial topology replaced by twist operators
- Calculation done in n-copies of the original CFT
- Twist operators emerge because of the existence of some internal symmetry when swapping these copies

# Free scalar 2d CFT & finite region A

Compute the Renyi entropy variation for $n = 2$
Strategy : Map $\Sigma_2$ into $\Sigma_1$ using the conformal transformation (uniformization)

$$\frac{\omega}{\omega - L} = z^2$$

The 4-pt function determining the Renyi entropy will equal

$$\begin{aligned}
&\langle \mathcal{O}^\dagger(\omega_1,\bar{\omega}_1)\mathcal{O}(\omega_2,\bar{\omega}_2)\mathcal{O}^\dagger(\omega_3,\bar{\omega}_3)\mathcal{O}(\omega_4,\bar{\omega}_4)\rangle_{\Sigma_2} \\
&= \prod_{i=1}^{4}\left|\frac{d\omega_i}{dz_i}\right|^{-2\Delta_{\mathcal{O}}} \langle \mathcal{O}^\dagger(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\mathcal{O}^\dagger(z_3,\bar{z}_3)\mathcal{O}(z_4,\bar{z}_4)\rangle_{\Sigma_1} \\
&= \prod_{i=1}^{4}\left|\frac{d\omega_i}{dz_i}\right|^{-2\Delta_{\mathcal{O}}} |z_{13}z_{24}|^{-4\Delta_{\mathcal{O}}}\, G(z,\bar{z})
\end{aligned}$$

where the cross-ratio $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, with $z_{ij} = z_i - z_j$

# Free scalar 2d CFT

Altogether

$$\frac{\langle \mathcal{O}^\dagger(\omega_1,\bar\omega_1)\mathcal{O}(\omega_2,\bar\omega_2)\mathcal{O}^\dagger(\omega_3,\bar\omega_3)\mathcal{O}(\omega_4,\bar\omega_4)\rangle_{\Sigma_2}}{\left(\langle \mathcal{O}^\dagger(\omega_1,\bar\omega_1)\mathcal{O}(\omega_2,\bar\omega_2)\rangle_{\Sigma_1}\right)^2} = |z|^{4\Delta_\mathcal{O}}|1-z|^{4\Delta_\mathcal{O}}\, G(z,\bar z)$$

We will consider two different excitations with $\Delta_\mathcal{O} = \frac{1}{8}$

- When $\mathcal{O}_1 = e^{i\phi/2}$, then

$$G_1(z,\bar z) = \frac{1}{\sqrt{|z||1-z|}}$$

- When $\mathcal{O}_2 = \frac{1}{2}\left(e^{i\phi/2} + e^{-i\phi/2}\right)$,

$$G_2(z,\bar z) = \frac{|z| + 1 + |1-z|}{2}\, G_1(z,\bar z)$$

# Specific details

Our points $(z_i, \bar{z}_i)$ equal

$$z_1 = -z_3 = \sqrt{\frac{\ell - t - i\epsilon}{\ell + L - t - i\epsilon}}\,,$$
$$z_2 = -z_4 = \sqrt{\frac{\ell - t + i\epsilon}{\ell + L - t + i\epsilon}}\,.$$

In the limit of small $\epsilon$ we obtain

- $(z, \bar{z}) \to (0, 0)$ when $0 < t < \ell$ or $t > L + \ell$

$$z \simeq \frac{L^2\epsilon^2}{4(\ell - t)^2(L + \ell - t)^2}\,, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(\ell + t)^2(L + \ell + t)^2}\,.$$

- $(z, \bar{z}) \to (1, 0)$ when $\ell < t < L + \ell$

$$z \simeq 1 - \frac{L^2\epsilon^2}{4(\ell - t)^2(L + \ell - t)^2}\,, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(\ell + t)^2(L + \ell + t)^2}\,.$$

# Results & interpretation

$$\Delta S_A^{(2)}(\mathcal{O}_1) = 0 \quad \text{all times}$$

$$\Delta S_A^{(2)}(\mathcal{O}_2) = \begin{cases} 0 & 0 < t < \ell\,, \text{ or } t > \ell + L \\ \log 2 & \ell < t < \ell + L \end{cases}$$

- $\Delta S_A^{(2)}(\mathcal{O}_1) = 0$ because it can be viewed as a direct product state

$$e^{i\phi_L/2}|0\rangle_L \otimes e^{i\phi_R/2}|0\rangle_R$$

- Since $\mathcal{O}_2$ creates a maximally entangled state at $x = -\ell$ propagating in opposite directions

$$\frac{1}{\sqrt{2}}\left(e^{i\phi_L/2}|0\rangle_L \otimes e^{i\phi_R/2}|0\rangle_R + e^{-i\phi_L/2}|0\rangle_L \otimes e^{-i\phi_R/2}|0\rangle_R\right)$$

  - Causality makes both pairs to be in the complement of $A$ for $0 < t < \ell$ and $t > \ell + L$
  - for $\ell < t < \ell + L$ one member of the pair lies in $A$.

# Excitations at finite temperature

Same set-up as before, but now

1. we perturb a thermal state :

$$\rho(t) \equiv \mathcal{N}\mathcal{O}(\omega_2, \bar{\omega}_2)\, e^{-\beta H}\, \mathcal{O}^\dagger(\omega_1, \bar{\omega}_1)$$

   with

$$\omega_1 = x_0 + t + t_\omega + i\epsilon \qquad \bar{\omega}_1 = x_0 - t - t_\omega - i\epsilon$$
$$\omega_2 = x_0 + t + t_\omega - i\epsilon \qquad \bar{\omega}_2 = x_0 - t - t_\omega + i\epsilon\,.$$

2. A pair of operators will be inserted on a cylinder, separated $2i\epsilon$

# Our calculation & notion of "scrambling"

- Consider a thermofield double set-up.
- Perturbed the system at $-t_\omega$ by a primary localised operator $\mathcal{O}$
- Evolve unitarily

Measure the amount of entanglement at $t = 0$ using the mutual information

$$I(A : B; t_\omega) = S_A + S_B - S_{A\cup B}$$

We can ask what the time scale $t_\omega$ has to be so that the perturbation can not be distinguished from the original thermal state (scrambling time)

$$\Delta I(A : B; t_\omega) = \Delta S_A + \Delta S_B - \Delta S_{A\cup B} = 0$$

# What our condition boils down to

Hartman & Maldacena showed that in the absence of perturbation :

- at early times, mutual information decreases linearly
- at late times, i.e. $t > \frac{L}{2}$, $S_{A\cup B} = S_A + S_B$ saturates and the mutual information vanishes.

Thus, if we assume $t^\star_\omega > \frac{L}{2}$, our condition reduces to

$$I(A : B; t^\star_\omega) = 0$$

This is what was analysed by Shenker & Stanford and what we will end up discussing today.

# Thermofield double set-up

Consider two non-interacting 2d CFTs, say $\text{CFT}_L$ and $\text{CFT}_R$, with isomorphic Hilbert spaces $\mathcal{H}_{L,R}$

Thermofield double state :

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R$$

- $|n\rangle_L$ is an eigenstate of the hamiltonian $H_L$ acting on $\mathcal{H}_L$ with eigenvalue $E_n$ (and similarly for $|n\rangle_R$).
- $|n\rangle_L$ is the CPT conjugate of the state $|n\rangle_R$
- Notation : $|n\rangle_L \otimes |n\rangle_R$ as $|n\rangle_L |n\rangle_R$.
- Thermal reduced density

$$\rho_R(\beta) = \text{tr}_{\mathcal{H}_L} \left( |\Psi_\beta\rangle \langle \Psi_\beta| \right) = \frac{1}{Z(\beta)} \sum_{n \in \mathcal{H}_R} e^{-\beta E_n} |n\rangle_R \langle n|_R \ ,$$

# Thermofield double : observables

- Single sided correlators are thermal

$$\langle\Psi_\beta|\,\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t_1)\ldots\mathcal{O}_R(x_n,t_n)\right)\,.$$

- Two sided correlators : by analytic continuation

$$\langle\Psi_\beta|\,\mathcal{O}_L(x_1,-t)\ldots\mathcal{O}_R(x_n',t_n')\,|\Psi_\beta\rangle = \mathrm{tr}_{\mathcal{H}_R}\left(\rho_R(\beta)\mathcal{O}_R(x_1,t-i\beta/2)\ldots\mathcal{O}_R(x_n',t_n')\right)\,.$$

Will use this observation when computing Renyi entropies

# CFT considerations

As discussed by Morrison & Roberts (see also Hartman & Maldacena) :

- *single* sided thermal correlation functions are computed on a *single* cylinder with periodicity $\tau \sim \tau + \beta$
- *two-sided* correlators involve a path integral over a cylinder with the same periodicity $\tau \sim \tau + \beta$, where *all* operators $\mathcal{O}_R$ are inserted at $\tau = i\beta/2$, whereas $\mathcal{O}_L$ are inserted at $\tau = 0$

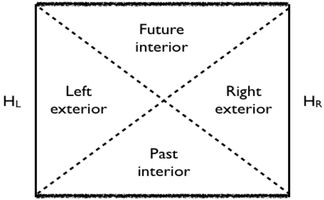
Set-up : Consider thermofield double state

- two finite intervals: $A = [L_1, L_2]$ in the left $\text{CFT}_L$ and $B = [L_1, L_2]$ in the right $\text{CFT}_R$
- perturb the TFD by the insertion of a local primary operator $\mathcal{O}_L$ acting on $\text{CFT}_L$ at $x = 0\,, t_- = -t_\omega$

# Bulk interpretation

1. *Single* BH in thermal equilibrium : evolution by a boost $(H_R - H_L)$

$$H_{\rm tf} = \mathbb{I}_L \otimes H_R - H_L \otimes \mathbb{I}_R \, .$$




- Time propagates upwards in $\mathcal{H}_R$ and downwards in $\mathcal{H}_L$.
- Thermofield double is (boost) invariant

2. Approximate description of the state at $t = 0$ of *two* AdS black holes $(H_R + H_L)$

$$H = \mathbb{I}_L \otimes H_R + H_L \otimes \mathbb{I}_R \equiv H_R + H_L \, .$$

Time propagates upwards in both boundaries

# Calculation of $S_A$

$$S_A = -\lim_{n\to 1} \frac{1}{n-1} \log\left(\mathrm{Tr}\, \rho_A^n(t)\right)$$

where

$$\mathrm{Tr}\, \rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_3,\bar{x}_3)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$x_1 = -i\epsilon\,, \quad x_2 = L_1 - t_\omega - t, \quad x_3 = L_2 - t_\omega - t\,, \quad x_4 = +i\epsilon$$
$$\bar{x}_1 = +i\epsilon\,, \quad \bar{x}_2 = L_1 + t_\omega + t, \quad \bar{x}_3 = L_2 + t_\omega + t\,, \quad \bar{x}_4 = -i\epsilon$$

with conformal dimensions

$$H_\psi = n h_\psi\,, \quad H_\sigma = n h_\sigma = n\,\frac{c}{24}\left(n - \frac{1}{n}\right)$$

# Conformal maps

1. From the cylinder to the plane

$$\omega(x) = e^{2\pi x/\beta}$$

2. Standard map : $\omega_1 \to 0$, $\omega_2 \to z$, $\omega_3 \to 1$ and $\omega_4 \to \infty$

$$z(\omega) = \frac{(\omega_1 - \omega)\omega_{34}}{\omega_{13}(\omega - \omega_4)}$$

where the cross-ratio satisfies

$$z = \frac{\omega_{12}\omega_{34}}{\omega_{13}\omega_{24}}$$

# Result

$$S_A^{(n)} = \frac{c(n+1)}{6} \log\left(\frac{\beta}{\pi\epsilon_{UV}} \sinh\frac{\pi(L_2 - L_1)}{\beta}\right) - \frac{1}{n-1}\log\left(|1-z|^{4H_\sigma} G(z,\bar{z})\right)$$

where

$$G(z,\bar{z}) = \langle\psi|\,\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle$$

Using the large c results derived by Fitzpatrick, Kaplan & Walters in the limit $n \to 1$

$$\Delta S_A = \frac{c}{6}\log\left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)}\bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)}(1-z_\psi^{\alpha})(1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi\bar{\alpha}_\psi(1-z)(1-\bar{z})}\right)$$

where $\alpha_\psi = \sqrt{1 - \frac{h_\psi}{c}}$.

# Cross-ratios

The cross-ratios are

$$z = \frac{\sinh\left(\frac{\pi x_{12}}{\beta}\right)\sinh\left(\frac{\pi x_{34}}{\beta}\right)}{\sinh\left(\frac{\pi x_{13}}{\beta}\right)\sinh\left(\frac{\pi x_{24}}{\beta}\right)}$$

$$\simeq 1 + \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi(L_2-L_1)}{\beta}}{\sinh\frac{\pi(L_2-t-t_\omega)}{\beta}\sinh\frac{\pi(L_1-t-t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\bar{z} = \frac{\sinh\left(\frac{\pi \bar{x}_{12}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{34}}{\beta}\right)}{\sinh\left(\frac{\pi \bar{x}_{13}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{24}}{\beta}\right)}$$

$$\simeq 1 - \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi(L_2-L_1)}{\beta}}{\sinh\frac{\pi(L_2+t+t_\omega)}{\beta}\sinh\frac{\pi(L_1+t+t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

# Final result

Analysing the imaginary parts, we reach the conclusions :

- $(z, \bar{z}) \to (1, 1)$ for $t + t_\omega < L_1$ and $t + t_\omega > L_2$
- $(z, \bar{z}) \to (e^{2\pi i}, 1)$ for $L_1 < t + t_\omega < L_2$

The importance of this monodromy has been emphasized by several groups including Asplund, Bernamonti, Galli & Hartman and Roberts & Stanford

$$\boxed{\Delta S_A = 0\,, \quad t_- + t_\omega < L_1 \text{ and } t_- + t_\omega > L_2}$$

$$\boxed{\begin{array}{l}\Delta S_A = \dfrac{c}{6} \log \left[ \dfrac{\beta}{\pi\epsilon} \dfrac{\sin \pi\alpha_\psi}{\alpha_\psi} \dfrac{\sinh\left(\frac{\pi(L - t_- - t_\omega)}{\beta}\right) \sinh\left(\frac{\pi(t_- + t_\omega)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)} \right] \\ \qquad L_1 < t_- + t_\omega < L_2 \end{array}}$$

where $L = L_2 - L_1$

# Calculation of $S_B$

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned} x_1 &= -i\epsilon\,, \quad & x_5 &= L_2 + i\frac{\beta}{2} - t, \quad & x_6 &= L_1 + i\tfrac{\beta}{2} - t\,, \quad & x_4 &= +i\epsilon \\ \bar{x}_1 &= +i\epsilon\,, \quad & \bar{x}_5 &= L_2 - i\frac{\beta}{2} + t, \quad & \bar{x}_6 &= L_1 - i\tfrac{\beta}{2} + t\,, \quad & \bar{x}_4 &= -i\epsilon \end{aligned}$$

We always obtain the expected thermal answer at all times

$$\boxed{S_B = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon_{\mathrm{UV}}}\sinh\frac{\pi L}{\beta}\right)}$$

# Calculation of $S_{A\cup B}$

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho^n_{A\cup B}(t) = \frac{\langle\psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_2\bar{x}_3)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle\psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned}
x_1 &= -i\epsilon, \quad x_2 = L_1 - t_\omega - t_-, \quad x_3 = L_2 - t_\omega - t_-, \quad x_4 = +i\epsilon\,, \\
\bar{x}_1 &= +i\epsilon, \quad \bar{x}_2 = L_1 + t_\omega + t_-, \quad \bar{x}_3 = L_2 + t_\omega + t_-, \quad \bar{x}_4 = -i\epsilon\,, \\
x_5 &= L_2 + i\frac{\beta}{2} - t_+, \quad x_6 = L_1 + i\frac{\beta}{2} - t_+\,, \\
\bar{x}_5 &= L_2 - i\frac{\beta}{2} + t_+, \quad \bar{x}_6 = L_1 - i\frac{\beta}{2} + t_+\,.
\end{aligned}$$

# Strategy

Using conformal maps

$$\mathrm{Tr}\,\rho^n_{A\cup B} = \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|z_{56}|^{4H_\sigma}$$
$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

where all cross-ratios $z$, $z_i$ are analytically known.

- $\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$ expected 6-pt function

# S-channel (I)

Let us introduce a resolution of the identity

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$
$$= \sum_{\alpha} \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

- $(z,\bar{z}) \to (1,1)$ for $t_- + t_\omega > L_2 \Rightarrow$ use OPE !!
- $\sigma(z,\bar{z})\tilde{\sigma}(1,1) \sim \mathbb{I} +$ corrections in $(z-1)^r\ \mathcal{O}_r$
- Orthogonality of 2-pt functions $\Rightarrow |\alpha\rangle = |\psi\rangle$ dominant

Thus,

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$
$$\simeq \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle\,\langle\psi|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

# S-channel (II)

Using conformal maps

$$\langle\psi|\,\sigma(z_5,\bar z_5)\tilde\sigma(z_6,\bar z_6)|\psi\rangle = |1-\tilde z_5|^{4H_\sigma}\,|z_{56}|^{-4H_\sigma}\,\langle\psi|\,\sigma(\tilde z_5,\bar{\tilde z}_5)\tilde\sigma(1,1)|\psi\rangle\,,$$

we obtain

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|1-\tilde z_5|^{4H_\sigma}\,G(z,\bar z)G(\tilde z_5,\bar{\tilde z}_5)+\ldots$$

Since $\tilde z_5 = z_5$, the cross-ratio determining $S_B$, we derive

$$S_{A\cup B} = S_A + S_B, \qquad \text{and} \qquad I_{A:B} = 0$$

This resembles the bulk calculation from two geodesics joining pairs of points in the same boundary !!

# T-channel (I)

We could introduce the resolution of the identity as follows

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$
$$= \sum_{\alpha}\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(z_6,\bar{z}_6)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(1,1)|\psi\rangle\,.$$

- $(z_5,\bar{z}_5)\to(1,1)$ for small $\epsilon\Rightarrow$ use OPE !!
- As before, $|\alpha\rangle=|\psi\rangle$ dominant contribution !!

# T-channel (II)

In this case,

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma}\left|\frac{x}{1-x}\right|^{4H_\sigma}|1-z_5|^{4H_\sigma}|1-\tilde{z}_2|^{4H_\sigma}$$
$$G(\tilde{z}_2,\bar{\tilde{z}}_2)G(z_5,\bar{z}_5)+\ldots$$

where $(x,\bar{x})$ are the cross-ratios computed out of the insertion points of the four twist operators

$$x=\frac{z_{23}z_{56}}{z_{25}z_{36}}=\frac{w_{23}w_{56}}{w_{25}w_{36}}=\frac{2\sinh^2\frac{\pi(L_2-L_1)}{\beta}}{\cosh\frac{2\pi(L_2-L_1)}{\beta}+\cosh\frac{2\pi(t_-+t_\omega-t_+)}{\beta}}=\bar{x}\,,$$

# T-channel (III)

For $t_- + t_\omega > L_2$, we derive

$$S_{A\cup B} \simeq \frac{2c}{3}\log\left|\frac{\beta}{\pi\epsilon_{\text{UV}}}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right)$$
$$+\frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_w)}{\beta}\cosh\frac{\pi t_+}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\frac{\sinh\frac{\pi(t_-+t_w-L)}{\beta}\cosh\frac{\pi(L-t_+)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right)$$

where we set $L_1 = 0$, $L_2 = L$ and $\Delta t = t_- + t_\omega - t_+$

- To derive this result we used Fitzpatrick, Kaplan & Walters

# Mutual information & "Scrambling" time (I)

In the regime $t_- + t_\omega > L_2 > L_1$,

$$\boxed{\begin{aligned} I_{A:B} \simeq\ & \frac{2c}{3}\log\left(\frac{\beta}{\pi\epsilon_{\rm UV}}\sinh\frac{\pi L}{\beta}\right) - \frac{2c}{3}\log\left|\frac{\beta}{\pi z_\infty}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| \\ & - \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha\psi}\right) \\ & - \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_\omega)}{\beta}\cosh\frac{\pi t_+}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\frac{\sinh\frac{\pi(t_-+t_\omega-L)}{\beta}\cosh\frac{\pi(L-t_+)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \end{aligned}}$$

- take $t_- = t_+ = 0$ and look for $t_\omega^\star$ satisfying

$$I_{A:B}(t_\omega^\star) = 0$$

# Mutual information & "Scrambling" time (II)

Assuming $t^\star_\omega/\beta \gg 1$, we currently obtain

$$t^\star_\omega = \frac{L}{4} - \frac{\beta}{2\pi}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right) + \frac{\beta}{4\pi}\log\left(\frac{8\sinh^4\frac{\pi L}{\beta}}{\cosh\frac{\pi L}{\beta}}\right)$$

- if $h_\psi \ll c$, then

$$t^\star_\omega = f(L,\beta) + \frac{\beta}{2\pi}\log\frac{S/L}{\pi\, E_\psi}$$

where we used

$$\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi} \sim \frac{\pi E_\psi}{S/L}$$

with $\frac{S}{L} = \frac{\pi c}{3\beta}$ and $E_\psi = \frac{h_\psi}{\epsilon}$

# Holographic considerations

Main idea & strategy :

- Static point particle at $r = 0$ in global $AdS_3$

$$ds^2 = -\left(r^2 + R^2 - \mu\right) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2 - \mu} + r^2 d\varphi^2 \,,$$

- Holographic entanglement entropy known

$$S_A = \frac{c}{6}\left[\log\left(\frac{r_\infty^{(1)} \cdot r_\infty^{(2)}}{R^2}\right) + \log\frac{2\cos\left(|\Delta\tau_\infty|\alpha_\mu\right) - 2\cos\left(|\Delta\varphi_\infty|\alpha_\mu\right)}{\alpha_\mu^2}\right]$$

- Map metric to Kruskal coordinates, while boosting the particle, to describe a free falling particle in eternal BTZ
  - Use an initial condition ensuring the particle carries the right energy, from CFT and stress tensor perspective
- Map endpoints & compute entanglement entropy

# Holographic comments

Calculations involve many explicit technical details, leading to

1. Exact matching of dominant CFT contributions with the holographic model geodesic calculations
   - S-channel and T-channel contributions precisely match the two dominant geodesics computing $S_{A \cup B}$
2. In the limit of large $t_\omega$ :
   - free falling particle becomes almost null with energy localised at the horizon
   - matches the schock-wave descriptions proposed/used by Shenker, Stanford, Roberts, Susskind

# Final remarks

- Stringy corrections (Shenker & Stanford)
- Our results in the CFT follow from properties of 2d correlators in the large c limit
  - they may exist in other slicings of AdS, i.e. hyperbolic slicing responsible for AdS-Rindler physics
  - this may be related to the bulk expectation that scrambling occurs more generally than for event horizons (Susskind, Fischler et al)
- Statistics of OPE coefficients : thermalisation, typicality of correlators in CFTs and validity of ensembles in CFT