HIGHER SPIN CORRECTIONS TO ENTANGLEMENT ENTROPY FROM CFT

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INTRODUCTION AND MOTIVATION

• Entanglement entropy is emerging as an important physical observable in quantum systems.

• Our interest in EE is due to its connection with holography.

It is a good observable to study in theories which admit a gravity dual.

• Studying it in gravity will teach us how entanglement is encoded in geometry.

• Definition: Consider a Hilbert space described by a set of commuting observables.

Partition the observables into 2 set of disjoint observables:

A and B.

Let the density matrix of the system be ρ.

Define the reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho$$

Entanglement entropy is the Von-Neumann entropy given by

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

By definition entanglement entropy reduces to the thermal entropy if ρ corresponds to the thermal state and when the sub-system A tends to the full system.

• There is a closely associated measure of entanglement: Rényi entropy.

$$S_A^n = \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n)$$

In the $n \to 1$ limit, $S_A^n \to S_A$.

• In quantum field theories: RE/EE are usually very difficult to evaluate.

• However there exists some exact results for $1+1$ relativistic quantum field theories at critical points.

• Consider a $1+1$ dimensional conformal field theory with central charge c on a real line.

Let the sub space A be an interval of length Δ.

Then EE is given by

$$S_A = \frac{c}{3}\ln(\frac{\Delta}{\epsilon})$$

ϵ is a UV cut off.

• If the CFT is on a circle of size L

$$S_A = \frac{c}{3}\ln\left(\frac{L}{\pi\epsilon}\sin(\frac{\pi\Delta}{L})\right)$$

• If the CFT is on the real line, but at finite temperature: β

$$S_A = \frac{c}{3}\ln\left(\frac{\beta}{\pi\epsilon}\sinh(\frac{\pi\Delta}{\beta})\right)$$

• There are similar expressions for the Rényi entropy eg.

$$S_A = \frac{c}{6}(n + \frac{1}{n}) \ln(\frac{\Delta}{\epsilon})$$

• Apart from the central charge c, EE/RE of a single interval does not have any other information of the theory.

It is 'universal'.

• In this respect it is similar to the high temperature behaviour of the thermal entropy of CFT, given by the Stefan-Boltzman-Cardy formula

$$s = \frac{\pi^2}{3} \frac{c}{\beta}$$

• Since the result for the EE is universal it must be possible to obtain the result in holography.

• There is a very simple geometric proposal to evaluate entanglement entropy for field theories which admit a gravitational dual
Ryu, Takayanagi 2006.

For the $1+1$ dimensional case, one has to consider AdS_3 geometry.

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2)$$

Examine the geometry at a constant time slice.

Consider a geodesic which originates at the boundary at point P and goes back to the boundary at Q.

$$P : (\rho_0, \theta = 0) \qquad Q : (\rho_0, \theta = \frac{2\pi\Delta}{L})$$

Boundary CFT in on a circle of circumference L.

The proposal states that the entanglement entropy is given by

$$S_A = \frac{1}{4G_N} \times (\text{Geodesic length between } P \text{ and } Q)$$

G_N is the 3 dimensional Newton's constant.

The geodesic length is given by

$$\cosh\left(\frac{l_{\rm geodesic}}{R}\right) = 1 + 2\sinh^2\rho_0 \sin^2\frac{\pi\Delta}{L}$$

Push $\rho_0 \to \infty$, the boundary

$$\frac{l_{\rm geodesic}}{R} = 2\ln\left(e^{\rho_0}\sin\frac{\pi\Delta}{L}\right)$$

Then the proposal states

$$\begin{aligned} S_A &= \frac{l_{\rm geodesic}}{4G_N} \\ &= \frac{1}{4}\times\frac{2c}{3R}\times 2R\ln\left(e^{\rho_0}\sin\frac{\pi\Delta}{L}\right) \\ &= \frac{c}{3}\ln\left(e^{\rho_0}\sin\frac{\pi\Delta}{L}\right) \end{aligned}$$

Note e^{ρ_0} plays the role of the UV cut off.

• This proposal reproduces the known result for the entanglement entropy of a single interval from CFT.

It also satisfies certain known properties of entanglement entropy (eg. strong sub-additivity).

• There is no such simple proposal for the evaluation of the Rényi entropy in holography.

It fails to capture more detailed properties of entanglement entropy.

• How one get more information about the CFT from RE/EE? (detailed properties).

We will list 3 methods.

Method1: Consider more than one disjoint intervals in the CFT. Say consider 2 intervals with end points

$$A: \ (y_1, y_2) \qquad B: \ (y_3, y_4)$$

Then the more detailed behaviour of $S(A \cup B)$ is captured by the cross ratio $x(y_1, y_2, y_3, y_4)$.

As an expansion in the cross ratio there is information about the spectrum of the theory.

The RT proposal is blind to this information of the CFT.

Recently methods have been developed on how to obtain these corrections in holography and precise matching with CFT results have been obtained.

• Method 2: Place the system on a ring of radius R and at finite temperature β.

The CFT is on a torus. The EE of an interval of size Δ

$$S_A = \frac{c}{3} \ln\left(\frac{\beta}{\epsilon} \sinh \frac{\pi \Delta}{\beta}\right) + e^{-RT} f(\Delta)$$

These are finite size corrections.

Finite size corrections capture information of the spectrum of the theory.

The RT prescription does not reproduce the finite size corrections.

Finite size corrections of RE/EE for free boson and free fermion CFT's on a torus has been evaluated.

The leading finite size corrections have been compared with more refined methods of evaluating these in the gravity dual developed by Faulkner, Barrella et. al 2013

and precise agreement obtained.
Datta, JRD JHEP 1404 (2014) 081

• Method 3: Consider CFT's deformed by holomorphic currents and held at temperature β.

The action is given by

$$S_{CFT} = S^{(0)}_{CFT} + \mu \int d^2 z (W(z) + \bar{W}(\bar{z}))$$

We consider currents of dimension $(3, 0)$ and $(0, 3)$, spin 3 current.

μ is the chemical potential.

• For any CFT which admits a spin 3 current we show that the entanglement entropy for a single interval is corrected

$$S_A(\beta,\mu,\Delta)=\frac{c}{3}\ln\left(\frac{\beta}{\epsilon}\sinh\frac{\pi\Delta}{\beta}\right)+\frac{\mu^2}{\beta^2}S_A^{(2)}(\beta,\Delta)+O(\mu^4)$$

We evaluate $S_A^{(2)}(\beta,\Delta)$ and also prove it is universal.

- Whats the motivation for this question ?

CFTs with higher spin symmetry is of interest due to the proposal of Gopakumar, Gaberdiel 2011.

Certain coset minimal model CFT's with $\mathcal{W}_\infty[\lambda]$ are dual to the Vasiliev theory in AdS_3 with an infinite tower of spins.

This theory can be written as a Chern-Simons theory based on the group $\mathrm{hs}[\lambda]$.

• The thermal entropy of CFTs deformed with a $(3,0)$ current has been evaluated in the high temperature limit

$$s(\mu,\beta,\lambda)=\frac{\pi^2 c}{3\beta}\left(1+\frac{32\pi^2}{3}\frac{\mu^2}{\beta^2}+\frac{\mu^4}{\beta^4}f(\lambda)+O(\mu^6)\right)$$

These are corrections to the Stefan-Botlzman-Cardy formula.

The $O(\mu^2)$ term is universal, does not depend on λ.
Gaberdiel, Hartman, Jin 2012

• In the dual Vasiliev theory a black hole with spin-3 charge has been constructed and an expression for its entropy $s(\mu,\beta,\lambda)$ obtained.

It agrees to order the $O(\mu^6)$ evaluated in the CFT.

• There is a proposal by deBoer, Jottar and Castro, Iqbal 2013 to evaluate entanglement entropy in holographic higher spin theories by generalizing the proposal of Ryu, Takanayagi.

It uses the Chern-Simons formulation of the higher spin theory.

The EE is written in terms of a Wilson line in the bulk joining the end points of the entangling interval.

A formula for the EE $S_A(\mu, \beta, \lambda = 3)$,
($\lambda = 3$ theory contains only spins 3 and spin 2),
has been written down resulting from this proposal.

• The CFT calculation and the fact the $O(\mu^2)$ term is universal will enable a precise check on the holographic proposal for entanglement entropy in these theories.

EVALUATION OF ENTANGLEMENT ENTROPY IN CFT

The Replica Trick

• Consider the evaluation of $\mathrm{Tr}\exp(-\beta H)$ in the path integral language.

One evolves the CFT living on the real line and then sews up after evolving to β because of the trace.

• The reduced density matrix $\rho_A = \mathrm{Tr}_B \exp(-\beta H)$ is obtained by evolving to β,

then sewing the CFT over the region B, the complement of A.

There is a cut along the interval A

- Then

$$\mathrm{Tr}(\rho_A^n) = \mathrm{Tr}(\rho_A \rho_A \cdots \rho_A)$$

is obtained by sewing n-copies of the CFT on the cut along the interval A.

The last copy joined to the original sheet.

• We therefore have

$$\mathrm{Tr}\rho_A^n = \frac{Z_n[A]}{(Z[1])^n}$$

The partition function on the n -sheeted Riemann surface joined along the interval A.

• The path integral representation is given by

$$Z_n[A] = \int [d\varphi_i] \exp(-S[\varphi_1] - S[\varphi_2] - \cdots S[\varphi_n])$$

with the boundary conditions

$$\varphi_k(\sigma, \tau = \beta^-) = \varphi_{k+1}(\sigma, \tau = 0^+), \qquad \sigma \in A = (y_1, y_2)$$

• It is easy to see by performing a discrete Fourier transform of the fields φ_i that the points y_1, y_2 are fixed points of a Z_n orbifold.

- It can be shown that

$$\mathrm{Tr}(\rho_A^n) = \frac{Z_n[A]}{Z^n}$$

is equal to the 2 point function of branch point twist operator τ_n of conformal dimensions

$$d_n = \bar{d}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

- The branch point twist operator implements the boundary conditions

$$\varphi_k(\tau + \beta^-, \sigma)\tau_n(y_1) \sim \varphi_{k+1}(\tau + 0^-, \sigma)$$

where $\sigma \in A$.

- Lets calculate

$$\mathrm{Tr}(\rho_A^n) = \langle \tau_n(\Delta)\bar{\tau}_n(0)\rangle = \left(\frac{\epsilon}{\Delta}\right)^{4d_n}$$

Then

$$\frac{1}{1-n}\ln \mathrm{Tr}\rho_A^n = \frac{c}{6}\left(n+\frac{1}{n}\right)\ln\frac{\Delta}{\epsilon}$$

The $n \to 1$ limit results in the entanglement entropy.

• To evaluate the partition function on the n-sheeted Riemann surface is to use the uniformization map

from the n-sheeted surface to the complex plane.

$$w = \left(\frac{z - y_1}{z - y_2} \right)^{\frac{1}{n}}$$

z is the co-ordinate on the multi-sheeted Riemann surface R^n,

w is the co-ordinate on the complex plane.

- Consider the following correlator

$$\frac{\langle T(z)\tau_n(y_1)\bar{\tau}_n(y_2)\rangle}{\langle \tau_n(y_1)\bar{\tau}_n(y_2)\rangle} = \frac{\langle T(z)\rangle_{R^n}}{\langle 1\rangle_{R^n}}$$

By using the map to the w-plane we obtain

$$\frac{\langle T(z)\rangle_{R^n}}{\langle 1\rangle_{R^n}} = \left(\frac{\partial w}{\partial z}\right)^2 \langle T(w)\rangle + \{w, z\}$$

where

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

is the Schwarzian.

By translation invariance on the complex w plane we see that $\langle T(w)\rangle = 0$.

Then the contribution to the correlator is entirely due to the Schwarzian.

Evaluating the Schwarzian one can show

$$\langle T(z)\rangle_{R^n} = \langle T(z)\tau_n(y_1)\bar{\tau}_n(y_2)\rangle_C$$

Thus the 1-point function of the stress tensor in the multi-sheeted plane

is a 3-point function of the stress tensor with operators of conformal dimensions

$$d_n = \bar{d}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

- Thus the correlator

$$\mathrm{Tr}(\rho_A^n) = \langle 1\rangle_{R^n} = \langle \tau_n(y_1)\bar{\tau}_n(y_2)\rangle$$

the two point function of the twist operator.

- For a theory of complex free fermions,

there is an explicit construction of the twist operator in terms of the bosonized fields

$$\psi = \exp(i\varphi)$$

- For a theory of complex free bosons,

though there is no explicit realization of the twist fields, all correlators involving it can be evaluated using OPEs and knowledge of the singularities.

Dixon, Friedan, Martinec, Shenker 1987

• The theory of N free fermions realizes a $\mathcal{W}_{1+\infty}$ algebra.

After removing the over all $U(1)$ it realizes the $W_\infty[\lambda = 1]$ algebra.

Therefore it admits a spin-3 current.

$$J = \psi_a^* \psi^a , \qquad T = \frac{1}{2} \left(\partial \bar{\psi}_a^* \psi^a - \psi_a^* \partial \psi^a \right) ,$$

$$W = i \frac{\sqrt{5}}{12\pi} \left(\partial^2 \psi_a^* \psi^a - 4\, \partial \psi_a^* \partial \psi^a + \psi_a^* \partial^2 \psi^a \right) .$$

- The theory of N free bosons realizes the $\mathcal{W}_\infty[\lambda = 0]$ algebra.

$$
\begin{aligned}
T(z) &= -\partial X_a \partial \bar{X}^a, \\
W(z) &= \sqrt{\frac{5}{12\pi^2}} (\partial^2 \bar{X}_i \partial X_i - \partial \bar{X}_i \partial^2 X_i).
\end{aligned}
$$

CONFORMAL PERTURBATION THEORY

• We are now ready to set up the the perturbation expansion of the Rényi entropy, EE in terms of the chemical potential μ.

• We need to evaluate

$$\mathrm{Tr}(\rho_A^n) = \frac{1}{Z^n(\mu)} \int_{R^n} [d\varphi_i] \exp\left(-\sum_{i=1}^{n} S[\varphi_i, \mu] \right)$$

where

$$S[\varphi, \mu] = S^{(0)}[\varphi] + \mu \int d^2 z (W(z) + \bar{W}(\bar{z}))$$

Note that there is a spin 3 current perturbation for each copy of the sheet.

• Expanding in μ we need to evaluate

$$\frac{1}{Z^n(\mu)}\left(\langle\tau_n(y_1)\bar{\tau}_n(y_2)\rangle_{(0)}+\mu\langle\tau_n(y_1)\bar{\tau}_n(y_2)\int d^2(W(z)+\bar{W}(\bar{z}))\rangle_{(0)}\right.$$
$$\left.+\frac{1}{2}\mu^2\langle\tau_n(y_1)\bar{\tau}_n(y_2)\left[\int d^2z(W(z)+\bar{W}(\bar{z}))\right]^2\rangle_{(0)}\right)$$

These correlators need to be evaluated on the cylinder and the integrals need to be performed.

At $O(\mu^2)$ there is a double integral to be performed.

• The linear term in μ vanishes. This can be seen by using the uniformization map

$$\begin{aligned} \langle \tau_n \bar{\tau}_n W \rangle &= \langle W \rangle_{R^n}, \\ &= \left(\frac{\partial w}{\partial z} \right)^3 \langle W \rangle_{w-\mathrm{plane}} \\ &= 0 \end{aligned}$$

The last step uses translational invariance in the *w*-plane.

This result can be explicitly checked either using the free fermion or the free boson realization.

• The holomorphic and the anti-holomorphic currents are decoupled,
the same argument can be used to show the correlators involving cross terms between holomorphic and anti-holomorphic spin-3 currents at $O(\mu^2)$ vanish.

• Therefore the only correlator left to evaluate is

$$\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle$$

• We first evaluated this correlator in the theory of free fermion and free boson theory.
We showed that they were same and agreed with the holographic proposal of d'Boer and Jottar to the μ^2 order.

• We will now present the outline of the calculation of this correlator for any CFT which admits a $\mathcal{W}_3$ current and show it is universal.

- By conformal invariance

$$\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle = -\frac{5c}{6\pi^2}\frac{F(x)}{z_{12}^6|y_{12}|^{4d_n}}$$

where

$$x = \frac{(z_1 - y_2)(z_2 - y_1)}{(z_1 - y_1)(z_2 - y_2)},$$

$$z_{12} = z_1 - z_2, \qquad y_{12} = y_1 - y_2 = \Delta$$

Determining $F(x)$

- From the fact the currents are holomorphic we have $F(x)$ is a holomorphic function in x.

- $z_1 \to z_2$ is a symmetry of the correlator, therefore

$$F(x) = F\left(\frac{1}{x}\right)$$

- We fix the normalization so that $z_1 \to z_2$ limit

$$F(x = 1) = 1$$

- Look at limits

$$z_1 \to y_1 \, , z_2 \to y_2 \, , \qquad x \to \infty$$

or the limits

$$z_2 \to y_2 \, , z_2 \to y_1 \, , \qquad x \to 0$$

In these limits the W current come close to the twist operator.

- We need the information of the following OPE

$$W(z)\tau_n(y) = \frac{\mathcal{O}}{(z-y)^M}$$

Note that $\mathcal{O}$ belongs to the twisted sector.

By definition τ_n creates the ground state of the twisted sector.

But $\mathcal{O}$ cannot be τ_n since $\langle W(z)\tau_n\bar{\tau}_n\rangle = 0$.

Therefore

$$\dim \mathrm{O} \; > \dim \tau_n = d_n$$

By dimensional analysis

$$M = 3 + d_n - \dim \mathrm{O} < 3$$

• Note that the W current involves the sum over the W currents over all the copies.

Therefore

$$W\tau_n = \sum_i W_i \tau_n$$

should not have branch cuts.

The insertion of the twist operator τ_n just cyclically permutes the copy label i and W is invariant under this.

Therefore M is an integer.

- This together with the fact $F(x) = F(1/x)$ results in

$$F(x) = a_0 + a_1(x + \frac{1}{x}) + a_2(x^2 + \frac{1}{x^2})$$

such that $a_0 + 2a_1 + 2a_2 = 1$.

We can re-write this by introducing the modified cross ratio

$$\begin{aligned}\eta &= x + \frac{1}{x} - 2, \\ &= \frac{(z_1 - z_2)^2(y_1 - y_2)^2}{(z_1 - y_1)(z_1 - y_2)(z_2 - y_1)(z_2 - y_2)}\end{aligned}$$

In terms of η we have

$$F(\eta) = 1 + f_1\eta + f_2\eta^2$$

• To fix f_1, f_2 use the $W(z_1)W(z_2)$ OPE.

$$
\begin{aligned}
-\frac{1}{\pi^2}W(z_1)W(z_2) \quad = \quad & \frac{5c}{6z_{12}^6} + \frac{5T(z_2)}{z_{12}^4} + \frac{5T'(z_2)}{z_{12}^3} \\
& + \frac{1}{z_{12}^2}\left(4U^{(4)}(z_2) + \frac{16\Lambda^{(4)}(z_2)}{c+\frac{22}{5}} + \frac{3}{4}T''(z_2)\right) \\
& \frac{1}{z_{12}}\left(2\partial U(z_2) + \frac{8}{c+\frac{22}{5}}\partial\Lambda^{(4)} + \frac{1}{6}T'''(z_2)\right)
\end{aligned}
$$

Where

$$
\Lambda^{(4)} =: TT : -\frac{3}{10}\partial^2 T
$$

and $U^{(4)}$ is the spin 4 current, it is a conformal primary.

The coefficient of this term contains λ dependence.

- To fix f_1, f_2 substitute the OPE into the correlator.

then one obtains an expansion in $(z_1 - z_2)$ with coefficients involving the three point functions

$$\langle U^{(4)} \tau_n \bar{\tau}_n \rangle = \langle \partial U^{(4)} \tau_n \bar{\tau}_n \rangle = 0,$$

$$\langle T \tau_n \bar{\tau}_n \rangle, \quad \langle \Lambda^{(4)} \tau_n \bar{\tau}_n \rangle, \quad \ldots$$

All of which can be evaluated by using the uniformization map.

• We can then perform the expansion in $(z_1 - z_2)$ in RHS of the expression

$$\begin{aligned}\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle &= -\frac{5c}{6\pi^2}\frac{F(x)}{z_{12}^6|y_{12}|^{4d_n}}, \\ &= -\frac{5c}{6\pi^2}\frac{1}{z_{12}^6|y_{12}|^{4d_n}}\left(1 + f_1\eta + f_2\eta^2\right)\end{aligned}$$

Matching the coefficients fixes f_1, f_2.

In fact there are 4 equations but only 2 parameters.

But there exists a unique solution

$$f_1 = \frac{n^2-1}{4n^2}, \qquad f_2 = \frac{(n^2-1)^2}{120n^4} - \frac{n^2-1}{40n^4}$$

• This determines the required correlation function of the spin 3 currents in presence of the twists.

• We have performed the following cross-checks on the result for the 4-point function.

∗ The 4-point function agrees with that obtained for the free field theories.

∗ Using the same procedure one can obtain the correlator of the stress tensor in presence of the twists $\langle TT\tau_n\bar{\tau}_n\rangle$.

We can also obtain this correlator using the conformal Ward identity on the 3-point function $\langle T\tau_n\bar{\tau}_n\rangle$.

Both results agree.

* Finally one can also evaluate the 4 point function $\langle WW\tau_n\bar{\tau}_n\rangle$ using the uniformization map.

$$\begin{aligned}\langle WW\tau_n\bar{\tau}_n\rangle &= \sum_{i,j}\langle W_iW_j\tau_n\bar{\tau}_n\rangle,\\ &= \sum_{i,j}\langle W_iW_j\rangle_{R^n}\end{aligned}$$

Using the uniformization map, this reduces to the two point function of the spin 3 currents inserted at the images of the respective Riemann sheet in the w-plane.

Then one needs to sum over the images.

The resultant correlator agrees with the OPE method.

• Now that one has the expression for the correlator, we can go over to the cylinder using the conformal map

$$u = \frac{\beta}{2\pi} \ln z$$

The we obtain

$$\frac{\mathrm{Tr}\rho_{\mathrm{A}}^{\mathrm{n}}}{1-n} = \frac{c(1+n)}{6n} \ln\left(\sinh\frac{\pi\Delta}{\beta}\right) + S_n^{(2)}$$

where

$$S_n^{(2)} = \frac{5\pi^4 c\mu^2}{6\beta^6(n-1)} \int d^2u_1 d^2u_2 \frac{f_1\eta_\beta + f_2\eta_\beta^2}{\sinh^6(\frac{\pi u_{12}}{\beta})}$$

$$\eta_\beta = \frac{\sinh^2[\frac{\pi}{\beta}(u_1-u_2)]\sinh^2\frac{\pi\Delta}{\beta}}{\sinh\frac{\pi}{\beta}(u_1-y_1)\sinh\frac{\pi}{\beta}(u_1-y_2)\sinh\frac{\pi}{\beta}(u_2-y_1)\sinh\frac{\pi}{\beta}(u_2-y}$$

• The integral can be done on the cylinder using the following prescription.

Integrate along the spatial direction infinite direction first.

Separate possible coincident point by insertion of an $i\epsilon$.

eg. one might encounter

$$\int_{-\infty}^{\infty} d\sigma \int_0^{\beta} d\tau \frac{1}{\sinh^2(\sigma + i\tau - a)} =$$

$$-\int_0^{\beta} d\tau \coth(\sigma + i\tau - a)|_{-\infty}^{\infty} = -2\beta$$

• One can show, the result depends only on the

Residue of the double poles of the integrand

Residue of the simple poles and its location of the integrand.

• Note that integrating along the spatial direction first reduces the current insertion to the conserved charge

$$\begin{aligned}\int_{\text{cylinder}} d^2 z W(z) &= \int_0^\beta d\tau \int_{-\infty}^{\infty} d\sigma W(\sigma,\tau), \\ &= \int_0^\beta d\tau Q\end{aligned}$$

Therefore this prescription captures the deformation of the Hamiltonian of the theory by μQ.

- The result for the correction to the Rényi entropy to $O(\mu^2)$ is given by

$$S_n^{(2)} = \frac{5c\mu^2 n}{6\pi^2(n-1)}(f_1\mathcal{I}_1 + f_2\mathcal{I}_2)$$

$$\mathcal{I}_1(\Delta) = \frac{4\pi^4}{3\beta^2}\left(\frac{4\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right) - 1\right) +$$
$$+\frac{4\pi^4}{\beta^2}\sinh^{-2}\left(\frac{\pi\Delta}{\beta}\right)\left\{\left(1-\frac{\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right)^2 - \left(\frac{\pi\Delta}{\beta}\right)^2\right\}$$

and

$$\mathcal{I}_2(\Delta) = \frac{8\pi^4}{\beta^2}\left(5 - \frac{4\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right) +$$
$$+\frac{72\pi^4}{\beta^2}\sinh^{-2}\left(\frac{\pi\Delta}{\beta}\right)\left\{\left(1-\frac{\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right)^2 - \frac{1}{9}\left(\frac{\pi\Delta}{\beta}\right)^2\right\}.$$

• Taking the limit $n \to 1$ results in the entanglement entropy.

If one takes further the limit $\frac{\Delta}{\beta} >> 1$, then one obtains the correction to the expected correction to the thermal entropy as evaluated by

Gaberdiel, Hartman, Jin 2012.

• The result agrees with the $O(\mu^2)$ correction evaluated using the Wilson line proposal of deBoer, Jottar.

• It is quite striking that the entire functional dependence (Δ, β) of the μ^2 correction is captured in holography.

This usually indicates that there is an underlying symmetry reason.

Summary of recent work

• We have studied the corrections to thermal entropy, single interval Rényi entropy and Entanglement entropy for the theory of the free fermions on the torus. The theory is deformed by:

$U(1)$ chemical potential

Stress tensor

Spin 3-chemical potential.

• This was done to further test and establish our prescription for doing the integrals.
It was also done to obtain finite size corrections: $O(e^{-TR})$.

• Such prescriptions were developed in the context the description of $2d$ Yang-Mills as a theory of free fermions
Dijkgraaf 1993, Douglas 1996

• The case of the $U(1)$ deformation all results agreed with the expectation:

The theory obeys twisted boundary conditions around the thermal circle.

• For the case of the stress tensor deformation, the results agreed with the expectation the temperature β of the theory is shifted.

• The results for the corrections to thermal entropy can be written in terms of quasi-modular forms of a definite weight (eg. spin 3 case , weight 6).

• The results for the corrections to the Rényi entropy can be written in terms of a quasi-elliptic functions of a definite weight.

- All these corrections were shown to satisfy several non-trivial consistency checks. eg. cylinder limit, thermal entropy limit.

- The correction to the partition function:

$$\ln \mathcal{Z} = \ln \mathcal{Z}_{\mathrm{CFT}} - \beta F^{(2)} + \dots$$

$$-\beta F^{(2)} = \frac{1}{2}\mu^2 \int_{\mathbb{T}^2} d^2 z_1 \int_{\mathbb{T}^2} d^2 z_2 \, \langle W(z_1) W(z_2) \rangle + \mathrm{h.c.}$$

$$
\begin{aligned}
-\beta F^{(2)} =& \\
& 80 M \mu^2 \beta^2 \frac{\pi^4}{L^4} \left[\frac{1}{2^5 \cdot 3^4 \cdot 5} \left(10 E_2^3 - 6 E_4 E_2 - 4 E_6 \right) \right. \\
& \qquad - \frac{1}{2^5 \cdot 3^2} (E_4 - E_2^2) \frac{\vartheta_\nu''}{\vartheta_\nu} \\
& \qquad \left. - \frac{1}{2^6 \cdot 3^2} \left(\frac{\vartheta_\nu^{(6)}}{\vartheta_\nu} + 2 E_2 \frac{\vartheta_\nu^{(4)}}{\vartheta_\nu} + E_2^2 \frac{\vartheta_\nu''}{\vartheta_\nu} \right) \right] .
\end{aligned}
$$

$\nu = 2, 3$ corresponds to periodic, anti-periodic fermions on the spatial circle.

In the Hamiltonian picture we have

$$\ln \mathcal{Z} \;=\; 2M \sum_{m=1}^{\infty} \left[\ln\left(1 \,+\, e^{2\pi i \tau m + b\mu m^2}\right) + \ln\left(1 \,+\, e^{2\pi i \tau m - b\mu m^2}\right) \right]$$

$$b \;=\; \frac{4\sqrt{6}\,\pi^3}{L} \tau\gamma\,. \qquad \gamma = i\sqrt{\frac{5}{6\pi^2}}$$

Expanding to order μ^2, the partition sum is

$$\ln \mathcal{Z} \;=\; 2M \ln\left[\frac{\vartheta_2}{\eta(\tau)}\right] + 160 M \mu^2 \beta^2 \frac{\pi^4}{L^4} \sum_{m=1}^{\infty} \frac{m^4\, q^m}{(1+q^m)^2} + \dots$$

M is number of free fermions.
The q expansions in both approaches agree .
($\nu = 2$)

The Rényi entropy when the size of the interval equals the system size $\Delta = L$, must agree in the $n = 1$ limit to the thermal entropy.

The Rényi entropy correction at order μ^2, with $\Delta = L$ is given by

$$
\begin{aligned}
&S^{(2)}_{\mathrm{RE}}(\Delta, n)\Big|_{\Delta=L} \\
&= \frac{\mu^2 M}{(1-n)} \frac{80\pi^4\beta^2}{L^4} \sum_k \left[\frac{1}{2^5\cdot 3^2}\left(E_2^2 - E_4\right)\left(\frac{\vartheta''_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta''_\nu}{\vartheta_\nu}\right) \right. \\
&- \frac{1}{2^6\cdot 3^2}\left\{ E_2^2 \left([\ln \vartheta_\nu(x)]'' - \frac{\vartheta''_\nu}{\vartheta_\nu}\right) \right. \\
&+ 2E_2\left(\frac{\vartheta^{(4)}_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta^{(3)}_\nu(x)}{\vartheta_\nu(x)}\frac{\vartheta'_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta^{(4)}_\nu}{\vartheta_\nu}\right) \\
&\left.\left. + \frac{\vartheta^{(6)}_\nu(x)}{\vartheta_\nu(x)} - \left(\frac{\vartheta^{(3)}_\nu(x)}{\vartheta_\nu(x)}\right)^2 - \frac{\vartheta^{(6)}_\nu}{\vartheta_\nu} \right\} \right]_{x=\frac{k\pi}{n}} .
\end{aligned}
$$

The sum over k runs from $-(n-1)/2$ to $(n-1)/2$ in steps of unity.

We have

$$S^{(2)}_{\mathrm{RE}}(\Delta, n=1)\Big|_{\Delta=L} = \beta^2 \frac{\partial F^{(2)}}{\partial \beta}$$

(Non-trivial mathematical identity)