

Renormalized 3-point functions in conformal field theory

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Outline

1 Introduction

2 Scalar 2-point functions

3 Scalar 3-point functions

4 Conclusions

Introduction

- Conformal invariance imposes strong constraints on correlation functions.
- It determines two- and three-point functions of scalars, conserved vectors and the stress-energy tensor [Polyakov (1970)] ... [Osborn, Petkou (1993)]. For example,

$$\begin{aligned} & \langle \mathcal{O}_1(\mathbf{x}_1) \mathcal{O}_2(\mathbf{x}_2) \mathcal{O}_3(\mathbf{x}_3) \rangle \\ &= \frac{c_{123}}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1 + \Delta_2 - \Delta_3} |\mathbf{x}_2 - \mathbf{x}_3|^{\Delta_2 + \Delta_3 - \Delta_1} |\mathbf{x}_3 - \mathbf{x}_1|^{\Delta_3 + \Delta_1 - \Delta_2}}. \end{aligned}$$

- It determines the form of higher point functions up to functions of cross-ratios.

Introduction

- These results (and many others) were obtained in **position space**.
- This is in stark contrast with general QFT where Feynman diagrams are typically computed in **momentum space**.
- While position space methods are powerful, typically they
 - provide results that hold only at separated points ("bare" correlators).
 - are hard to extend beyond CFTs
- The purpose of this work is to provide a first principles analysis of CFTs in momentum space.

Introduction

- Momentum space results were needed in several recent applications:
 - Holographic cosmology [McFadden, KS](2010)(2011) [Bzowski, McFadden, KS (2011)(2012)] [Pimentel, Maldacena (2011)][Mata, Raju, Trivedi (2012)] [Kundu, Shukla, Trivedi (2014)].
 - Studies of 3d critical phenomena [Sachdev et al (2012)(2013)]

References

- Adam Bzowski, Paul McFadden, KS
[Implications of conformal invariance in momentum space](#)
1304.7760
- Adam Bzowski, Paul McFadden, KS
[CFT renormalized scalar 3-point functions](#)
15xx.xxxx
- Adam Bzowski, Paul McFadden, KS
[CFT renormalized tensor 3-point functions](#)
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Conformal invariance

- Conformal transformations consist of dilatations and special conformal transformations.
- Dilatations $\delta x^\mu = \lambda x^\mu$, are linear transformations, so their implications are easy to work out.
- Special conformal transforms, $\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x$, are non-linear, which makes them difficult to analyse (and also more powerful).
- The corresponding Ward identities are **partial differential equations** which are difficult to solve.

Conformal invariance

- In **position space** one overcomes the problem by using the fact that special conformal transformations can be obtained by combining **inversions** with translations and then analyzing the implications of inversions.
- In **momentum space** we will see that one can actually directly solve the special conformal Ward identities.

Conformal Ward identities

- These are derived using the conformal transformation properties of conformal operators. For scalar operators:

$$\langle \mathcal{O}_1(\mathbf{x}_1) \cdots \mathcal{O}_n(\mathbf{x}_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle \mathcal{O}_1(\mathbf{x}'_1) \cdots \mathcal{O}_n(\mathbf{x}'_n) \rangle$$

- For (infinitesimal) dilatations this yields

$$0 = \left[\sum_{j=1}^n \Delta_j + \sum_{j=1}^n x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(\mathbf{x}_1) \cdots \mathcal{O}_n(\mathbf{x}_n) \rangle.$$

- In momentum space this becomes

$$0 = \left[\sum_{j=1}^n \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle \mathcal{O}_1(\mathbf{p}_1) \cdots \mathcal{O}_n(\mathbf{p}_n) \rangle,$$

Special conformal Ward identity

- For (infinitesimal) special conformal transformations this yields

$$0 = \left[\sum_{j=1}^n \left(2\Delta_j x_j^\kappa + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\kappa}} \right) \right] \langle \mathcal{O}_1(\mathbf{x}_1) \dots \mathcal{O}_n(\mathbf{x}_n) \rangle$$

- In momentum space this becomes

$$0 = \mathcal{K}^\mu \langle \mathcal{O}_1(\mathbf{p}_1) \dots \mathcal{O}_n(\mathbf{p}_n) \rangle,$$

$$\mathcal{K}^\mu = \left[\sum_{j=1}^{n-1} \left(2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^\kappa} + (p_j)_\kappa \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\alpha}} \right) \right]$$

Special conformal Ward identities

- To extract the content of the special conformal Ward identity we expand \mathcal{K}^μ is a basis of linear independent vectors, **the $(n - 1)$ independent momenta**,

$$\mathcal{K}^\kappa = p_1^\kappa \mathcal{K}_1 + \dots + p_{n-1}^\kappa \mathcal{K}_{n-1}.$$

- Special conformal Ward identities constitute $(n - 1)$ differential equations.

Conformal Ward identities

- Poincaré invariant n -point function in $d \geq n$ spacetime dimensions depends on $n(n-1)/2$ kinematic variables.
- Thus, after imposing $(n-1) + 1$ conformal Ward identities we are left with

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

undetermined degrees of freedom.

- This number equals the number of **conformal ratios** in n variables in $d \geq n$ dimensions.
- ➡ **It is not known however what do the cross ratios become in momentum space.**

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- 2 Scalar 2-point functions
- 3 Scalar 3-point functions
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Scalar 2-point function

- The dilatation Ward identity reads

$$0 = \left[d - \Delta_1 - \Delta_2 + p \frac{\partial}{\partial p} \right] \langle O_1(\mathbf{p}) O_2(-\mathbf{p}) \rangle$$

- The 2-point function is a homogeneous function of degree $(\Delta_1 + \Delta_2 - d)$:

$$\langle O_1(\mathbf{p}) O_2(-\mathbf{p}) \rangle = c_{12} p^{\Delta_1 + \Delta_2 - d}.$$

where c_{12} is an integration constant.

Scalar 2-point function

- The special conformal Ward identity reads

$$0 = \mathcal{K} \langle O_1(\mathbf{p}) O_2(-\mathbf{p}) \rangle, \quad \mathcal{K} = \frac{d^2}{dp^2} - \frac{2\Delta_1 - d - 1}{p} \frac{d}{dp}$$

- Inserting the solution of the dilatation Ward identity we find that we need

$$\Delta_1 = \Delta_2$$

Scalar 2-point function

The general solution of the conformal Ward identities is:

$$\langle O_{\Delta}(\mathbf{p})O_{\Delta}(-\mathbf{p}) \rangle = c_{12}p^{2\Delta-d}.$$

➤ This solution is **trivial** when

$$\Delta = \frac{d}{2} + k, \quad k = 0, 1, 2, \dots$$

because then correlator is local,

$$\langle O(\mathbf{p})O(-\mathbf{p}) \rangle = cp^{2k} \rightarrow \langle O(\mathbf{x}_1)O(\mathbf{x}_2) \rangle \sim \square^k \delta(x_1 - x_2)$$

➤ Let ϕ_0 be the source of O . It has dimension $d - \Delta = d/2 - k$.
 The term

$$\phi_0 \square^k \phi_0$$

has dimension d and can act as a local counterterm.

Position space [Petkou, KS (1999)]

- In position space, it seems that none of these are an issue:

$$\langle \mathcal{O}(\mathbf{x}) \mathcal{O}(0) \rangle = \frac{C}{x^{2\Delta}}$$

- This expression however is valid **only at separated points**, $x^2 \neq 0$.
- Correlation functions should be **well-defined distributions** and they should have well-defined Fourier transform.
- Fourier transforming we find:

$$\int d^d \mathbf{x} e^{-i\mathbf{p} \cdot \mathbf{x}} \frac{1}{x^{2\Delta}} = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d},$$

- This is well-behaved, **except when $\Delta = d/2 + k$** , where k is a positive integer.

Strategy

- Regularize the theory.
 - Solve the Ward identities in the regulated theory.
 - Renormalize by adding appropriate counterterms.
- ➡ The renormalised theory may be anomalous.

Regularization

- We use dimensional regularisation to regulate the theory

$$d \mapsto d + 2u\epsilon, \quad \Delta_j \mapsto \Delta_j + (u + v)\epsilon$$

- In the regulated theory, the solution of the Ward identities is the same as before but the integration constant may depend on the regulator,

$$\langle O(\mathbf{p})O(-\mathbf{p}) \rangle_{\text{reg}} = c(\epsilon, u, v) p^{2\Delta - d + 2v\epsilon}.$$

Regularization and Renormalization

$$\langle O(\mathbf{p})O(-\mathbf{p}) \rangle_{\text{reg}} = c(\epsilon, u, v) p^{2\Delta - d + 2v\epsilon}.$$

➤ Now, in **local CFTs**:

$$c(\epsilon, u, v) = \frac{c^{(-1)}(u, v)}{\epsilon} + c^{(0)}(u, v) + O(\epsilon)$$

➤ This leads to

$$\langle O(\mathbf{p})O(-\mathbf{p}) \rangle_{\text{reg}} = p^{2k} \left[\frac{c^{(-1)}}{\epsilon} + c^{(-1)} v \log p^2 + c^{(0)} + O(\epsilon) \right].$$

➤ We need to renormalise

Renormalization

- Let ϕ_0 the source that couples to O ,

$$S[\phi_0] = S_0 + \int d^{d+2u\epsilon} \mathbf{x} \phi_0 O.$$

- The divergence in the 2-point function can be removed by the addition of the counterterm action

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^{d+2u\epsilon} \mathbf{x} \phi_0 \square^k \phi_0 \mu^{2v\epsilon},$$

- Removing the cut-off we obtain the renormalised correlator:

$$\langle O(\mathbf{p})O(-\mathbf{p}) \rangle_{\text{ren}} = p^{2k} \left[C \log \frac{p^2}{\mu^2} + C_1 \right]$$

Anomalies

- The counter term breaks scale invariance and as result the **theory has a conformal anomaly**.
- The 2-point function depends on a scale [Petkou, KS (1999)]

$$\mathcal{A}_2 = \mu \frac{\partial}{\partial \mu} \langle O(\mathbf{p}) O(-\mathbf{p}) \rangle = c p^{2\Delta-d},$$

- The integrated anomaly is Weyl invariant

$$A = \int d^d \mathbf{x} \phi_0 \square^k \phi_0$$

On a curved background, \square^k is replaced by the "k-th power of the conformal Laplacian", P^k .

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Scalar 3-point functions

We would now like to understand 3-point functions at the same level:

- What is the general solution of the conformal Ward identities?
- What is the analogue of the condition

$$\Delta = \frac{d}{2} + k, \quad k = 0, 1, 2, \dots$$

- Are there new conformal anomalies associated with 3-point functions and if yes what is their structure?

Conformal Ward identities

➤ Dilatation Ward identity

$$0 = \left[2d - \Delta_t + \sum_{j=1}^3 p_j \frac{\partial}{\partial p_j} \right] \langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle$$

$$\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$$

- The correlation is a **homogenous function of degree** $(2d - \Delta_t)$.
- The special conformal Ward identities give rise to two scalar 2nd order PDEs.

Special conformal Ward identities

- Special conformal WI

$$0 = K_{12} \langle \mathcal{O}_1(\mathbf{p}_1) \mathcal{O}_2(\mathbf{p}_2) \mathcal{O}_3(\mathbf{p}_3) \rangle = K_{23} \langle \mathcal{O}_1(\mathbf{p}_1) \mathcal{O}_2(\mathbf{p}_2) \mathcal{O}_3(\mathbf{p}_3) \rangle,$$

where

$$K_{ij} = K_i - K_j,$$

$$K_j = \frac{\partial^2}{\partial p_j^2} + \frac{d+1-2\Delta_j}{p_j} \frac{\partial}{\partial p_j}, \quad (i, j = 1, 2, 3).$$

- This system of differential equations is precisely that defining **Appell's F_4 generalised hypergeometric function of two variables**. [Coriano, Rose, Mottola, Serino][Bzowski, McFadden, KS] (2013).

Scalar 3-point functions

- There are **four linearly independent solutions** of these equations.
- **Three of them have unphysical singularities** at certain values of the momenta leaving one physically acceptable solution.
- **We thus recover the well-known fact that scalar 3-point functions are determined up to a constant.**

Scalar 3-pt functions and triple- K integrals

- The physically acceptable solution has the following *triple- K integral* representation:

$$\langle \mathcal{O}_1(\mathbf{p}_1) \mathcal{O}_2(\mathbf{p}_2) \mathcal{O}_3(\mathbf{p}_3) \rangle = C_{123} p_1^{\Delta_1 - \frac{d}{2}} p_2^{\Delta_2 - \frac{d}{2}} p_3^{\Delta_3 - \frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2} - 1} K_{\Delta_1 - \frac{d}{2}}(p_1 x) K_{\Delta_2 - \frac{d}{2}}(p_2 x) K_{\Delta_3 - \frac{d}{2}}(p_3 x),$$

where $K_\nu(p)$ is a Bessel function and C_{123} is a constant.

- This is the general solution of the conformal Ward identities.

Triple K -integrals

- Triple- K integrals,

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1, p_2, p_3) = \int_0^\infty dx x^\alpha \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x),$$

are the building blocks of all 3-point functions.

- The integral converges provided

$$\alpha > \sum_{j=1}^3 |\beta_j| - 1$$

- The integral can be defined by **analytic continuation** when

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2k,$$

where k is any non-negative integer.

Renormalization and anomalies

- If the equality holds,

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2k,$$

the integral cannot be defined by analytic continuation.

- Non-trivial subtractions and renormalization may be required and this may result in conformal anomalies.
- Physically when this equality holds, there are new terms of dimension d that one can add to the action (counterterms) and/or new terms that can appear in T_{μ}^{μ} (conformal anomalies).

Scalar 3-pt function

- For the triple- K integral that appears in the 3-pt function of scalar operators the condition becomes

$$\frac{d}{2} \pm (\Delta_1 - \frac{d}{2}) \pm (\Delta_2 - \frac{d}{2}) \pm (\Delta_3 - \frac{d}{2}) = -2k$$

- There are four cases to consider, according to the signs needed to satisfy this equation. We will refer to the 4 cases as the $(- - -)$, $(- - +)$, $(- + +)$ and $(+ + +)$ cases.
- Given Δ_1, Δ_2 and Δ_3 these relations may be satisfied with more than one choice of signs and k .

Procedure

- To analyse the problem we will proceed by using dimensional regularisation

$$d \mapsto d + 2u\epsilon, \quad \Delta_j \mapsto \Delta_j + (u + v)\epsilon$$

- In the regulated theory the solution of the conformal Ward identity is given in terms of the triple-K integral but now **the integration constant C_{123} in general will depend on the regulator ϵ, u, v .**
- We need to understand the singularity structure of the triple-K integrals and then renormalise the correlators.
- We will discuss each case in turn.

The (---) case

$$\Delta_1 + \Delta_2 + \Delta_3 = 2d + 2k$$

- This is the analogue of the $\Delta = d/2 + k$ case in 2-point functions.
- There are possible **counterterms**

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^d \mathbf{x} \phi_1^{k_1} \phi_2^{k_2} \phi_3^{k_3}$$

where $k_1 + k_2 + k_3 = k$. **The same terms may appear in T_μ^μ as new conformal anomalies.**

- After adding the contribution of the counterterms one may remove the regulator to obtain the renormalised correlator.

Example: $\Delta_1 = \Delta_2 = \Delta_3 = 2, d = 3$

- The source ϕ for an operator of dimension 2 has dimension 1, so ϕ^3 has dimension 3.
- Regularizing:

$$\begin{aligned} \langle \mathcal{O}(\mathbf{p}_1) \mathcal{O}(\mathbf{p}_2) \mathcal{O}(\mathbf{p}_3) \rangle &= C_{123} \left(\frac{\pi}{2} \right)^{3/2} \int_0^\infty dx x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)} \\ &= C_{123} \left(\frac{\pi}{2} \right)^{3/2} \left[\frac{1}{\epsilon} - (\gamma_E + \log(p_1 + p_2 + p_3)) + O(\epsilon) \right]. \end{aligned}$$

Renormalization and anomalies

- We add the counterterm

$$S_{ct} = -\frac{C_{123}}{3!\epsilon} \left(\frac{\pi}{2}\right)^{3/2} \int d^{3+2\epsilon} \mathbf{x} \phi^3 \mu^{-\epsilon}$$

- This leads to the renormalized correlator,

$$\langle \mathcal{O}(\mathbf{p}_1) \mathcal{O}(\mathbf{p}_2) \mathcal{O}(\mathbf{p}_3) \rangle = -C_{123} \left(\frac{\pi}{2}\right)^{3/2} \log \frac{p_1 + p_2 + p_3}{\mu}$$

- The renormalized correlator is not scale invariant

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}(\mathbf{p}_1) \mathcal{O}(\mathbf{p}_2) \mathcal{O}(\mathbf{p}_3) \rangle = C_{123} \left(\frac{\pi}{2}\right)^{3/2}$$

- ➡ There is a new conformal anomaly:

$$\langle T \rangle = -\phi \langle \mathcal{O} \rangle + \frac{1}{3!} C_{123} \left(\frac{\pi}{2}\right)^{3/2} \phi^3.$$

The $(- - +)$ case

$$\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$$

- In this case the new local term one can add to the action is

$$S_{\text{ct}} = a_{\text{ct}} \int d^d x \square^{k_1} \phi_1 \square^{k_2} \phi_2 O_3$$

where $k_1 + k_2 = k$.

- In this case we have **renormalization of sources**,

$$\phi_3 \rightarrow \phi_3 + a_{\text{ct}} \square^{k_1} \phi_1 \square^{k_2} \phi_2$$

- The renormalised correlator will satisfy a **Callan-Symanzik equation with beta function terms**.

Callan-Symanzik equation

- The quantum effective action \mathcal{W} (the generating functional of renormalised connected correlators) obeys the equation

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_i \int d^d \vec{x} \beta_i \frac{\delta}{\delta \phi_i(\vec{x})} \right) \mathcal{W} = \int d^d \vec{x} \mathcal{A},$$

- This implies that for 3-point functions we have

$$\mu \frac{\partial}{\partial \mu} \langle O_i(p_1) O_j(p_2) O_j(p_3) \rangle = \beta_{j,ji} (\langle O_j(p_2) O_j(-p_2) \rangle + \langle O_j(p_3) O_j(-p_3) \rangle) + \mathcal{A}_{ijj}^{(3)},$$

$$\beta_{i,jk} = \frac{\delta^2 \beta_i}{\delta \phi_j \delta \phi_k} \Big|_{\{\phi_l\}=0}, \quad \mathcal{A}_{ijk}^{(3)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = - \frac{\delta^3}{\delta \phi_i(\vec{x}_1) \delta \phi_j(\vec{x}_2) \delta \phi_k(\vec{x}_3)} \int d^d \vec{x} \mathcal{A}(\{\phi_l(\vec{x})\})$$

Example: $\Delta_1 = 4, \Delta_2 = \Delta_3 = 3$ in $d = 4$

- $\Delta_1 + \Delta_2 + \Delta_3 = 10 = 2d + 2k$, which satisfies the $(- - -)$ -condition with $k = 1$.
- ➡ There is an anomaly

$$\int d^d x \phi_0 \phi_1 \square \phi_1$$

- $\Delta_1 + \Delta_2 - \Delta_3 = 4 = d + 2k$, which satisfies the $(- - +)$ condition with $k = 0$. The following counterterm is needed,

$$\int d^4 x \phi_0 \phi_1 O_3$$

- ➡ There is a beta function for ϕ_1 .

$\langle O_4 O_3 O_3 \rangle$

$$\begin{aligned}
 \langle O_4(\mathbf{p}_1) O_3(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle &= \alpha \left(2 - p_1 \frac{\partial}{\partial p_1} \right) I^{(\text{non-local})} \\
 &+ \frac{\alpha}{8} \left[(p_2^2 - p_3^2) \log \frac{p_1^2}{\mu^2} \left(\log \frac{p_3^2}{\mu^2} - \log \frac{p_2^2}{\mu^2} \right) \right. \\
 &\quad \left. - (p_2^2 + p_3^2) \log \frac{p_2^2}{\mu^2} \log \frac{p_3^2}{\mu^2} \right. \\
 &\quad \left. (p_1^2 - p_2^2) \log \frac{p_3^2}{\mu^2} + (p_1^2 - p_3^2) \log \frac{p_2^2}{\mu^2} + p_1^2 \right]
 \end{aligned}$$

$\langle O_4 O_3 O_3 \rangle$

$$I^{(\text{non-local})} = -\frac{1}{8} \sqrt{-J^2} \left[\frac{\pi^2}{6} - 2 \log \frac{p_1}{p_3} \log \frac{p_2}{p_3} \right. \\
 + \log \left(-X \frac{p_2}{p_3} \right) \log \left(-Y \frac{p_1}{p_3} \right) \\
 \left. - Li_2 \left(-X \frac{p_2}{p_3} \right) - Li_2 \left(-Y \frac{p_1}{p_3} \right) \right],$$

$$J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3),$$

$$X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2 p_3}, \quad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1 p_3}.$$

Callan-Symanzik equation

- It satisfies

$$\mu \frac{\partial}{\partial \mu} \langle O_4(\mathbf{p}_1) O_3(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle =$$

$$\frac{\alpha}{2} \left(p_2^2 \log \frac{p_2^2}{\mu^2} + p_3^2 \log \frac{p_3^2}{\mu^2} - p_1^2 + \frac{1}{2}(p_2^2 + p_3^2) \right).$$

- This is indeed the correct Callan-Symanzik equation.
 (Recall that $\langle O_3(\mathbf{p}) O_3(\mathbf{p}) \rangle = p^2 \log \frac{p^2}{\mu^2}$)

The $(+++)$ and $(-++)$ cases

- In these cases it is **the representation of the correlator in terms of the triple- K integral that is singular**, not the correlator itself,

$$\langle \mathcal{O}_1(\mathbf{p}_1) \mathcal{O}_2(\mathbf{p}_2) \mathcal{O}_3(\mathbf{p}_3) \rangle = C_{123} p_1^{\Delta_1 - \frac{d}{2}} p_2^{\Delta_2 - \frac{d}{2}} p_3^{\Delta_3 - \frac{d}{2}} \\ \times I_{d/2-1, \{\Delta_1-d/2, \Delta_2-d/2, \Delta_3-d/2\}}$$

Taking the integration constant $C_{123} \sim \epsilon^m$ for appropriate m and sending $\epsilon \rightarrow 0$ results in an expression that satisfies the original (non-anomalous) Ward identity.

- In other words, the Ward identities admit a solution that is finite.

The (+ + +) case

$$\Delta_1 + \Delta_2 + \Delta_3 = d - 2k$$

- For example, for $k = 0$ the finite solution to the Ward identities is

$$\langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle = c p_1^{(\Delta_1 - \Delta_2 - \Delta_3)} p_2^{(\Delta_2 - \Delta_1 - \Delta_3)} p_3^{(\Delta_3 - \Delta_1 - \Delta_2)}$$

- When the operators have these dimensions there are "multi-trace" operators which are classically marginal

$$\mathcal{O} = \square^{k_1} O_1 \square^{k_2} O_2 \square^{k_3} O_3$$

where $k_1 + k_2 + k_3 = k$.

The $(-++)$ case

$$\Delta_1 - \Delta_2 - \Delta_3 = 2k$$

- For $k = 0$, there are "extremal correlators". In position, the 3-point function is a product of 2-point functions

$$\langle O_1(\mathbf{x}_1)O_2(\mathbf{x}_2)O_3(\mathbf{x}_3) \rangle = \frac{c_{123}}{|\mathbf{x}_2 - \mathbf{x}_1|^{2\Delta_2} |\mathbf{x}_3 - \mathbf{x}_1|^{2\Delta_3}}$$

- In momentum space, the finite solution to the Ward identities is

$$\langle O_1(\mathbf{p}_1)O_2(\mathbf{p}_2)O_3(\mathbf{p}_3) \rangle = c p_2^{(2\Delta_2-d)} p_3^{(2\Delta_3-d)}$$

- When the operators have these dimensions there are "multi-trace" operators of dimension Δ_1

$$\mathcal{O} = \square^{k_2} O_2 \square^{k_3} O_3$$

where $k_2 + k_3 = k$.

Tensorial correlators

- New issues arise for tensorial correlation functions, such as those involving stress-energy tensors and conserved currents.
- Lorentz invariance implies that the tensor structure will be carried by tensors constructed from the momenta p^μ and the metric $\delta_{\mu\nu}$.
- **After an appropriate parametrisation**, the analysis becomes very similar to the one we discussed here.
- **In particular, these correlator are also given in terms of triple- K integrals.**

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Conclusions/Outlook

- We obtained the implications of conformal invariance for **three-point functions** working in **momentum space**.
- We discussed **renormalization and anomalies**.
- The presence of "beta function" terms in the Callan - Symanzik equation for CFT correlators is new.
- It would be interesting to understand how to extend the analysis to **higher point functions**. What is the momentum space analogue of cross-ratios?
- Bootstrap in momentum space?

