A c-theorem for Two-dimensional Boundaries and Defects

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THE ROYAL SOCIETY

Holography, Strings, and Higher Spins
University of Swansea
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WORK IN PROGRESS

with

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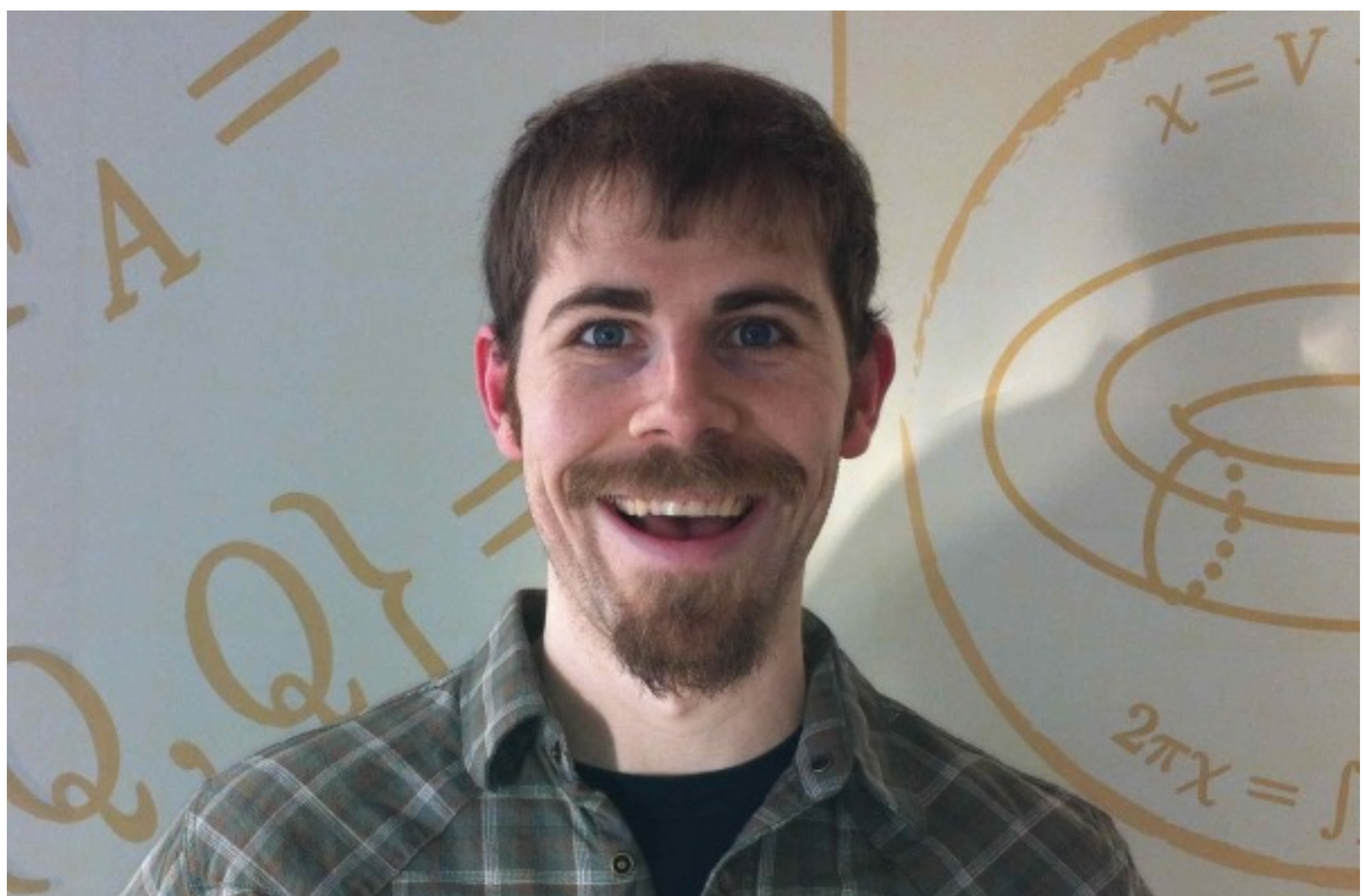
Stony Brook University



DISCLAIMERS

No Holography, Strings, or Higher Spins

Apologies for missed references

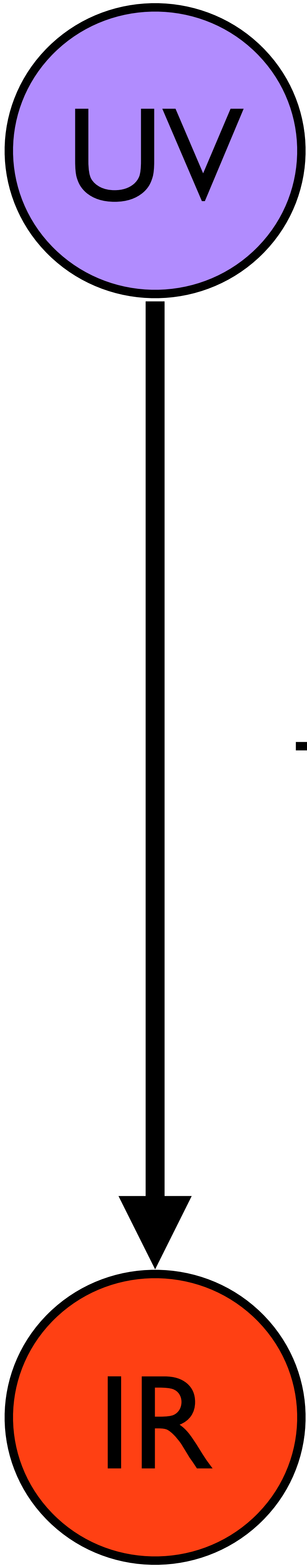
WORK IN PROGRESS

caveat emptor (buyer beware)

Suggestions welcome!

Outline:

- Review: Monotonicity Theorems
- The Systems
- The Weyl Anomaly
- The Proof
- Summary and Outlook

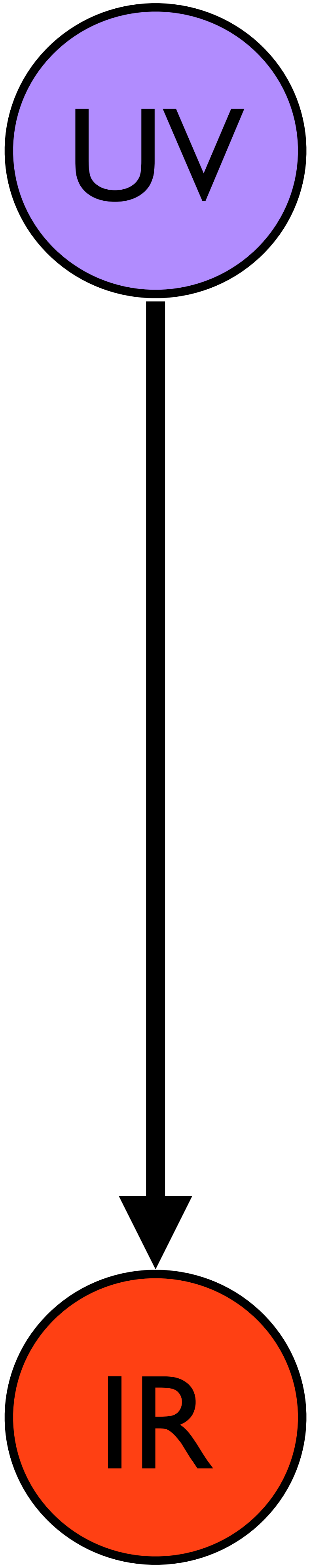


Microscopic/High-energy scales

Intuition:

The “number of degrees of freedom (DOF)” will DECREASE

Macroscopic/Low-energy scales



Example

Quantum Field Theory (QFT)

Wilsonian Renormalization Group (RG)

"integrate out" DOF

Below a mass threshold,
integrate out massive DOF

massive DOF
"decouples"

Monotonicity Theorems

Make our intuition precise, for RG flows in QFT

Provide a precise way to count number of DOF

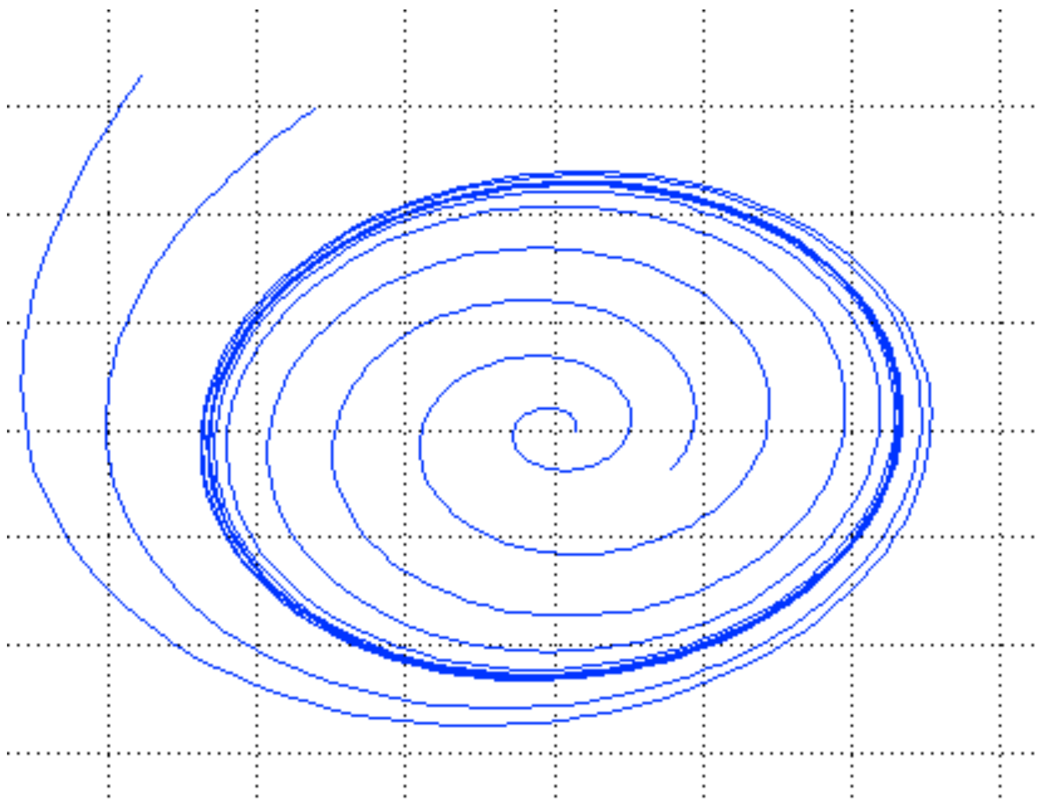
Provide rigorous proof that the number of DOF DECREASES along RG flow

Monotonicity Theorems

Monotonicity Theorems are important!

They place stringent theoretical constraints on what is possible in RG flows, without relying on perturbation theory

For example, they can eliminate limit cycles



The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

for RG flows in

RENORMALIZABLE EUCLIDEAN

QFTs in $d = 2$

Assumptions

① Euclidean Symmetry

② Locality

③ Unitarity

1. Euclidean Symmetry

Non-dynamical background metric $g_{\mu\nu}(x)$

Action functional $S(g_{\mu\nu}, \lambda)$

Coupling constants $\lambda = (\lambda_1, \lambda_2, \ldots)$

Generating functional $Z[g_{\mu\nu}, \lambda]$

1. Euclidean Symmetry

Stress-Energy Tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \ln Z[g_{\mu\nu}, \lambda]$$

$$T_{\mu\nu} = T_{\nu\mu}$$

$$g_{\mu\nu} \to \delta_{\mu\nu}$$

Translational and Rotational Symmetry

$$\partial_\mu T_{\mu\nu} = 0$$

1. Euclidean Symmetry

Stress-Energy Tensor

complex coordinates $z, \bar{z}$

$$[T_{\mu\nu}] = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix}$$

$$T_{z\bar{z}} = T_{\bar{z}z}$$

$$T_{\mu}{}^{\mu} = T_{z\bar{z}} = T_{\bar{z}z}$$

2. Locality

$$S(\lambda) = \int d^2z \, L(\lambda, z, \bar{z})$$

RG flow triggered by relevant local scalar operator

$$L(\lambda, z, \bar{z}) \to L(\lambda, z, \bar{z}) + \lambda_{\mathcal{O}} \mathcal{O}(z, \bar{z})$$

$$\Delta_{\mathcal{O}} < 2$$

3. Unitarity

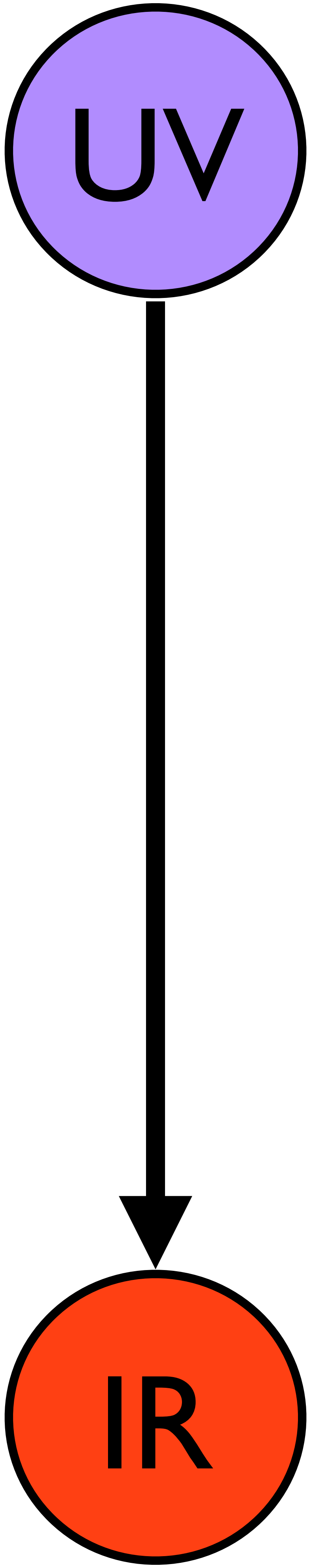
$$||\psi\rangle|^2 = \langle\psi|\psi\rangle \geq 0$$

Reflection Positivity

Two point function of local scalar operator must be non-negative

$$\langle\mathcal{O}^\dagger(x)\mathcal{O}(0)\rangle \geq 0$$

Euclidean “time evolution” preserves norm ≥ 0



UV CFT

The c-theorem

RG flow between fixed points

Conformal Field Theories
(CFTs)

IR CFT

Conformal Field Theory

Non-dynamical background metric $g_{\mu\nu}(x)$

Conformal Transformation

Diffeomorphism

$$x^{\mu} \to x'^{\mu}(x)$$

such that

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$T_\mu{}^\mu = \frac{1}{\sqrt{g}} \frac{\delta}{\delta\Omega} \ln Z$$

Conformal invariance

$$T_\mu{}^\mu = 0$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d > 2$$

Rotations $\quad x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$

Translations $\quad x^\mu \to x^\mu + c^\mu$

Dilatations $\quad x^\mu \to \lambda \, x^\mu$

Special Conformal $\quad x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$

$$SO(d+1,1)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic DECOUPLE

$$T_{\mu}{}^{\mu} = T_{z\bar{z}} = T_{z\bar{z}} = 0$$

$$\partial_\mu T_{\mu\nu} = 0$$

$$\partial_{\bar{z}} T_{zz} = 0 \qquad \partial_z T_{\bar{z}\bar{z}} = 0$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d = 2$$

Conformal Transformation

$$z \to w(z)$$

$$\bar{z} \to \bar{w}(\bar{z})$$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$SO(d+1,1) = SO(3,1)$ **subgroup** $L_{\pm 1}$ and L_0

$\bar{L}_{\pm 1}$ and $\bar{L}_0$

Conformal Field Theory

Central Charge

Counts the number of DOF in the CFT

Add a single, real, free, massless scalar field or Dirac fermion

$$c \to c + 1$$

Conformal Field Theory

Central Charge

Thermodynamic entropy

Cardy NPB 270 (186) 1986

System size L Temperature T

$$T \gg 1/L$$

$$S_{\rm thermo} = \frac{\pi}{3} c L T + \ldots$$

Conformal Field Theory

Central Charge

Entanglement Entropy (EE)

Holzhey, Larsen, Wilczek hep-th/9403108 Calabrese + Cardy hep-th/0405152

Short-distance cutoff a Interval of length ℓ

$$S_{\mathrm{EE}} = \frac{c}{3} \ln \frac{2\ell}{a} + \ldots$$

Conformal Field Theory

Central Charge

$$||\psi\rangle|^2 = \langle\psi|\psi\rangle \geq 0$$

Vacuum $|0\rangle$

$$L_m|0\rangle = 0 \qquad \forall\ m \geq 0$$

$$|L_{-m}|0\rangle|^2 = \langle 0|[L_m, L_{-m}]|0\rangle = \frac{c}{12}m(m^2-1) \geq 0$$

$$c \geq 0$$

The c-theorem

$$\langle T_{zz}(z,\bar{z})T_{zz}(0,0)\rangle = \frac{F(z\bar{z})}{z^4}$$

$$\langle T_\mu{}^\mu(z,\bar{z})T_{zz}(0,0)\rangle = \frac{G(z\bar{z})}{z^3\,\bar{z}}$$

$$\langle T_\mu{}^\mu(z,\bar{z})T_\mu{}^\mu(0,0)\rangle = \frac{H(z\bar{z})}{z^2\,\bar{z}^2}$$

Fixed point $\Longrightarrow$ $F = c/2 \quad G = 0 \quad H = 0$

The c-theorem

$$\langle T_{zz}(z,\bar{z})T_{zz}(0,0)\rangle = \frac{F(z\bar{z})}{z^4}$$

$$\langle T_{\mu}{}^{\mu}(z,\bar{z})T_{zz}(0,0)\rangle = \frac{G(z\bar{z})}{z^3\,\bar{z}}$$

$$\langle T_{\mu}{}^{\mu}(z,\bar{z})T_{\mu}{}^{\mu}(0,0)\rangle = \frac{H(z\bar{z})}{z^2\,\bar{z}^2}$$

Reflection Positivity $\Rightarrow \quad H \geq 0$

The c-theorem

c-function

$$C \equiv 2F - G - \frac{3}{8}H$$

Fixed point $\quad \Rightarrow \quad C = c$

$$\partial_\mu T_{\mu\nu} = 0$$

$$r \equiv \sqrt{z\bar{z}}$$

$$r\frac{\partial C}{\partial r} = -\frac{3}{2}H \leq 0$$

The c-theorem

$$\frac{\partial C}{\partial r} \leq 0 \qquad \Rightarrow \qquad c_{\mathrm{UV}} \geq c_{\mathrm{IR}}$$

Strong form

Weak form

Other Proofs

Holography

Freedman, Gubser, Pilch, Warner hep-th/9904017
Myers and Sinha 1006.1263, 1011.5819

Null energy Condition — Strong Form

Entanglement Entropy (EE)

Casini and Huerta hep-th/0405111

Strong Sub-Additivity (SSA) — Strong Form

Weyl Anomaly Matching

Komargodski and Schwimmer 1107.3987 — Komargodski 1112.4538

Reflection Positivity — Weak form

Generalizations?

non-local and/or non-unitary QFTs?

QFTs with less symmetry?

higher dimensions?

QFTs without Euclidean symmetry?

What if $d_{\rm UV} \neq d_{\rm IR}$?

What if the relevant operator is not a scalar?

What if the fixed points have Lifshitz scaling?

$\vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t \qquad z =$ dynamical exponent

What if $z_{\rm UV} \neq z_{\rm IR}$?

Higher Dimensions

F-theorem
3-dimensional QFT

Jafferis, Klebanov, Pufu, and Safdi 1103.1181

Casini and Huerta 1202.5650

a-theorem
4-dimensional QFT

Cardy PLB 215 (1988) 749

Komargodski and Schwimmer 1107.3987

weak forms only!

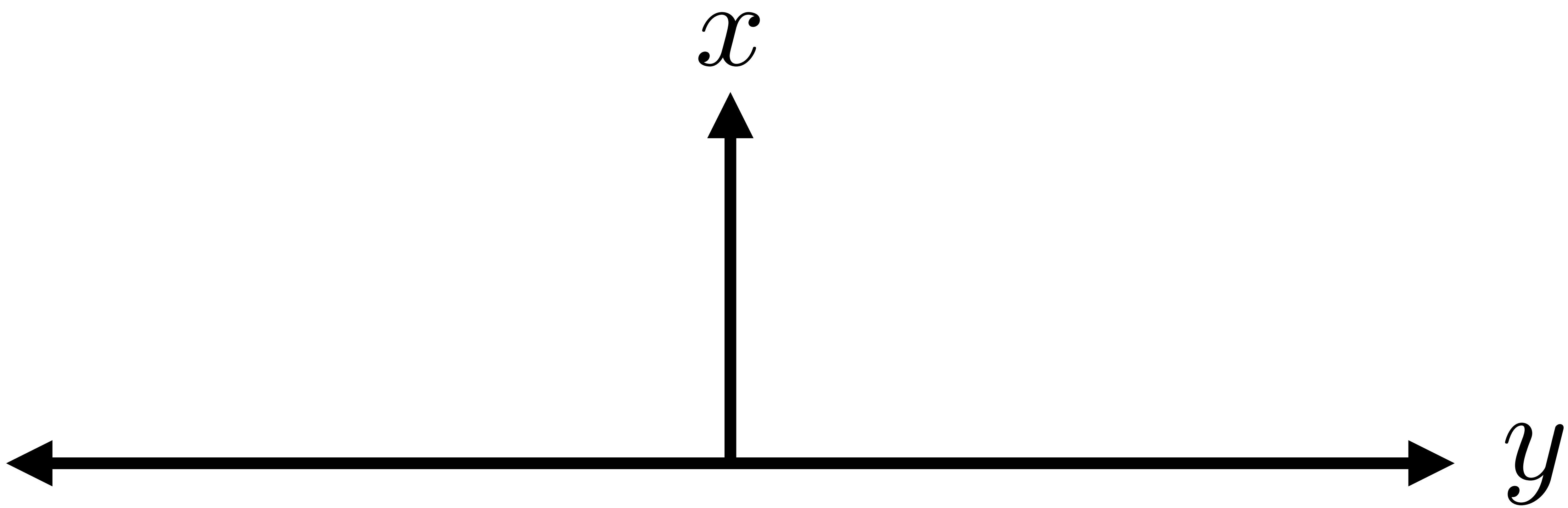
The g-theorem

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

Local, unitary CFT in $d = 2$

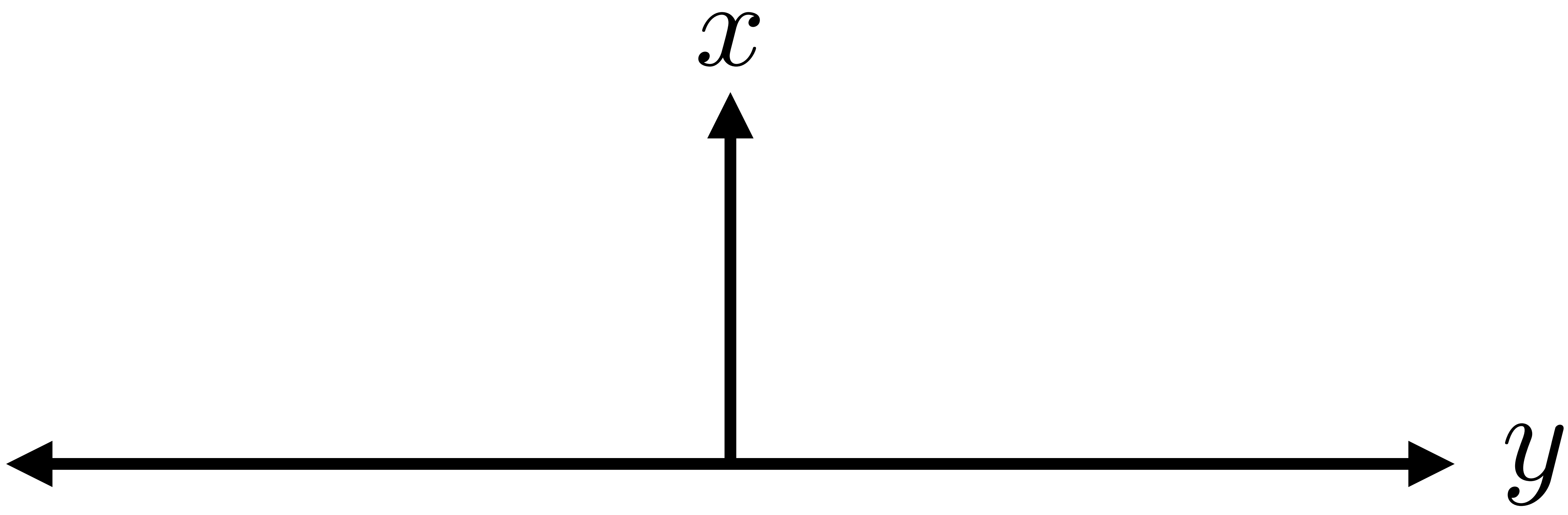
on a space with a boundary



The g-theorem

Conformal boundary conditions

Boundary CFT (BCFT)



The g-theorem

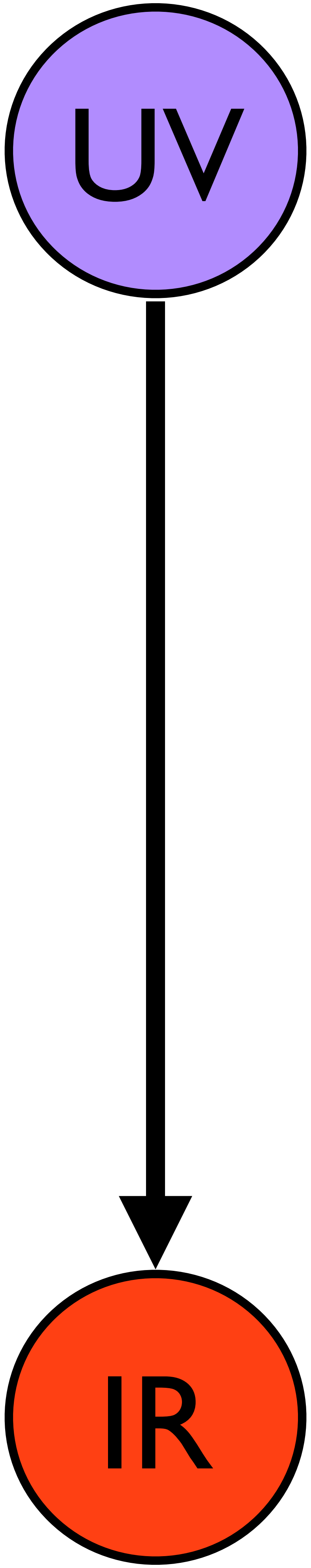
Boundary RG flows

$$S(\lambda) \to S(\lambda) + \int dx\, dy\, \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y) \qquad \Delta_{\mathcal{O}} < 1$$

Bulk theory remains conformal

$$T_{\mu\nu} = [T_{\mu\nu}]_{\text{bulk}} + \delta(x)\, [T_{\mu\nu}]_{\partial}$$

$$\left[T_{\mu}{}^{\mu}\right]_{\text{bulk}} = 0 \qquad T_{\mu}{}^{\mu} = \delta(x) \left[T_{\mu}{}^{\mu}\right]_{\partial} \neq 0$$



UV BCFT

CFT with boundary condition "α"

Boundary RG flow

$$S(\lambda) \to S(\lambda) + \int dx\, dy\, \delta(x) \lambda_{\mathcal{O}} \mathcal{O}(y) \qquad \Delta_{\mathcal{O}} < 1$$

IR BCFT

CFT with boundary condition "β"

The g-theorem

At fixed point: conformally map to disk

$$g_\alpha \equiv \langle \mathbf{1} \rangle_\alpha^{\text{ren.}}$$

"Boundary entropy"

$$\ln g_\alpha$$

Counts DOF localized at boundary

The g-theorem

Thermodynamic entropy

Affleck and Ludwig PRL 67 (1991) 161

System size L Temperature T

$$T \gg 1/L$$

$$S_{\mathrm{thermo}} = \frac{\pi}{3} c L T + \ln g_\alpha + \ldots$$

The g-theorem

Entanglement Entropy (EE)

Calabrese + Cardy hep-th/0405152

Interval including the boundary

$$S_{\mathrm{EE}} = \frac{c}{6} \ln \frac{2\ell}{a} + \ln g_\alpha + \ldots$$

$$\ln g_\alpha = S_{\mathrm{EE}}^{\mathrm{BCFT}} - \frac{1}{2} S_{\mathrm{EE}}^{\mathrm{CFT}}$$

The g-theorem

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

Define a g-function G

Euclidean symmetry, locality, unitarity

$$\frac{\partial G}{\partial r} \leq 0 \qquad \Rightarrow \qquad g_{\mathrm{UV}} \geq g_{\mathrm{IR}}$$

Strong form

Weak form

The g-theorem

Affleck and Ludwig PRL 67 (1991) 161

Friedan and Konechny hep-th/0312197

Define a g-function G

Euclidean symmetry, locality, unitarity

$$\frac{\partial G}{\partial r} \leq 0 \qquad \Rightarrow \qquad g_{\mathrm{UV}} \geq g_{\mathrm{IR}}$$

They can't prove that $\ln g$ is bounded from below!

Generalizations?

Higher-dimensional g-theorems?

Proposals

Yamaguchi	hep-th/0207171
Takayanagi et al.	1105.5165, 1108.5152, 1205.1573
Estes, Jensen, O'B., Tsatis, Wrase	1403.6475
Gaiotto	1403.8052

Many tests in particular examples

No proofs yet!

GOAL

Prove a g-theorem for
Local, unitary BCFT in $d = 3$

Proposals

Nozaki, Takayanagi, Ugajin 1205.1573

Estes, Jensen, O'B., Tsatis, Wrase 1403.6475

Method

Komargodski and Schwimmer 1107.3987

Komargodski 1112.4538

Examples

Graphene with a boundary

Critical Ising model in $d = 3$ with a boundary

M-theory: M2-branes with a boundary

String theory: various brane intersections

Holographic BCFTs

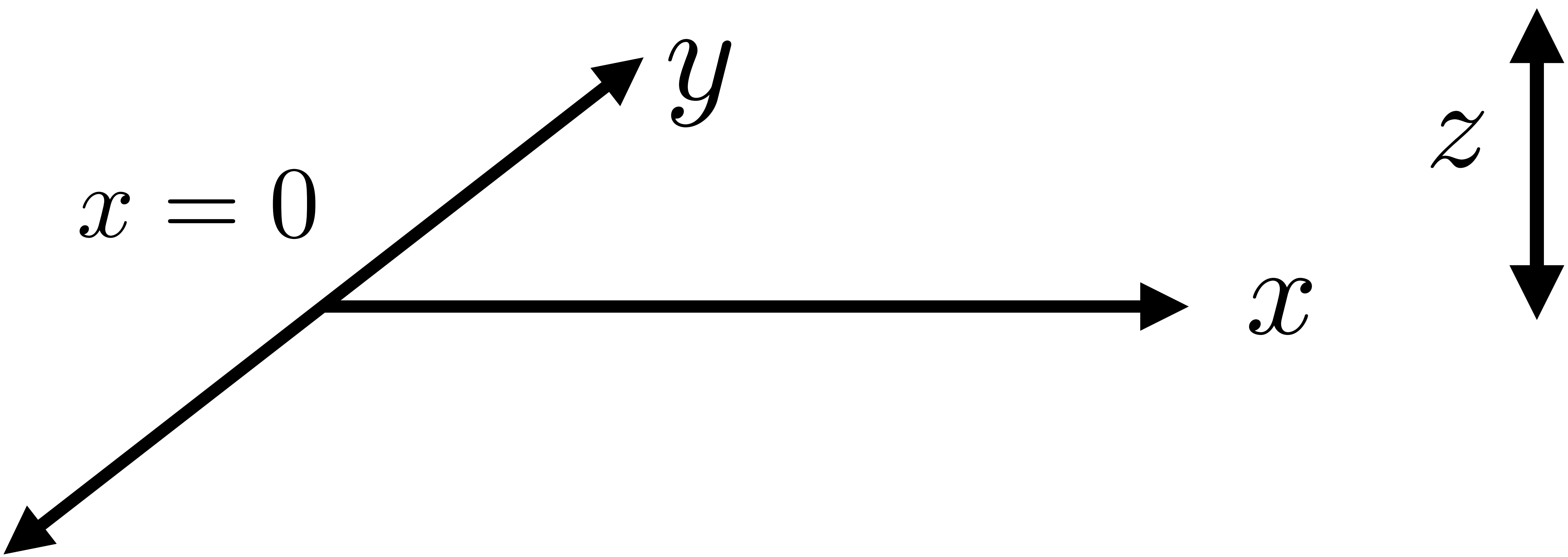
Outline:

- Review: Monotonicity Theorems
- The Systems
- The Weyl Anomaly
- The Proof
- Summary and Outlook

The Systems

BCFT in $d = 3$

With a planar boundary



The Systems

$$SO(d+1,1) = SO(4,1)$$

Rotations	$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$
Translations	$x^\mu \to x^\mu + c^\mu$
Dilatations	$x^\mu \to \lambda \, x^\mu$
Special Conformal	$x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$

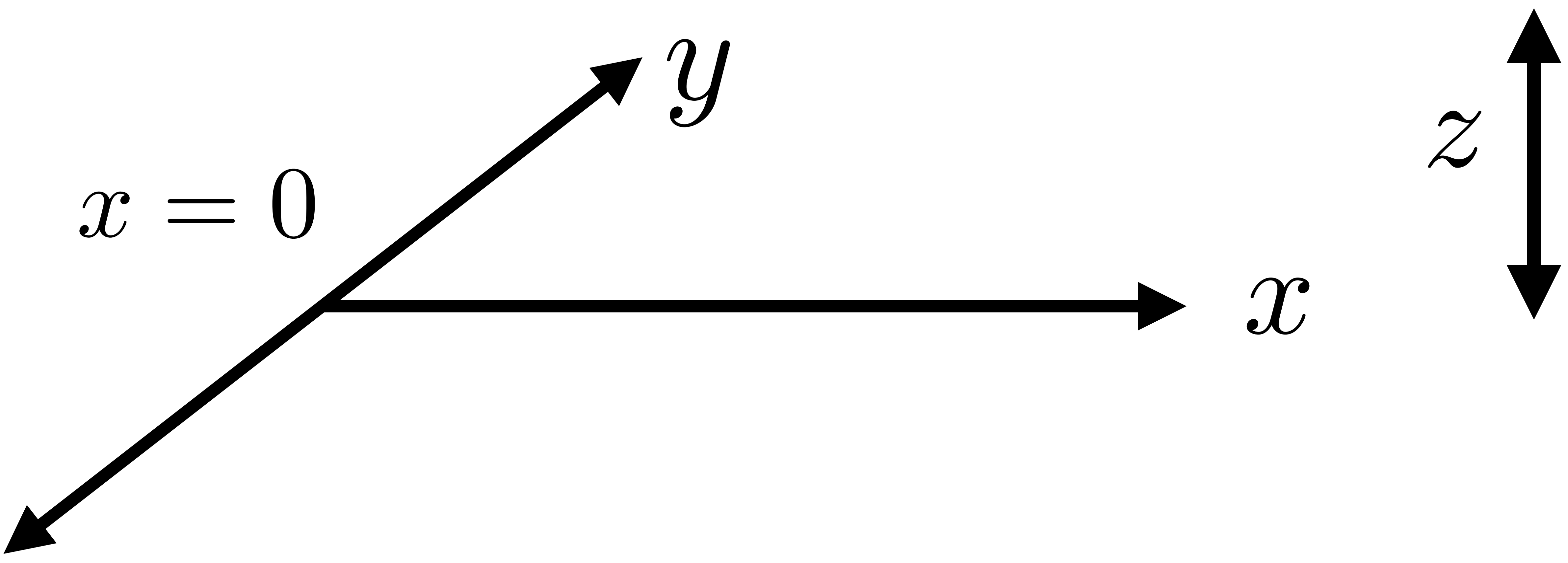
Broken to subgroup that preserves $x = 0$

The Systems

Rotations

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$$

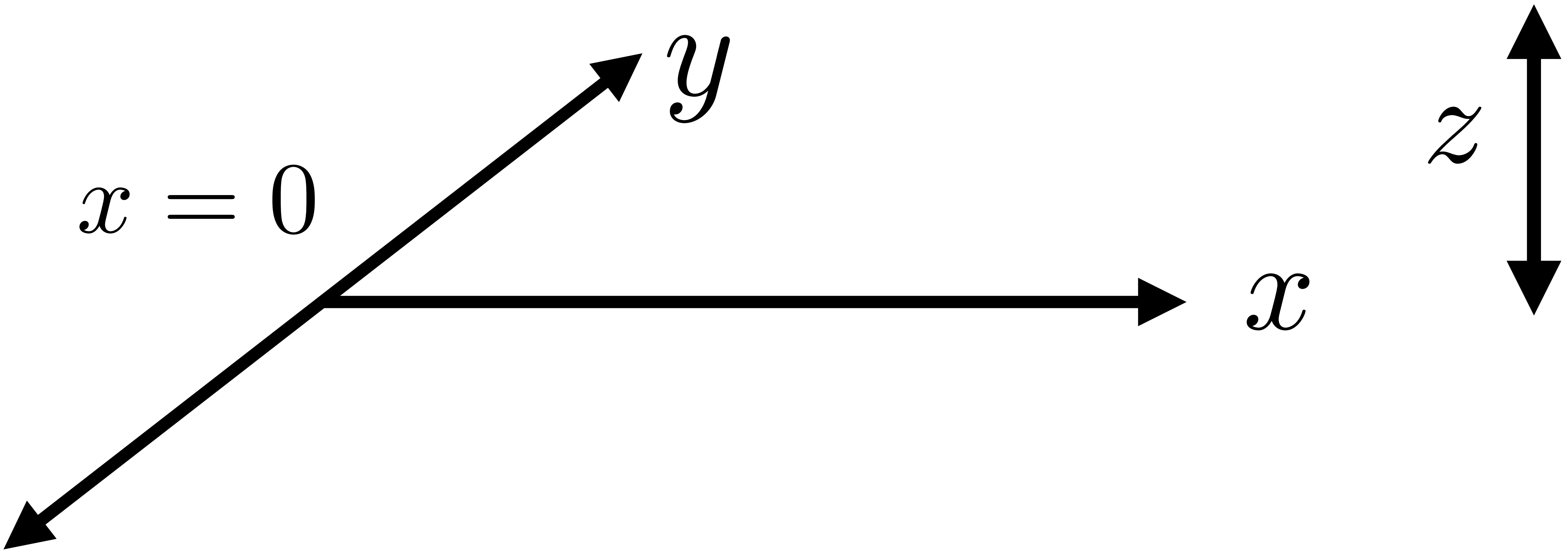
Broken to rotations in (y, z)



The Systems

Translations $x^\mu \to x^\mu + c^\mu$

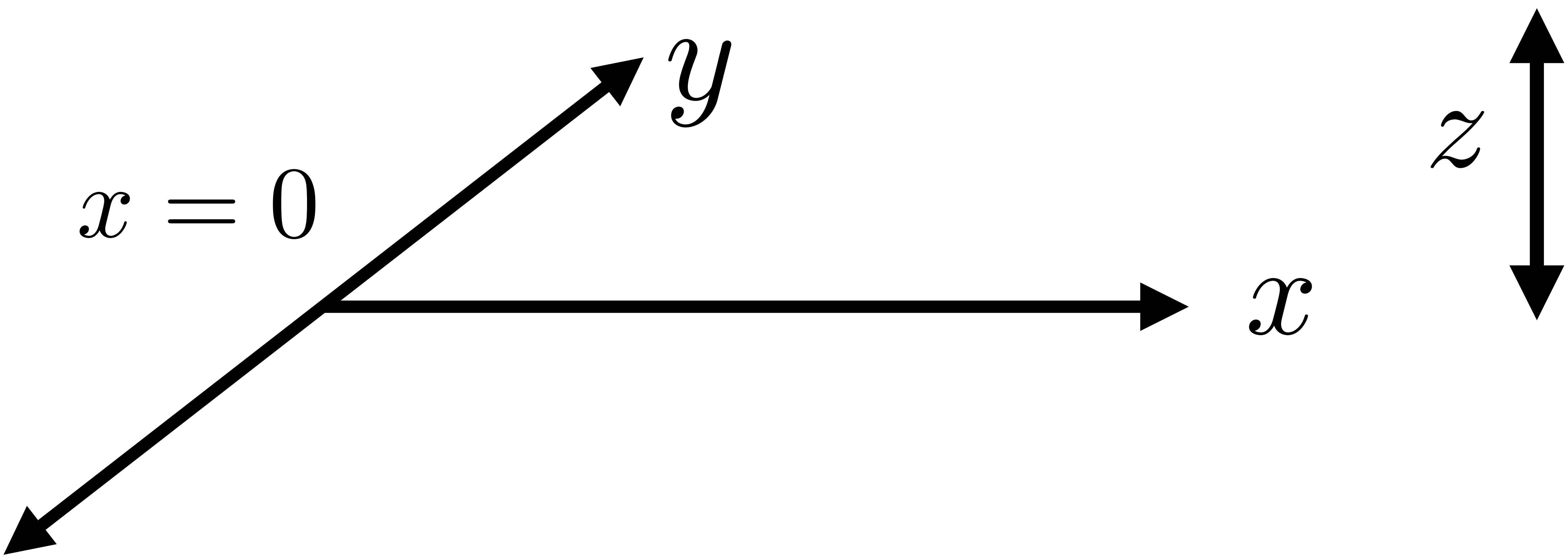
Broken to translations along (y, z)



The Systems

Dilatations $\quad x^\mu \to \lambda\, x^\mu$

Unbroken

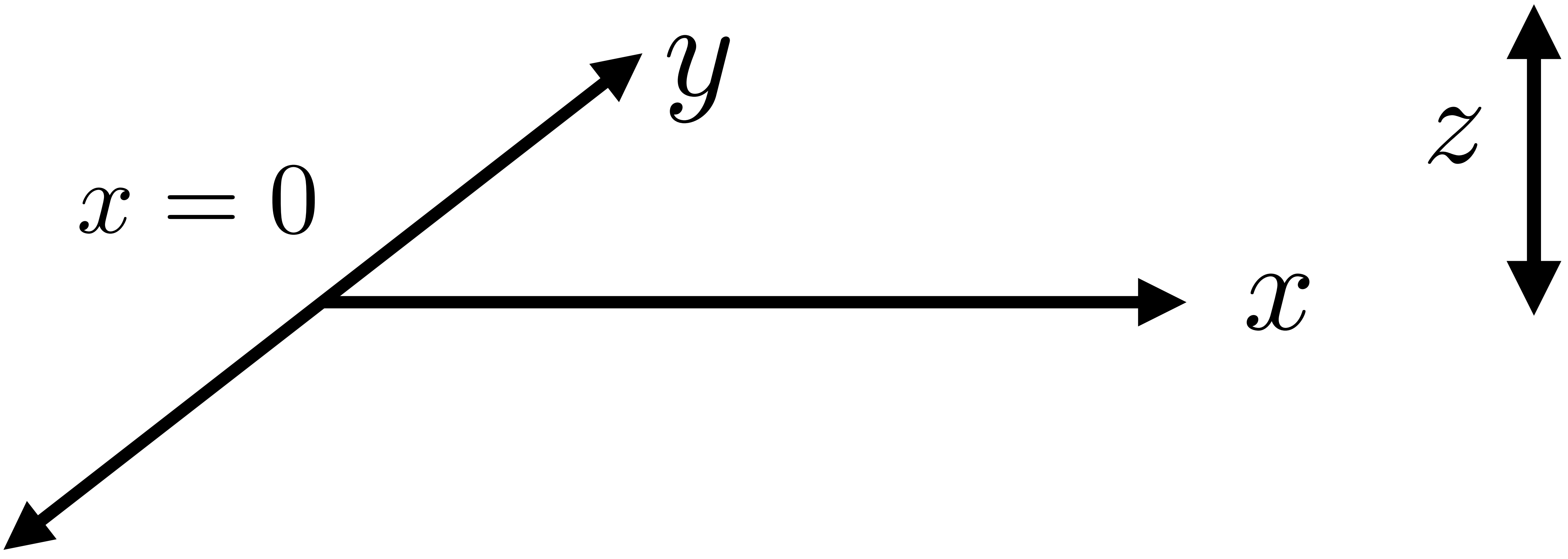


The Systems

Special Conformal

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

Broken to $b^x = 0$

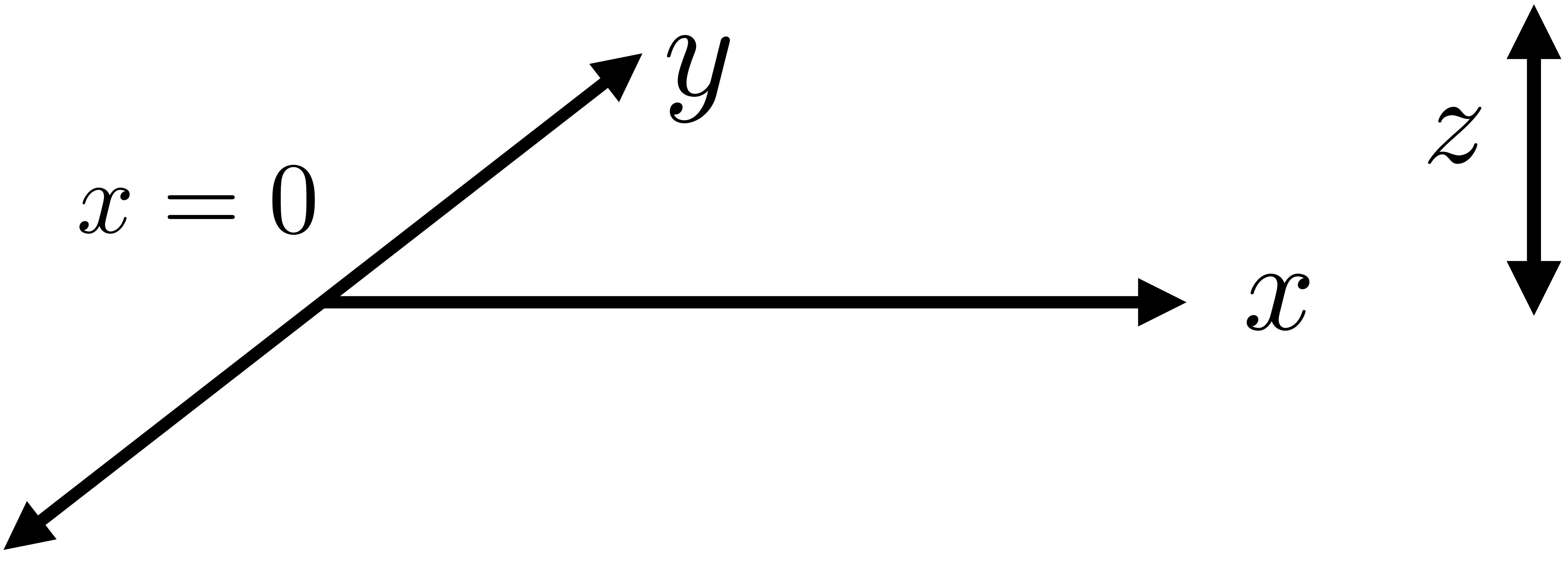


The Systems

Unbroken symmetry

$$SO(d+1,1) \to SO(d,1)$$

$$SO(4,1) \to SO(3,1)$$



The Systems

Boundary RG Flows

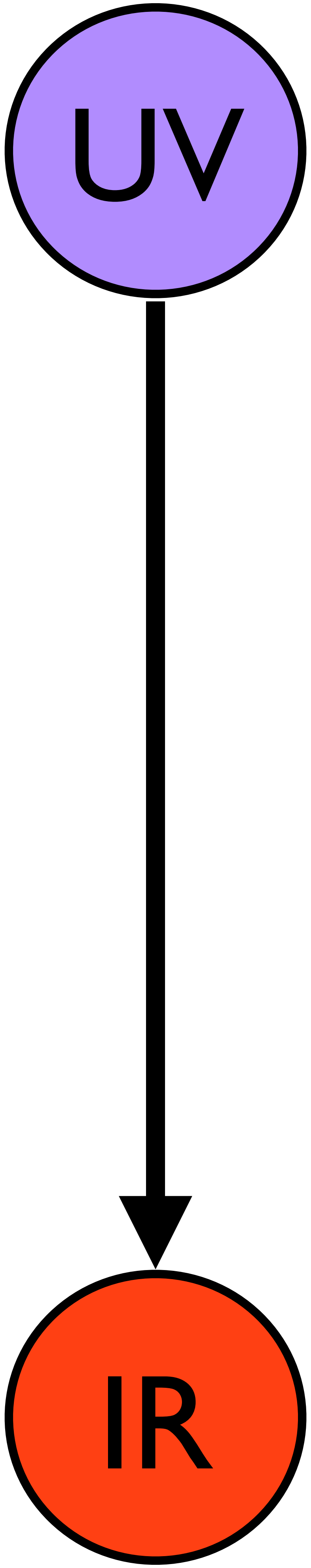
$$S(\lambda) \to S(\lambda) + \int dx\, dy\, dz\, \delta(x)\, \lambda_{\mathcal{O}} \mathcal{O}(y,z)$$

$\mathcal{O}$ scalar of $SO(3,1)$ with $\Delta_{\mathcal{O}} < 2$

$$T_{\mu\nu} = [T_{\mu\nu}]_{\mathrm{bulk}} + \delta(x)\, [T_{\mu\nu}]_{\partial}$$

Bulk theory remains conformal

$$\left[T_{\mu}{}^{\mu}\right]_{\mathrm{bulk}} = 0 \qquad T_{\mu}{}^{\mu} = \delta(x)\, \left[T_{\mu}{}^{\mu}\right]_{\partial} \neq 0$$



UV BCFT

Single real, free, massless, scalar

Neumann B.C.

Boundary RG Flow

$$S \to S + \int d^3x\, \delta(x)\, m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1$$

IR BCFT

Single real, free, massless, scalar

Dirichlet B.C.

The Systems

Boundary RG Flows

$$S(\lambda) \to S(\lambda) + \int dx\, dy\, dz\, \delta(x)\, \lambda_{\mathcal{O}} \mathcal{O}(y, z)$$

Weyl Anomaly Matching

Komargodski and Schwimmer 1107.3987   Komargodski 1112.4538

Reflection Positivity

Weak form

Outline:

- Review: Monotonicity Theorems
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- The Weyl Anomaly
- The Proof
- Summary and Outlook

Weyl Anomaly

CFT in any d

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_\mu{}^\mu = 0$$

Weyl Anomaly

CFT in any d

Non-trivial $g_{\mu\nu}$

Quantum Effects
Break Conformal Invariance

$$T_\mu{}^\mu \neq 0$$

Weyl Anomaly

What is the general form of $T_\mu{}^\mu$?

Step #1

Write down all curvature invariants

built from $g_{\mu\nu}$

with the correct dimension

$$d = 4$$

$$T_\mu{}^\mu = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \ldots$$

Weyl Anomaly

What is the general form of $T_\mu{}^\mu$?

Step #2

Wess-Zumino consistency

$$g_{\mu\nu} \to e^{2\Omega_1} e^{2\Omega_2} g_{\mu\nu} \quad = \quad g_{\mu\nu} \to e^{2\Omega_2} e^{2\Omega_1} g_{\mu\nu}$$

Fixes some coefficients

$$T_\mu{}^\mu = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \ldots$$

Weyl Anomaly

What is the general form of $T_\mu{}^\mu$?

Step #3

Add local counterterms to $S(g_{\mu\nu}, \lambda)$

Determine how they enter $T_\mu{}^\mu$

Fixes more coefficients

$$T_\mu{}^\mu = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + \dots$$

Weyl Anomaly

CFT in any d

d odd $\qquad T_\mu{}^\mu = 0$

d even $\qquad T_\mu{}^\mu \neq 0$

$d = 2 \qquad T_\mu{}^\mu = \frac{c}{24\pi} R$

Weyl Anomaly

$$d = 4$$

$$T_{\mu}{}^{\mu} = a\,E - c\,W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$$

Euler density

$$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

Weyl tensor

$$W_{\mu\nu\rho\sigma}$$

"central charges" a and c

Weyl Anomaly

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x)$$

$\int d^d x \sqrt{g}\, T_\mu{}^\mu$ is invariant

$$T_\mu{}^\mu = a\, E - c\, W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Type A

$$\sqrt{g}\, E$$

Changes by a total derivative

Type B

$$\sqrt{g}\, W^2$$

Invariant

Weyl Anomaly

BCFT in $d = 3$

$$T_\mu{}^\mu = \left[T_\mu{}^\mu\right]_{\text{bulk}} + \delta(x)\,\left[T_\mu{}^\mu\right]_\partial$$

$$\left[T_\mu{}^\mu\right]_{\text{bulk}} = 0$$

What is the general form of $\left[T_\mu{}^\mu\right]_\partial$?

Geometry of Submanifolds

“worldsheet” σ^1, σ^2

“target space” x^μ

Embedding $x^\mu(\sigma^a)$

Induced metric

$$\hat{g}_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}$$

$$\hat{R}_{abcd} \qquad \hat{R}_{ab} \qquad \hat{R}$$

Geometry of Submanifolds

Extrinsic Curvature
"Second Fundamental Form"

Gaussian Normal Coordinates

$$K_{ab} = \frac{1}{2}\partial_x \hat{g}_{ab}(x,\sigma)$$

Mean curvature

$$K \equiv \hat{g}^{ab} K_{ab}$$

Weyl Anomaly

$$[T_\mu{}^\mu]_\partial = c_1\,\hat{R} + c_2(K_{ab}K^{ab} - \frac{1}{2}K^2)$$

Schwimmer + Theisen 0802.1017

See also:

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Weyl Anomaly

$$\left[T_\mu{}^\mu\right]_\partial = c_1\,\hat{R} + c_2\left(K_{ab}K^{ab} - \frac{1}{2}K^2\right)$$

Boundary "central charges"

c_1 and c_2

Weyl Anomaly

$$[T_\mu{}^\mu]_\partial = c_1\,\hat{R} + c_2(K_{ab}K^{ab} - \frac{1}{2}K^2)$$

Type A

Type B

$$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$$

$$\sqrt{\hat{g}}\hat{R} \to \sqrt{\hat{g}}\left[\hat{R} - 2\nabla^2\Omega\right]$$

Changes by a total derivative

$$\sqrt{\hat{g}}(K_{ab}K^{ab} - \frac{1}{2}K^2)$$

Invariant

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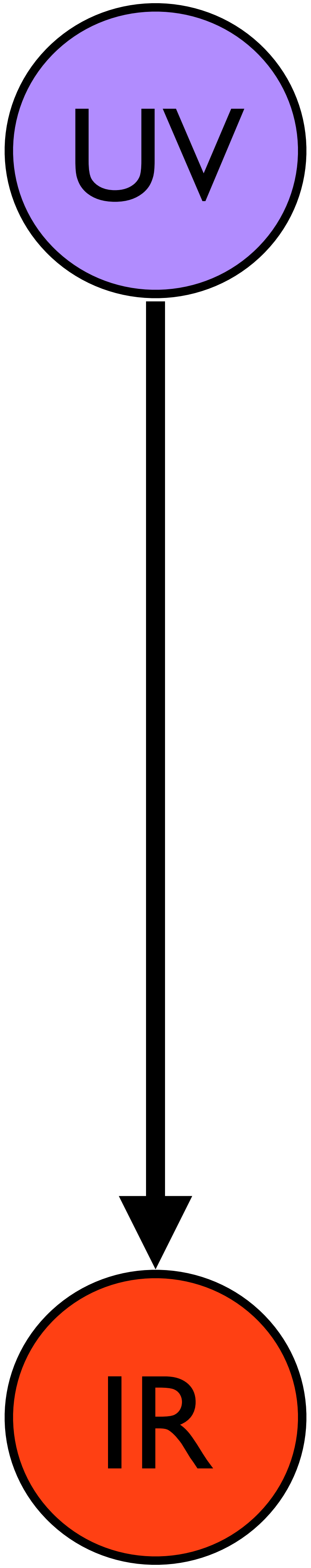


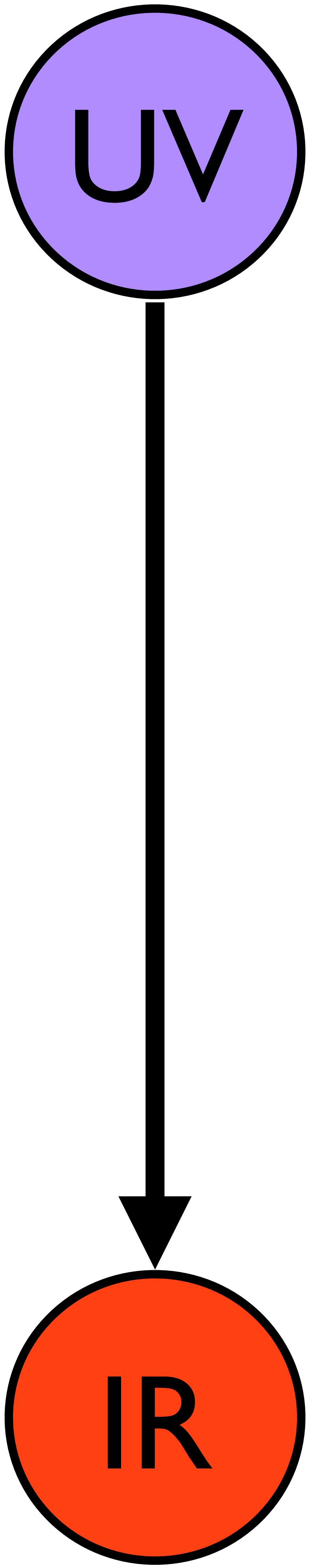
The Proof

Komargodski and Schwimmer 1107.3987

Komargodski 1112.4538

local, unitary QFT in any d

RG flow between fixed point CFTs



$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} = 0$$

$$T_\mu{}^\mu \neq 0$$

RG flow between fixed point CFTs

$$\left[T_\mu{}^\mu\right]^{\mathrm{IR}} = 0$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

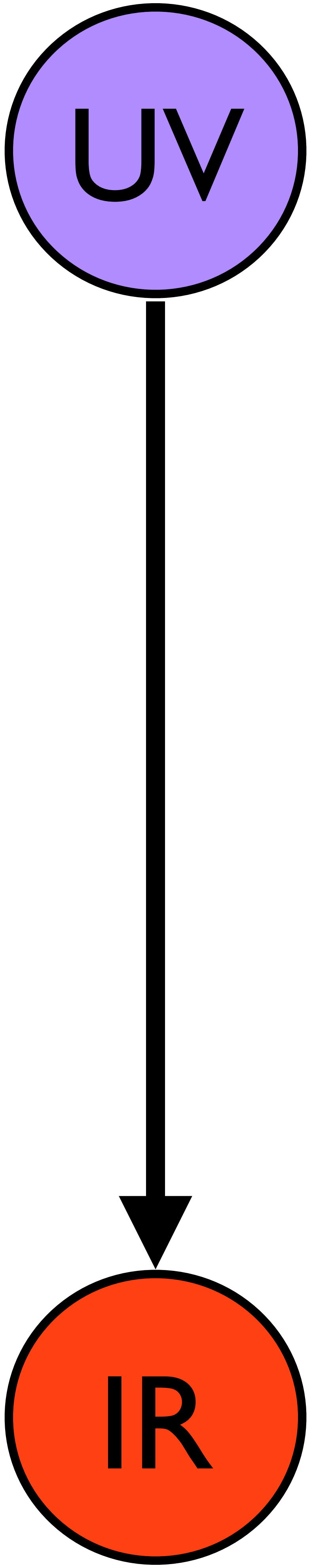
$$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu} \qquad \tau \to \tau + \Omega$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

$$\lambda_{\mathcal{O}}\mathcal{O}(x) \to e^{(\Delta_{\mathcal{O}}-d)\tau(x)}\lambda_{\mathcal{O}}\mathcal{O}(x)$$



$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} = 0$$

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$$\tau \neq 0$$

$$T_\mu{}^\mu = \left[T_\mu{}^\mu\right]_{\tau=0} + \left[T_\mu{}^\mu\right]_\tau = 0$$

$$\left[T_\mu{}^\mu\right]^{\mathrm{IR}} = 0$$

Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

$$S(\lambda)_\tau = \int d^d x \, \sqrt{g} \, L(\lambda, \vec{x})_\tau$$

$$= \int d^d x \sqrt{g} \left[L(\lambda, \vec{x})_{\tau=0} + \tau \left[T_\mu{}^\mu \right]_{\tau=0} + \mathcal{O}(\tau^2) \right]$$

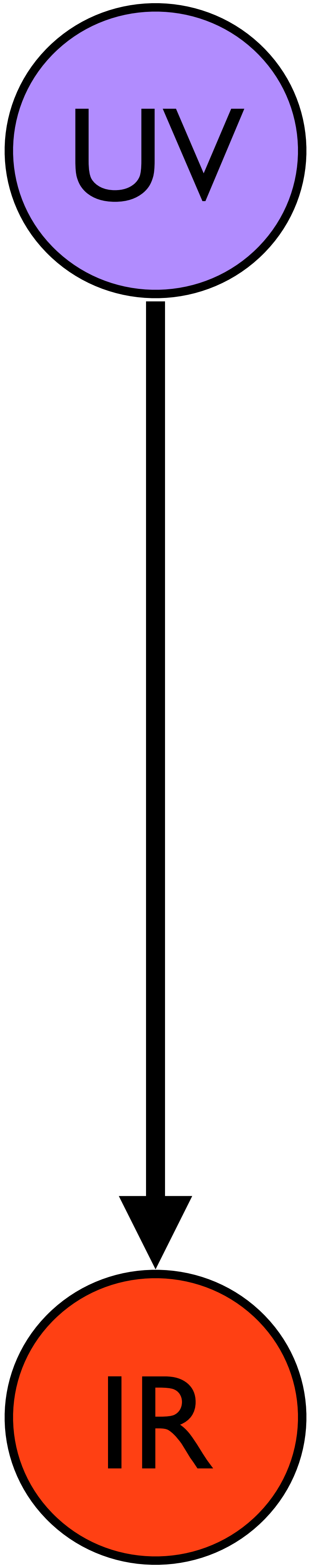
Dilaton

Non-dynamical background metric $g_{\mu\nu}(x)$

Non-dynamical background scalar $\tau(x)$

Weyl Anomaly Matching

d even



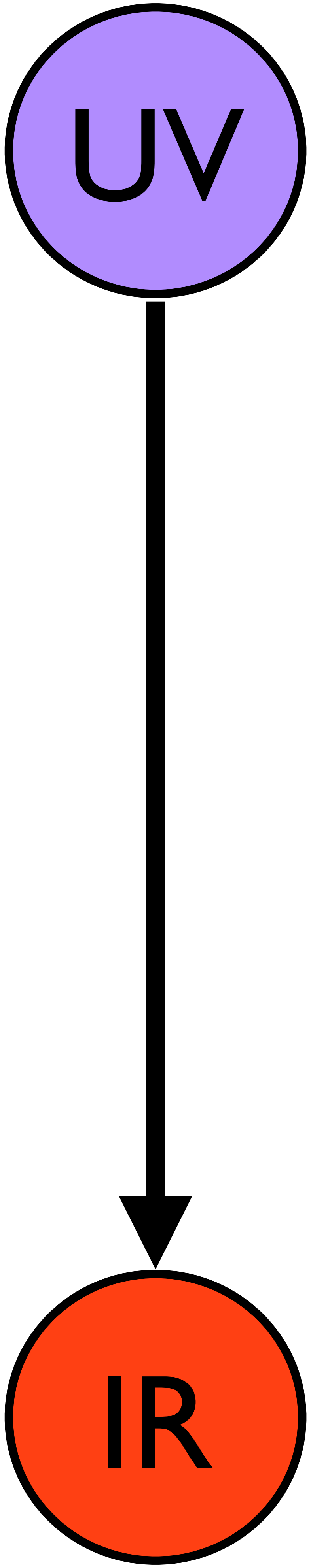
$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} \neq 0$$

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

$$\tau = 0$$

$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} \neq \left[T_\mu{}^\mu\right]^{\mathrm{IR}}$$

$$\left[T_\mu{}^\mu\right]^{\mathrm{IR}} \neq 0$$



$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} \neq 0$$

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

$$\tau \neq 0$$

Weyl Anomaly Matching

$$\left[T_\mu{}^\mu\right]^{\mathrm{UV}} = \left[T_\mu{}^\mu\right]^{\mathrm{IR}}_{\tau=0} + \left[T_\mu{}^\mu\right]_\tau$$

Dilaton

Integrate out massive DOF

Obtain low-energy effective action

$$S_{\text{eff}} \equiv -\ln Z$$

Regular and local in $\mathcal{T}$

Dilaton

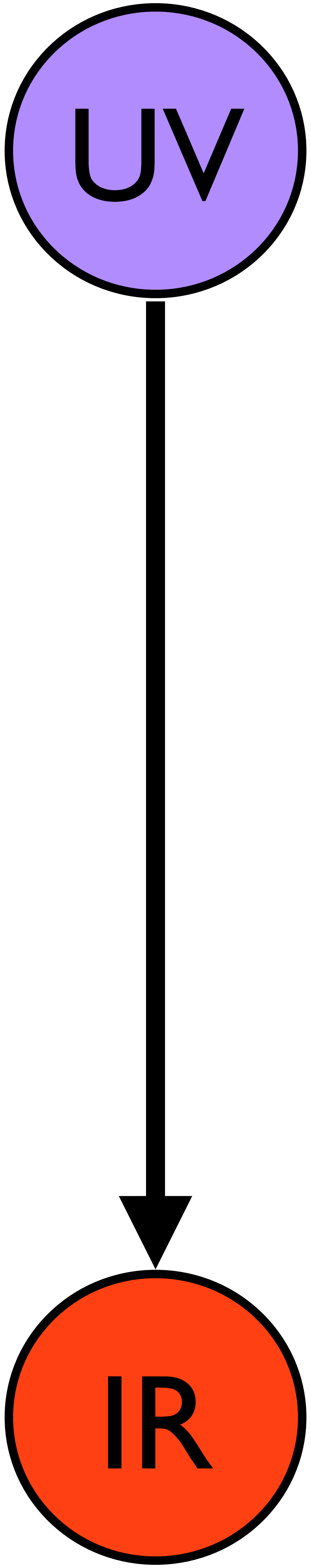
$$S_{\text{eff}} \equiv -\ln Z$$

$$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu} \qquad \tau \to \tau + \Omega$$

$$\frac{\delta S_{\text{eff}}}{\delta \Omega} = -\frac{\delta}{\delta \Omega} \ln Z = -\sqrt{g}\, T_\mu{}^\mu$$

τ's contribution to S_{eff} must produce

$$\left[T_\mu{}^\mu\right]^\tau = \left[T_\mu{}^\mu\right]^{\text{UV}} - \left[T_\mu{}^\mu\right]^{\text{IR}}$$



$$[T_\mu{}^\mu]_\partial^{\rm UV} = c_1^{\rm UV}\,\hat{R} + c_2^{\rm UV}(K_{ab}K^{ab} - \frac{1}{2}K^2)$$

$$T_\mu{}^\mu = [T_\mu{}^\mu]_{\rm bulk} + \delta(x)\,[T_\mu{}^\mu]_\partial$$

Boundary RG Flow

$$S(\lambda) \to S(\lambda) + \int dx\,dy\,dz\,\delta(x)\,\lambda_{\mathcal{O}}\,\mathcal{O}(y,z)$$

$$[T_\mu{}^\mu]_{\rm bulk} = 0$$

$$[T_\mu{}^\mu]_\partial^{\rm IR} = c_1^{\rm IR}\,\hat{R} + c_2^{\rm IR}(K_{ab}K^{ab} - \frac{1}{2}K^2)$$

Dilaton

$$S^{\tau}_{\text{eff}} = -\int d^3x\,\delta(x)\,\sqrt{g}\;\tau\left[(c_1^{\text{UV}} - c_1^{\text{IR}})\,\hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}})(K_{ab}K^{ab} - \frac{1}{2}K^2)\right]$$

$$+\mathcal{O}(\tau^2)$$

$$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu} \qquad \tau \to \tau + \Omega$$

$$\sqrt{\hat{g}}\hat{R} \to \sqrt{\hat{g}}\left[\hat{R} - 2\nabla^2\Omega\right]$$

$$\frac{\delta S^{\tau}_{\text{eff}}}{\delta\Omega} = -\left(\sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{UV}} - \sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{IR}}\right) + \sqrt{g}\,(c_1^{\text{UV}} - c_1^{\text{IR}})\,2\nabla^2\tau$$

Dilaton

$$S^{\tau}_{\text{eff}} = -\int d^3x\,\delta(x)\,\sqrt{g}\;\tau\left[(c_1^{\text{UV}} - c_1^{\text{IR}})\,\hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}})(K_{ab}K^{ab} - \frac{1}{2}K^2)\right]$$

$$- \int d^3x\,\delta(x)\,\sqrt{g}\;(c_1^{\text{UV}} - c_1^{\text{IR}})\,\tau\nabla^2\tau \;+ \ldots$$

$$\frac{\delta S^{\tau}_{\text{eff}}}{\delta\Omega} = -\left(\sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{UV}} - \sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{IR}}\right)$$

Dilaton

$$S_{\text{eff}}^{\tau} = -\int d^3x\,\delta(x)\,\sqrt{g}\;\tau\left[(c_1^{\text{UV}} - c_1^{\text{IR}})\,\hat{R} + (c_2^{\text{UV}} - c_2^{\text{IR}})(K_{ab}K^{ab} - \frac{1}{2}K^2)\right]$$

$$-\int d^3x\,\delta(x)\,\sqrt{g}\;(c_1^{\text{UV}} - c_1^{\text{IR}})\,\tau\nabla^2\tau \;+\ldots$$

↑

Survives the flat-space limit

$$g_{\mu\nu} \to \delta_{\mu\nu}$$

$$\frac{\delta S_{\text{eff}}^{\tau}}{\delta\Omega} = -\left(\sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{UV}} - \sqrt{g}\,\left[T_{\mu}{}^{\mu}\right]^{\text{IR}}\right)$$

Dilaton

Another form for the two-derivative term

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$$\ln Z = \int [\text{fields}]\, e^{-S(\lambda)_\tau} = \langle e^{-\int d^3x\, \tau(x) T_\mu{}^\mu(x) + \dots} \rangle_{\tau=0}$$

Dilaton

Another form for the two-derivative term

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$$\langle e^{-\int d^3x\, \tau(x) T_\mu{}^\mu(x) + \dots} \rangle = 1 - \int d^3x\, \tau(x)\, \langle T_\mu{}^\mu(x) \rangle$$

$$+ \frac{1}{2} \int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T_\mu{}^\mu(x) T_\mu{}^\mu(y) \rangle + \dots$$

↑

Taylor expand about x

Dilaton

Another form for the two-derivative term

$$\frac{1}{2}\int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T_\mu^\mu(x) T_\mu^\mu(y)\rangle$$

$$\supset \frac{1}{4}\int d^3x\, \tau(x)\partial_\rho\partial_\sigma\tau(x)\left[\int d^3y\, (y-x)^\rho (y-x)^\sigma \langle {T_\mu}^\mu(x) {T_\mu}^\mu(y)\rangle\right]$$

$${T_\mu}^\mu = \left[{T_\mu}^\mu\right]_{\mathrm{bulk}} + \delta(x)\left[{T_\mu}^\mu\right]_\partial$$

$$\left[{T_\mu}^\mu\right]_{\mathrm{bulk}} = 0$$

Dilaton

Another form for the two-derivative term

$$\frac{1}{2}\int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T_\mu^\mu(x) T_\mu^\mu(y)\rangle$$

$$\supset \frac{1}{4}\int d^3x\, \tau(x)\partial_\rho\partial_\sigma\tau(x)\left[\int d^3y\, (y-x)^\rho (y-x)^\sigma \langle T_\mu{}^\mu(x) T_\mu{}^\mu(y)\rangle\right]$$

$$\langle T_\mu{}^\mu(x) T_\mu{}^\mu(y)\rangle = \delta(x)\delta(y)\langle \left[T_\mu{}^\mu(x)\right]_\partial \left[T_\mu{}^\mu(y)\right]_\partial\rangle$$

Dilaton

Another form for the two-derivative term

$$\frac{1}{2}\int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T^{\mu}_{\mu}(x)T^{\mu}_{\mu}(y)\rangle$$

$$\supset \frac{1}{4}\int d^3x\, \tau(x)\partial_\rho\partial_\sigma\tau(x)\left[\int d^3y\, (y-x)^\rho (y-x)^\sigma \langle T_\mu{}^\mu(x)T_\mu{}^\mu(y)\rangle\right]$$

Reflection Positivity

$$\langle T_\mu{}^\mu(x)T_\mu{}^\mu(y)\rangle = \delta(x)\delta(y)\langle \left[T_\mu{}^\mu(x)\right]_\partial \left[T_\mu{}^\mu(y)\right]_\partial\rangle \geq 0$$

Dilaton

Another form for the two-derivative term

$$\frac{1}{2}\int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T_\mu^\mu(x) T_\mu^\mu(y)\rangle$$

$$\supset \frac{1}{4}\int d^3x\, \tau(x)\partial_\rho\partial_\sigma\tau(x)\Big[\int d^3y\,(y-x)^\rho(y-x)^\sigma\,\langle T_\mu{}^\mu(x)T_\mu{}^\mu(y)\rangle\Big]$$

Translation invariance along the boundary

$$\delta(x)\int d^3y\,\delta(y)\,(y-x)^\rho(y-x)^\sigma\langle\left[T_\mu{}^\mu(x)\right]_\partial\left[T_\mu{}^\mu(y)\right]_\partial\rangle$$

$$=\delta(x)\frac{1}{2}\delta^{\rho\sigma}\int d^3y\,\delta(y)\,y^2\langle\left[T_\mu{}^\mu(0)\right]_\partial\left[T_\mu{}^\mu(y)\right]_\partial\rangle$$

Dilaton

Another form for the two-derivative term

$$\frac{1}{2}\int d^3x\, d^3y\, \tau(x)\tau(y)\, \langle T^\mu_\mu(x) T^\mu_\mu(y)\rangle$$

$$\supset \frac{1}{4}\int d^3x\, \tau(x)\partial_\rho\partial_\sigma\tau(x)\Big[\int d^3y\, (y-x)^\rho (y-x)^\sigma \langle T_\mu{}^\mu(x) T_\mu{}^\mu(y)\rangle\Big]$$

Reflection Positivity

$$\delta(x)\int d^3y\, \delta(y)\, (y-x)^\rho (y-x)^\sigma \langle \left[T_\mu{}^\mu(x)\right]_\partial \left[T_\mu{}^\mu(y)\right]_\partial\rangle$$

$$= \delta(x)\frac{1}{2}\delta^{\rho\sigma}\int d^3y\, \delta(y)\, y^2 \langle \left[T_\mu{}^\mu(0)\right]_\partial \left[T_\mu{}^\mu(y)\right]_\partial\rangle \geq 0$$

Dilaton

$$\int d^3x\, \delta(x)\ \tau\nabla^2\tau\ (c_1^{\rm UV} - c_1^{\rm IR})$$

$$= \int d^3x\, \delta(x)\ \tau\nabla^2\tau\ \left[\frac{1}{8}\int d^3y\, \delta(y)\, y^2 \langle [T_\mu{}^\mu(0)]_\partial\ [T_\mu{}^\mu(y)]_\partial \rangle\right]$$

$$c_1^{\rm UV} - c_1^{\rm IR} = \left[\frac{1}{8}\int d^3y\, \delta(y)\, y^2 \langle [T_\mu{}^\mu(0)]_\partial\ [T_\mu{}^\mu(y)]_\partial \rangle\right] \geq 0$$

$$c_1^{\rm UV} \geq c_1^{\rm IR}$$

$$c_1^{\mathrm{UV}} \geq c_1^{\mathrm{IR}}$$

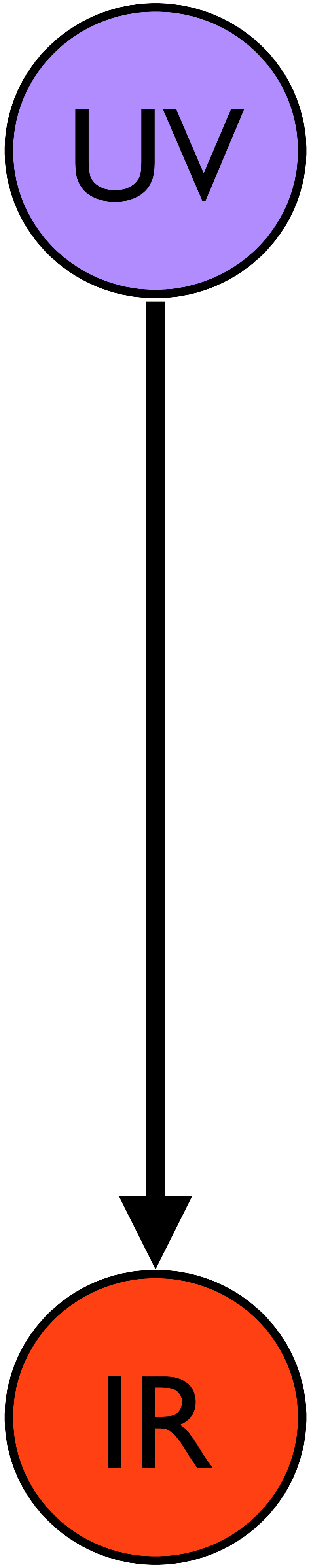
Does c_1 count DOF?

Add a single, real, free scalar field or Dirac fermion at the boundary

$$24\pi c_1 \to 24\pi c_1 + 1$$

c_2 unchanged

Both depend on boundary conditions of bulk fields



UV BCFT

Single real, free, massless, scalar
Neumann B.C.

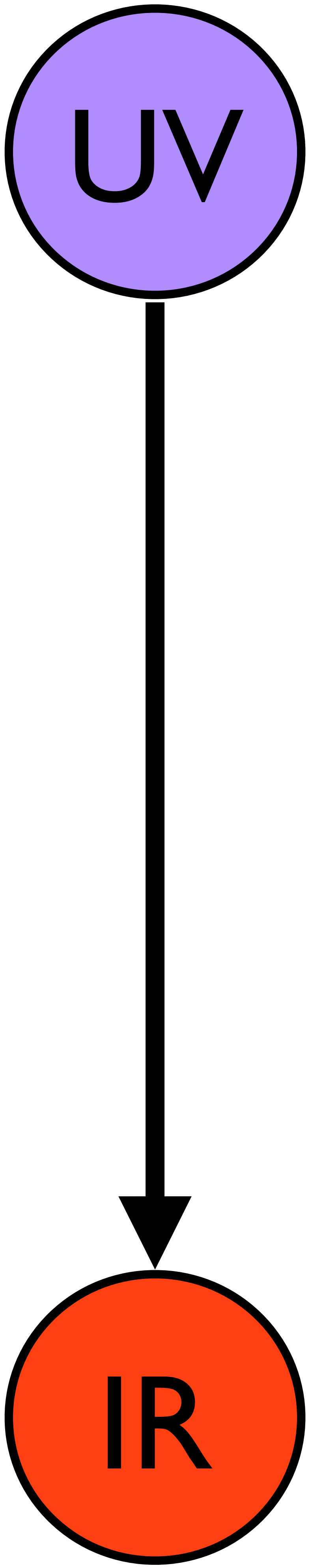
Boundary RG Flow

$$S \to S + \int d^3x\, \delta(x)\, m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1$$

IR BCFT

Single real, free, massless, scalar
Dirichlet B.C.



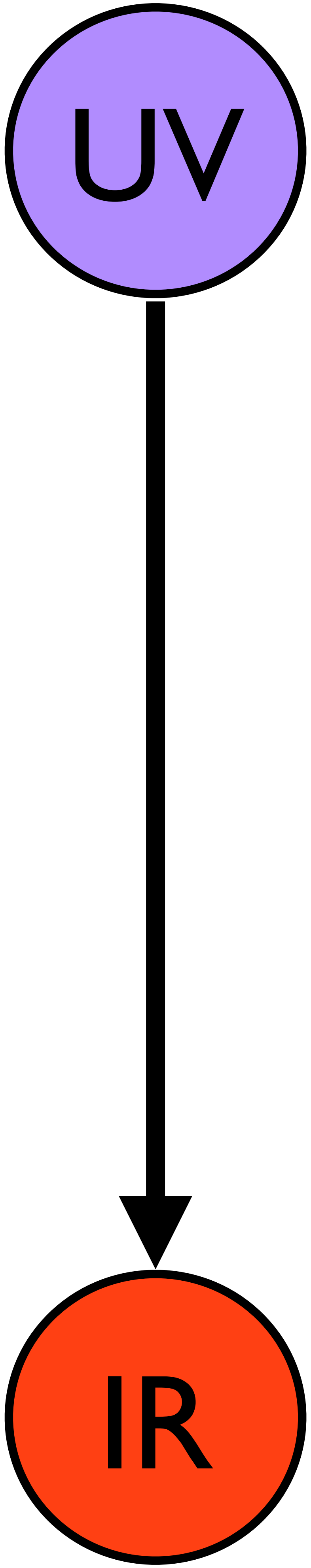
$$c_1^{\mathrm{UV}} = \frac{1}{24\pi}\frac{7}{16}$$

Neumann B.C.

Nozaki, Takayanagi, Ugajin
1205.1573

$$c_1^{\mathrm{IR}} = -\frac{1}{24\pi}\frac{1}{16}$$

Dirichlet B.C.



$$c_1^{\mathrm{UV}} = \frac{1}{24\pi}\frac{7}{16}$$

Neumann B.C.

$$c_1^{\mathrm{UV}} \geq c_1^{\mathrm{IR}}$$

Is c_1 bounded below?

$$c_1^{\mathrm{IR}} = -\frac{1}{24\pi}\frac{1}{16}$$

Dirichlet B.C.

Defects

Local, unitary CFT in any $d \geq 3$

With a two-dimensional planar defect

Conformal defect

$$SO(d+1,1) \to SO(3,1) \times SO(d-2)$$

conformal transformations preserving the defect

rotations about the defect

"Defect CFT" (DCFT)

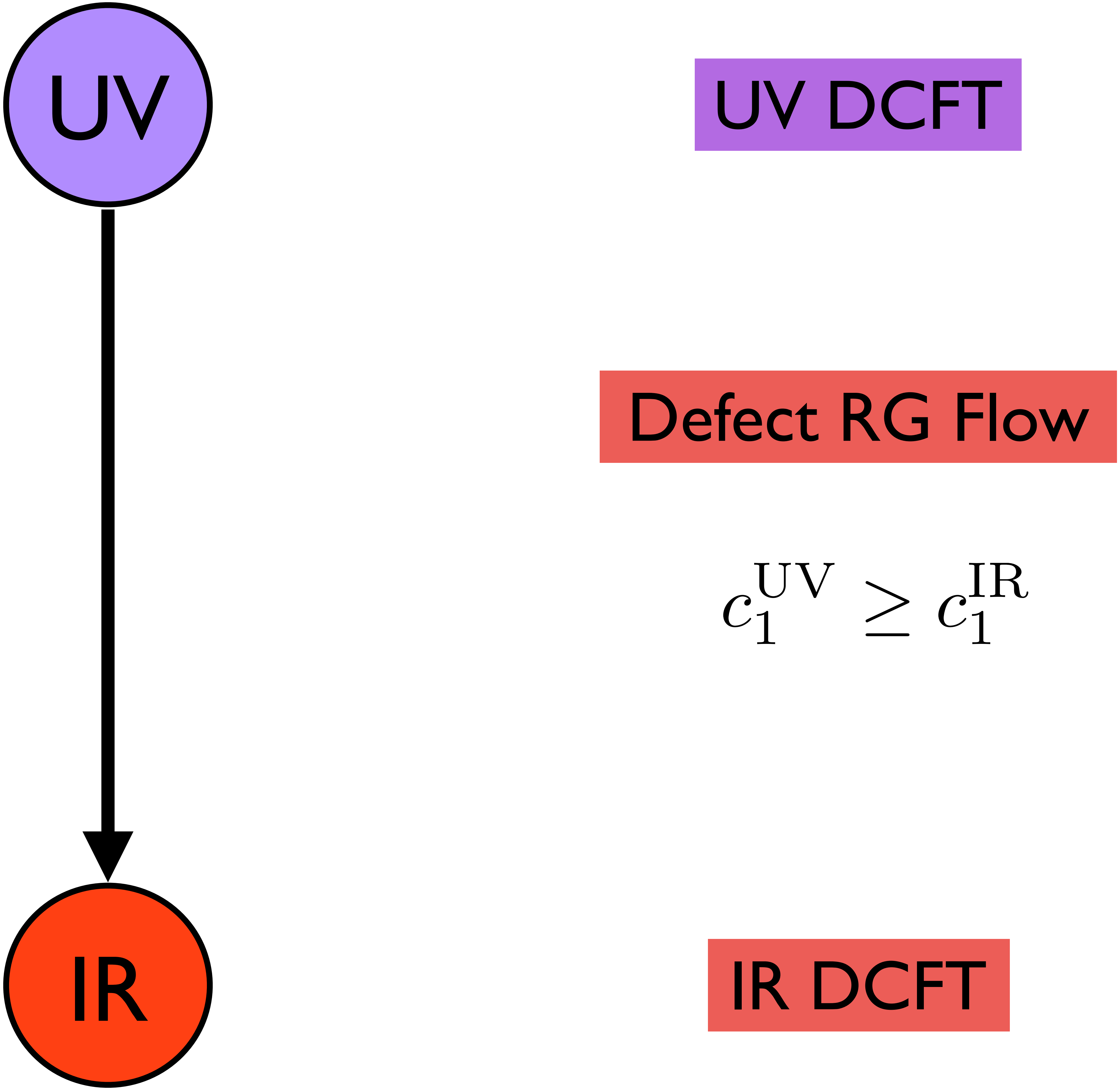
Defects

$$[T_\mu{}^\mu]_{\text{defect}} = c_1\,\hat{R} + c_2(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) + c_3\hat{g}^{ac}\hat{g}^{bd}W_{abcd}$$

Boundary “central charges”

c_1 c_2 c_3

$\hat{g}^{ac}\hat{g}^{bd}W_{abcd}$ is B-type


UV
IR
UV DCFT
Defect RG Flow
$c_1^{\mathrm{UV}} \geq c_1^{\mathrm{IR}}$
IR DCFT

Outline:

- Review: Monotonicity Theorems
- The Systems We Study
- The Trace Anomaly
- The Proof
- Summary and Outlook

Summary

Local, unitary BCFT in $d = 3$

Local, unitary DCFT in $d \geq 3$
with two-dimensional defect

$$[T_\mu{}^\mu]_{\text{defect}} = c_1 \hat{R} + c_2 (K^\mu_{ab} K^{ab}_\mu - \frac{1}{2} K^\mu K_\mu) + c_3 \hat{g}^{ac} \hat{g}^{bd} W_{abcd}$$

Boundary or Defect RG Flows

$$c_1^{\text{UV}} \geq c_1^{\text{IR}}$$

Summary

$$c_1^{\mathrm{UV}} \geq c_1^{\mathrm{IR}}$$

Higher-dimensional g-theorem

Generalization of the weak form
of Zamolodchikov's c-theorem
to include coupling to higher-dimensional CFT

Proof used only existing ingredients!

Outlook

Immediate questions

Can we define a c_1-function?

Is c_1 bounded below?

Other methods of proof?

What about EE? Or holography?

Examples

Graphene with a boundary

Critical Ising model in $d = 3$ with a boundary

M-theory: M2-branes with a boundary

String theory: various brane intersections

Holographic BCFTs

Outlook

Prove more boundary/defect monotonicity theorems

Yamaguchi	hep-th/0207171
Estes, Jensen, O'B., Tsatis, Wrase	1403.6475
Gaiotto	1403.8052

Find a "universal" proof of monotonicity theorems?

Myers and Sinha	Giombi and Klebanov
1006.1263, 1011.5819	1409.1937

Do monotonicity theorems always survive coupling to a higher-dimensional CFT?

Thank You.