Scattering in large Susy large N Chern Simons Theory

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References

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- Last three years; fair amount of work on $U(N)$ Chern Simons theories with fundamental matter.
- Initial motivations. Vasiliev dualities. Limit of ABJ theories.
- Theories turn out to be solvable in the large N limit. Effectively vector like (Chern Simons almost smoke and mirrors).
- Intensive study has revealed many interesting properties. First strong weak (Bose-Fermi) duality even without susy. Second, many unusual aspects to dynamics.

- In particular: recent intensive study of 2×2 scattering in purely bosonic and purely fermionic theories.
- Results in perfect agreement with duality. However had some unusual structural features. Explained by conjectures motivated by intuitive arguments.
- This talk: exact scattering in most general susy $N = 1$ Chern Simons theory.
- Two principal motivations. First to test the universal conjectures in a different setting. Second to move towards contact with the extensive work on ABJ scattering. Third to make precise the statement that Giveon Kutasov type dualities are Bose Fermi dualities.

Theories

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$$S = -\int d^3x d^2\theta \left[\frac{\kappa}{2\pi} Tr \left(-\frac{1}{4} D_\alpha \Gamma^\beta D_\beta \Gamma^\alpha - \frac{1}{6} D^\alpha \Gamma^\beta \{\Gamma_\alpha, \Gamma_\beta\} \right. \right.$$

$$(1)$$

$$\left. -\frac{1}{24} \{\Gamma^\alpha, \Gamma^\beta\} \{\Gamma_\alpha, \Gamma_\beta\} \right)$$

$$-\frac{1}{2}(D^\alpha \bar{\Phi} + i\bar{\Phi}\Gamma^\alpha)(D_\alpha \Phi - i\Gamma_\alpha \Phi) + m_0 \bar{\Phi}\Phi$$

$$(2)$$

$$\left. +\frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \right],$$

$$(3)$$

- Parameters: N, κ (integers). w (dimensionless). m_0 (dimensionful). New effectively continuous dimensionless parameter $\lambda = \frac{N}{\kappa}$ in the large N limit.

Conjectured Duality

- Motivated by computations of the thermal partition function on $S^2 \times S^1$, about two years ago Jain, Yokoyama and myself conjectured that this class of theories is self dual.

- Conjectured dualiy map as follows. Define

$$m(m_o, w) = \frac{2m_0}{2 + (-1 + w)\lambda \, \mathrm{Sgn(m)}} \, . \tag{4}$$

-
$$\lambda' = \lambda - \mathrm{Sgn}(\lambda) \, , \; w' = \frac{3 - w}{1 + w} \, , \, m' = -m \tag{5}$$

-
$$N' = |\kappa| - N + 1, \quad \kappa' = -\kappa$$

Generalization of level rank duality.

- Bose-Fermi or bosonization duality. Sounds interesting. Would like to understand in more detail. Best case: bosonisation formula like $\psi = e^\phi$. Too much to ask for in current context as ψ and ϕ are not gauge invariant.

- While arbitrary insertions of ψ and ϕ are not meaningful, insertions that are taken to infinity along lines of equations of motion - i.e. S matrices - are well defined. For this reason the map between S matrices is a 'poor mans bosonization' map between these two theories.

- This talk: We indpependently compute the S matrix for the bosons and fermions, and examine their interrelationship under duality. Also uncover interesting structural features along the way.

- Solution

$$\Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2} \bigg[\bigg(a(\mathbf{p})(1 + m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p}) \bigg) e^{ip.x}$$
$$+ \bigg(a^{c\dagger}(\mathbf{p})(1 + m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p}) \bigg) e^{-ip.x} \bigg]$$

- Susy

$$-iQ_\alpha = u_\alpha(\mathbf{p}_i) \left(a\partial_\alpha + a^c \partial_{\alpha^c} \right) + u_\alpha^*(\mathbf{p}_i) \left(-\alpha\partial_a + \alpha^c \partial_{a^c} \right)$$
$$+ v_\alpha(\mathbf{p}_i) \left(a^\dagger \partial_\alpha^\dagger + (a^c)^\dagger \partial_{(\alpha^c)^\dagger} \right) + v_\alpha^*(\mathbf{p}_i) \left(\alpha^\dagger \partial_a^\dagger + (\alpha^c)^\dagger \partial_{(a^c)^\dagger} \right)$$

$$(6)$$

-

$$A_i(\mathbf{p}) = a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i$$
$$A_i^\dagger(\mathbf{p}) = a_i^\dagger(\mathbf{p}) + \theta_i \alpha_i^\dagger(\mathbf{p}) \tag{7}$$

-

$$[Q_\alpha, A_i(\mathbf{p}_i, \theta_i)] = Q_\alpha^1 A_i(\mathbf{p}_i, \theta_i)$$
$$[Q_\alpha, A_i^\dagger(\mathbf{p}_i, \theta_i)] = Q_\alpha^2 A_i^\dagger(\mathbf{p}_i, \theta_i) \tag{8}$$

-

$$Q_\beta^1 = i \left(-u_\beta(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial \theta} - v_\beta(\mathbf{p})\theta \right)$$

$$Q_\beta^2 = i \left(v_\beta(\mathbf{p}) \frac{\overrightarrow{\partial}}{\partial \theta} - u_\beta(\mathbf{p})\theta \right) \tag{9}$$

Dual Supersymmetry

- Under Duality $a \leftrightarrow \alpha$
- Action of dual susy on $A, A^{\dagger}$

$$
\begin{aligned}
Q_{\beta}^{1} &= i\left(-v_{\beta}(\mathbf{p}, -m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_{\beta}(\mathbf{p}, -m)\theta\right) \\
Q_{\beta}^{2} &= i\left(u_{\beta}(\mathbf{p}, -m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_{\beta}(\mathbf{p}, -m)\theta\right)
\end{aligned}
\tag{10}
$$

- Using

$$
u(m, p) = -v(-m, p), \quad v(m, p) = -u(m, p)
$$

we see that the dual and original susy generators are proportional to each other. So an S matrix invariant under original susy is automatically invariant under dual susy.

- Consider the S matrix

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)(2\pi)^3 \delta(p^1 + p^2 - p^3 - p^4) =$$
$$\langle 0 | A_4(\mathbf{p}_4, \theta_4) A_3(\mathbf{p}_3, \theta_3) U A_2^\dagger(\mathbf{p}_2, \theta_2) A_1^\dagger(\mathbf{p}_1, \theta_1) | 0 \rangle$$

(11)

- Invariance under susy implies

$$\left(\overrightarrow{Q}_\alpha^1(\mathbf{p}_1, \theta_1) + \overrightarrow{Q}_\alpha^1(\mathbf{p}_2, \theta_2) \right.$$
$$\left. + \overrightarrow{Q}_\alpha^2(\mathbf{p}_3, \theta_3) + \overrightarrow{Q}_\alpha^2(\mathbf{p}_4, \theta_4) \right) S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = 0$$

(12)

- We have explicitly solved (12); the solution is given by

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = f_1 + F_2\ \theta_1\theta_2\theta_3\theta_4 +$$

$$\left(\frac{1}{2} C_{12} f_1 - \frac{1}{2} C_{34}^* F_2 \right)\ \theta_1\theta_2$$

$$+ \left(\frac{1}{2} C_{13} f_1 - \frac{1}{2} C_{24}^* F_2 \right)\ \theta_1\theta_3 + \left(\frac{1}{2} C_{14} f_1 + \frac{1}{2} C_{23}^* F_2 \right)\ \theta_1\theta_4$$

$$+ \left(\frac{1}{2} C_{23} f_1 + \frac{1}{2} C_{14}^* F_2 \right)\ \theta_2\theta_3 + \left(\frac{1}{2} C_{24} f_1 - \frac{1}{2} C_{13}^* F_2 \right)\ \theta_2\theta_4$$

$$+ \left(\frac{1}{2} C_{34} f_1 - \frac{1}{2} C_{12}^* F_2 \right)\ \theta_3\theta_4$$

$$(13)$$

- Where

$$\frac{1}{2}C_{12} = -\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) \quad \frac{1}{2}C_{23} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3)$$

$$\frac{1}{2}C_{13} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) \quad \frac{1}{2}C_{24} = -\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4)$$

$$\frac{1}{2}C_{14} = -\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) \quad \frac{1}{2}C_{34} = -\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4)$$

$$(14)$$

- Note that susy determines 6 of the 8 scattering amplitudes in terms of the other two. Indpendent data = four boson and four fermion scattering.

- We refer to a quantum (bosonic or ferionic) that transforms in the fundamental of $U(N)$ as a particle and a quantum that tranforms in the antifundamental of $U(N)$ as an antiparticle.

- We study the most general 2×2 scattering process. There are three such processes

$$P_i + P_m \rightarrow P_j + P_n$$
$$P_i + A^j \rightarrow P_n + A^n \qquad (15)$$
$$A^j + A^n \rightarrow A^i + A^j$$

- All these S matrices are contained in appropriate on shell limits of the single correlator

$$< \phi_i \phi_m \bar{\phi}^j \bar{\phi}^n >$$

- It follows from $U(N)$ invariance that

$$< \phi_i \phi_m \bar{\phi}^j \bar{\phi}^n > = a\delta_i^j \delta_m^n + b\delta_i^n \delta_m^j$$

- Consequently, in each channel we need to compute 2 distinct scattering amplitudes. For particle-particle scattering it is convenient to work in the following basis for $U(N)$ singlets

$$< \phi_i \phi_m \bar{\phi}^j \bar{\phi}^n > = T_{sym} \left(\delta_i^j \delta_m^n + \delta_i^n \delta_m^j \right) + T_{as} \left(\delta_i^j \delta_m^n - \delta_i^n \delta_m^j \right)$$

$T_{Sym} = \frac{a+b}{2}$ is the S matrix in the channel of symmetric exchange, while $T_{as} = \frac{a-b}{2}$ is the S matrix in the channel of antisymmetric exchange.

- On the other hand for particle - antiparticle exchange the following basis is most useful

$$< \phi_i \phi_m \bar{\phi}^j \bar{\phi}^n > = T_{Sing} \frac{\delta_i^j \delta_m^n}{N} + T_{Adj} \left(\delta_i^n \delta_m^j - \frac{\delta_i^j \delta_m^n}{N} \right)$$

$T_{Sing} = Na + b$ is the scattering matrix in the singlet channel, while $T_{Adj} = b$ is the scattering matrix in the adjoint channel.

- It is not difficult to convince oneself that, at leading order in the $\frac{1}{N}$ expansion

$$a \sim b \sim T_{sym} \sim T_{as} \sim T_{Adj} \sim \frac{1}{N}$$

On the other hand

$$S_{Sing} \sim \mathcal{O}(1)$$

- We have chosen the colour structures multiplying T_{Sing} etc to be simply projectors onto the exchange representations. As projectors square to unity it follows from unitarity that

$$i(T_c - T_c^\dagger) = T_c T_c^\dagger$$

T_c is any one of the four T matrices described above. We have used the fact that e.g. $2 \rightarrow 4$ production contributes to unitarity only at subleading order in $\frac{1}{N}$.

- The unitarity equation is rather trivial for T_{sym}, T_{as}, T_{adj}, as the RHS is subleading in $\frac{1}{N}$. However it imposes a highly nontrivial nonlinear constraint on T_{sing}, the most nontrivial of the four scattering matrices.

- The S matrix for non relativistic particles interacting via Chern Simons exchange was worked out in the early 90s, most notably by Bak, Jackiw and collaborators.

- The main result is strikingly simple. Consider the scattering of two bosonic particles in representations R_1 and R_2, in exchange channel R. It was demonstrated that the S matrix equals the scattering matrix of a $U(1)$ charged particle of unit charge scattering off a point like flux tube of magnetic field strength $\nu = \frac{c_2(R_1) + c_2(R_2) - c_2(R)}{\kappa}$.

- This quantum mechanical S matrix was computed originally by Aharonov and Bohm and generalized by Bak and Camillio to take account of possible point like interactions between the scattering particles.

- $$T_{NR} = -16\pi i c_B \left(\cos\left(\pi\nu\right) - 1\right)\delta(\theta) + 8i c_B \sin(\pi\nu)\text{Pv}\left(\cot\frac{\theta}{2}\right)$$

$$+ 8c_B|\sin\pi\nu|\frac{1 + e^{i\pi|\nu|}\frac{A_{NR}}{k^{2|\nu|}}}{1 - e^{i\pi|\nu|}\frac{A_{NR}}{k^{2|\nu|}}},$$

$$A_{NR} = \frac{-1}{w}\left(\frac{2}{R}\right)^{2|\nu|}\frac{\Gamma(1 + |\nu|)}{\Gamma(1 - |\nu|)}.$$

$$(16)$$

- Here $wR^{|2\nu|}$ is a measure of the strength of the Bak-Camillio contact interaction between the scattering particles. In the limit $w(kR)^{|2\nu|} \to 0$ $A_{NR} \to \infty$ and the second line of the scattering amplitude simplifies to

$$8c_B|\sin\pi\nu|,$$

the original Aharonov Bohm result.

- The non relativistic amplitude above has a very unusual feature, a piece in the scattering amplitude proportional to the $\delta(\theta)$.

- This term in the scattering amplitude was missed in the original paper of Aharonov and Bohm. The amplitude was corrected with the addition of this piece in the early 80s. In the early 90s Jackiw and collabortors emphasized that is term is necessary to unitarize Aharonov Bohm scattering.

- Infact this term has a simple physical interpretation (white board). The physical interpretation makes no reference to the non relativistic limit, so we assume that the δ function piece is present unmodified even in relativistic scattering.

- In the large N limit there is a simple formula for the quadratic Casimir of representations with a finite number of boxes plus a finite number of antiboxes.

$$c_2(R) = \frac{N(n_b + n_a)}{2}$$

- using this result in the formula for ν we find

$$\nu_{sym} \sim \nu_{as} \sim \nu_{Adj} \sim \mathcal{O}(1/N)$$

On the other hand

$$\nu_{Sing}| = \lambda$$

- Now the non relativistic limit is obtained by taking $k \to 0$ at fixed ν. If $\nu \sim \mathcal{O}(1/N)$ and if we work to leading order in $\frac{1}{N}$ we effectively take $\nu \to 0$ first. In other words the results of Aharonov Bohm Bak Camillio yield a sharp prediction for the non relativistic limit only of T_{sing}.

- Work in Lorentzian space. Set gauge $\Gamma_- 0$. In particular implies $A_- = 0$. Supersymmetric generalization of lightcone gauge. Main advantage: No gauge boson self interactions.

- If we want to understand scattering we first have understand free propagation.

- No gauge self interactions plus planarity gives a simple Schwinger Dyson equation (like t' Hooft model). Nonlinear integral equation. Quite remarkably admits simple exact solution

- The bare scalar superfield propagator in momentum space is

$$\langle \bar{\Phi}(\theta_1, p)\Phi(\theta_2, -p')\rangle = \frac{D_{\theta_1,p}^2 - m_0}{p^2 + m_0^2}\delta^2(\theta_1 - \theta_2)(2\pi)^3\delta^3(p - p').$$

$$(17)$$

where m_0 is the bare mass

- The gauge superfield propagator in momentum space is

$$\langle \Gamma^-(\theta_1, p)\Gamma^-(\theta_2, -p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1 - \theta_2)}{p_{--}}(2\pi)^3\delta^3(p - p'),$$

$$(18)$$

-

Figure : Integral equation for self energy

$$\Sigma_{(p,\theta_1,\theta_2)} = \quad + \quad$$

-

$$
\begin{aligned}
\Sigma(p, \theta_1, \theta_2) = {} & 2\pi\lambda w \int \frac{d^3 r}{(2\pi)^3} \delta^2(\theta_1 - \theta_2) P(r, \theta_1, \theta_2) \\
& - 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} D_-^{\theta_2, -p} D_-^{\theta_1, p} \left(\frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} P(r, \theta_1, \theta_2) \right) \\
& + 2\pi\lambda \int \frac{d^3 r}{(2\pi)^3} \frac{\delta^2(\theta_1 - \theta_2)}{(p-r)_{--}} D_-^{\theta_1, r} D_-^{\theta_2, -r} P(r, \theta_1, \theta_2)
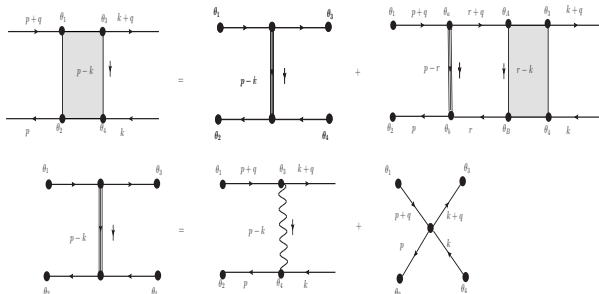\end{aligned}
$$

$$(19)$$

- The solution for the exact propagator is incredibly simple. It is given by

$$P(p, \theta_1, \theta_2) = \frac{D^2 - m}{p^2 + m^2} \delta^2(\theta_1 - \theta_2) \qquad (20)$$

-

$$m = \frac{2m_0}{2 + (-1 + w)\lambda \, \text{Sgn(m)}} \, . \qquad (21)$$

Figure : Four point function: Schwinger-Dyson equation for offshell four point function. The second line represents the tree level contributions from the gauge superfield interaction and the quartic interactions, The first line is the pictorial representation of the integral equation

- Now turn to evaluating the 4 point function needed for computing scattering. Linear integral equation. Shamefully unable to solve in general. Exact solution for the special case $q_{\pm} = 0$.

- Defect has different implications in different channels

- In the singlet exchange channel it turns out that q_μ is the centre of mass energy. If $q_{\pm} = 0$ then momentum centre of mass momentum is spacelike. Incompatable with putting onshell. Solution not useful for directly computing T_{sing}.

- In the other three channels q_μ is the momentum transfer. In the scattering of two particles of equal mass, the momentum exchange is always spacelike. Setting $q_{\pm} = 0$ is simply a choice of Lorentz frame. Assuming the answer is covariant, there is no loss of information. Can read off full results for T_{Adj}, T_{sym}, T_{as}.

- Evaluating the onffshell 4 pt function subject to this restriction and taking the onshell limit we find

-

$$T_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + J_B(|q_3|,\lambda) , \quad (22)$$

$$T_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + +J_F(|q_3|,\lambda) , \quad (23)$$

where the J functions are

$$J_B(|q|,\lambda) = \frac{4\pi|q|}{\kappa}\frac{N_1 N_2 + M_1}{D_1 D_2}$$

$$J_F(|q|,\lambda) = \frac{4\pi|q|}{\kappa}\frac{N_1 N_2 + M_2}{D_1 D_2} \quad (24)$$

$$N_1 = \left(\left(\frac{2m + iq}{2m - iq} \right)^{-\lambda} (w - 1)(2m + iq) + (w - 1)(2m - iq) \right)$$

$$N_2 = \left(\left(\frac{2m + iq}{2m - iq} \right)^{-\lambda} (q(w + 3) + 2im(w - 1)) + (q(w + 3) \right.$$

$$\left. - 2im(w - 1)) \right) \tag{25}$$

$$M_1 = -8mq((w + 3)(w - 1) - 4w) \left(\frac{2m + iq}{2m - iq} \right)^{-\lambda}$$

$$M_2 = -8mq(1 + w)^2 \left(\frac{2m + iq}{2m - iq} \right)^{-\lambda}$$

$$D_1 = \left(i \left(\frac{2m + iq}{2m - iq} \right)^{-\lambda} (w - 1)(2m + iq) - 2im(w - 1) + q(w + 3) \right)$$

- It is not too difficult to verify that the S matricces above map to themselves under the duality map, if we also make the identifications

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$$T_{Adj}^B = T_{Adj}^F, \quad T_{sym}^B = T_{as}^F, \quad T_{as}^B = T_{sym}^F$$

This is the result we should have expected both from level rank duality as well as from basic statisitics.

- This matching is an impressive verification of the duality conjecture.

- The results in the adjoint, symmetric and antisymmetric channels may be rewritten in a way that makes it manivest that they transform into each under under crossing symmetry. Though we do not have an explicit computation of T_{sing}, according to standard QFT lore one should be able to obtaine it from the naive analytic continuation.

- Performing the naive analytic continuation gives a result that cannot be right. To start with it does not have the delta function piece that we know must be there on physical grounds. Indeed no analytic continuation can give this piece as it is not analytic.

- Clearly the rules for crossing symmetry must be modified. In order to resolve a similar paradox in the study of scattering in pure boson and pure fermion theories, we had proposed that

$$T_{sing} = \frac{\sin(\pi\lambda_B)}{\pi\lambda_B} T_{sing}^{ac} - i(\cos(\pi\lambda_B) - 1)I(p_1, p_2, p_3, p_4).$$

where T_S^{trial} is the singlet amplitude one obtains from naive analytic continuation

- We had also proposed a physical interpretation for this formula. Review. Since this physical interpretation is very general, it should also apply to the susy case. So we are forced to make the same conjecture in the susy case as well. We have no wiggle room. Either T_{sing} defined passes all consistency tests or it does not.

$$N_1 = (w-1)(2m+\sqrt{s}) + (w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda$$

$$N_2 = (-i\sqrt{s}(w+3) + 2im(w-1)) + (-i\sqrt{s}(w+3) \quad (27)$$
$$- 2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda$$

$$M_1 = 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda$$

$$M_2 = 8mi\sqrt{s}(1+w)^2 e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda$$

$$(28)$$

-

$$D_1 = i(w-1)(2m+\sqrt{s}) - (2im(w-1)$$
$$\quad + i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda$$
$$D_2 = (\sqrt{s}(w+3) - 2im(w-1))$$
$$\quad + (w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)^\lambda \quad (29)$$

In order to check supersymmetry, it is useful to define the product of two S matrix superfields.

$$S_1 \star S_2 \equiv \int d\Gamma \, S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2)$$

$$\exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2)$$

$$S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) =$$

$$\exp(\theta_1\theta_3 + \theta_2\theta_4) 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)$$

$$(30)$$

where the measure $d\Gamma$ is

$$d\Gamma = \frac{d^2 k_3}{2k_3^0(2\pi)^2} \frac{d^2 k_4}{2k_4^0(2\pi)^2} \frac{d^2 k_1}{2k_1^0(2\pi)^2} \frac{d^2 k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4$$

$$(31)$$

- Define

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) = \exp(\theta_1\theta_3 + \theta_2\theta_4)$$
$$2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) \tag{32}$$

- It is easy to check that

$$S \star I = I \star S = S \tag{33}$$

And I is supersymmetric.

- Also define

$$S^\dagger(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2)$$

$$\tag{34}$$

If S is susy can show $S^\dagger$ is susy

- With these definitions, unitarity is the equation

$$S \star S^{\dagger} - I = 0$$

- Since the LHS of this equation is supersymmetric, in order to establish it is sufficient to show that the θ^0 and the θ^4 tersm vanish.
- These two conditions are easily written in terms of F and f.

$$\int d\Gamma' \bigg[f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) f_1^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)$$

$$- Y(\mathbf{p}_3, \mathbf{p}_4) \bigg(f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_1, \mathbf{p}_2) f_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \bigg)$$

$$\bigg(f_1^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_3, \mathbf{p}_4) f_2^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \bigg) \bigg]$$

$$= 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)$$

$$\int d\Gamma' \bigg[-16 Y^2(\mathbf{p}_3, \mathbf{p}_4) f_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) f_2^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)$$

$$+ Y(\mathbf{p}_3, \mathbf{p}_4) \bigg(f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_1, \mathbf{p}_2) f_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) \bigg)$$

$$\bigg(f_1^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_3, \mathbf{p}_4) f_2^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \bigg) \bigg] =$$

$$- 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4).$$

Very complicated relation. Does not work at all for naive analytic continuation. However relation obeyed exactly for S matric obtained out of the modified crossing relations.

- The conjecture is also consistent with duality.
- It also has the correct nonrelativistic limit, as we explain below.

- The straight forward non relativistic limit is taken by taking $\sqrt{s} \to 2m$. In this limit $F = \text{sgn}(\lambda_B)$ and

$$T_{sing} = 8\pi i \sqrt{s}(1 - \cos(\pi \lambda_B))\delta(\theta) + 4i\sqrt{s}\sin(\pi \lambda_B)\text{Pv}\left(\cot\left(\frac{\theta}{2}\right)\right)$$
$$+ 4\sqrt{s}|\sin(\pi|\lambda_B|)|$$

$$(35)$$

- This is in perfect agreement with the Aharonov Bohm result. Natural quesiton: is there any way to get the Bak-Camillio modification out of our formula? Ans: yes! There is a second, more sophisiticated non relativistic limit one can take.

- Can show that the S matrix (both 4 boson and 4 fermions) have a pole for $w \leq -1$.
- For $w = -1 - \epsilon$ the pole is very near threshhold
- As w is decreased the mass of the pole decreases from $2m$ downto zero. The pole mass hits zero at

$$w_c = -\frac{2}{\lambda} + 1 \tag{36}$$

- As w is decreased further the pole mass rises again, and approaches $2m$ as $w \to -\infty$.
- The appearance of a massless bound state in our aparently massive theory is fascinating. Would be nice to understand further.

- Consider the theory at $w = -1 - \delta w$ for δw small. It turns out that $E_B \sim (\delta w)^{1/|\lambda_B|})$ at small δw.

- This observation motivates study of the scaled non relativistic limit

$$\frac{\delta w}{m} \to 0, \quad \frac{k}{m} \to 0, \quad \frac{k}{m} \left(\frac{m}{\delta w} \right)^{\frac{1}{2|\lambda_B|}} = \text{ fixed.} \qquad (37)$$

- In this limit the 4 boson scattering amplitude may be shown to agree precisely with the Bak-Camillio form with the identification

$$A_{NR} = \frac{\delta w}{2} (2m)^{2\lambda}. \qquad (38)$$

- Interesting observation: $\frac{\sin \pi \lambda}{\pi \lambda}$ equals Witten's result for the expectation value of a circular Wilson loop on S^3 in pure Chern Simons theory in the large N limit.

- This observation suggests a possible explanation of the modified crossing symmetry rules (details on white board)

- If this is right it is the tip of the iceberg. Finite N and κ.

- Let us accept the conjecture for the moment. How did fermions turn into bosons? Different answers in different channels.

- Adjoint channel. Boring. No phase, no statistics. Boring.

- Sym and as channel. No phase but statistics. Level rank duality on Young Tableax of exchange representations

- Singlet channel. Most interesting. Anyonic phase. Neither bosons nor fermions. $e^{i\pi\lambda_B} = e^{i\pi\lambda_F}e^{-i\pi\mathrm{sgn}\lambda_F}$ implies $\lambda_B = \lambda_F - \mathrm{sgn}\lambda_F$. Equating anyonic phases in the singlet channel gives a derivation of the anyonic phase!

- Presented computations and conjectures for all orders scattering matrices in Chern Simons matter theories.
- Results in perfect agreement with duality
- Suggest novel transformations of S matrices under crossing symmetry. Current conjectures are large N results. Study of crossing at finite N and k a very interesting problem.
- Generalizations to susy theories underway. Should evenually make contact with scattering in ABJM theories.