(Entanglement)Entropy in 3d Higher Spin Theories

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2d CFT

AdS3 gravity

SL(2,R)xSL(2,R) Chern-Simons theory

Many universal results (Cardy formula, entanglement entropy,….)

2d CFT with extended symmetries

AdS3 higher-spin gravity

SL(N,R)xSL(N,R) Chern-Simons theory

Many universal results????? (Cardy formula, entanglement entropy,….)

Gaberdiel and Gopakumar proposed that the coset theory

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$$

is dual to Vasiliev higher spin theory with gauge group

$$hs[\lambda] \qquad \text{with} \qquad \lambda = \frac{N}{N+k}$$

For suitable choices of parameters the higher spin theory truncates to a finite number of higher spin fields and contains an SL(N,R)xSL(N,R) Chern-Simons theory.

Cardy formula:

$$S = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24})} + 2\pi\sqrt{\frac{c}{6}(\bar{L}_0 - \frac{c}{24})}$$

Finite temperature entanglement entropy:

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right)$$

a: UV cutoff

Universal results apply to all 2d CFTs.

To get something new, we want to consider deformations of 2d CFTs by higher spin operators

$$S = S_{\rm CFT} + \sum_s \int d^2z \mu_s(z,\bar z) W_s(z,\bar z) + \sum_s \int d^2z \bar\mu_s(z,\bar z) \bar W_s(z,\bar z)$$

The coefficients $\mu_2, \bar\mu_2$ couple to the stress tensor and simply correspond to deforming the metric of the 2d space on which the CFT lives.

Such theories have issues:

- Deformations are irrelevant. Not clear theory is well-defined.
- Can treat theory perturbatively in $\mu_s, \bar{\mu}_s$. Because currents are conserved we do not run into any apparent problems when doing so. This is perhaps the only meaningful definition of such theories.
- It is possible that all these theories have natural UV completions.(Ammon, Gutperle, Kraus, Perlmutter; Ferlaino, Hollowood, Prem Kumar)
- Restricting to zero modes gives generalized partition functions. Similar to `generalized Gibbs ensembles' used in integrable systems.

$$Z = \left\langle e^{2\pi i\tau + \sum_s \mu_s W_s^0 - 2\pi i\bar{\tau} + \sum_s \bar{\mu}_s \bar{W}_s^0} \right\rangle$$

It is also possible that such theories are perfectly OK and in fact still have an exact W-symmetry (Compere, Song)

These partition functions are supposedly dual to higher spin black holes. More evidence later.

3d gravity vs CS theory

$$S = \frac{k}{4\pi}\int \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) - \frac{k}{4\pi}\int \mathrm{Tr}(\bar{A}\wedge d\bar{A} + \frac{2}{3}\bar{A}\wedge \bar{A}\wedge \bar{A})$$

Relation to 3d gravity

$$A = \omega + e, \qquad \bar{A} = \omega - e$$

Obtain Einstein-Hilbert action with negative cosmological constant in first order form.

For AdS application, boundary conditions are very important.

Consider only A from now on. Pick coordinates x^+, x^-, ρ where ρ is the radial direction.

Define $b = \mathrm{Exp}\left[\rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$

Then the AdS boundary conditions are

$$A \longrightarrow b^{-1} a b + b^{-1} db \text{ as } \rho \to \infty$$

where a is up to corrections of order $e^{-\rho}$ equal to the following flat 2d connection

$$a = \left(\begin{array}{cc} 0 & 1 \\ T_{++} & 0 \end{array} \right) dx^+ + \left(\begin{array}{cc} \partial_+\mu/2 & \mu \\ \mu T_{++} - \partial_+^2\mu/2 & -\partial_+\mu/2 \end{array} \right) dx^-$$

normalizable

non-normalizable

Flatness is equivalent to

$$\partial_- T_{++} = 2T_{++}\partial_+\mu + \partial_+ T_{++}\mu - \partial_+^3\mu/2$$

which is simply expressing conservation of the stress tensor in the presence of an interaction $\int d^2x \mu T_{++}$

Gauge transformations that preserve this form of the connection have infinitesimal gauge parameter

$$\begin{pmatrix} \partial_+\epsilon/2 & \epsilon \\ \epsilon T_{++} - \partial_+^2\epsilon/2 & -\partial_+\epsilon/2 \end{pmatrix}$$

and change the stress tensor to

$$\delta T_{++} = 2T_{++}\partial_+\epsilon + \partial_+ T_{++}\epsilon + \partial_+^3\epsilon/2$$

which is exactly the correct behavior under diffeomorphisms.

It is important that we include suitable boundary terms so that the variation of the action looks like

$$\delta S = \int \text{normalizable } \delta(\text{non} - \text{normalizable})$$

In this way, normalizable modes can fluctuate while non-normalizable ones do not.

Will come back to these boundary terms in a moment.

Generalization to higher spin theories uses ideas from so-called Drinfeld-Sokolov reduction.

Take any SL(2) embedding in SL(N). These are classified by the way the fundamental representation decomposes in SL(2) representations. If N is irreducible this called the principal embedding.

These give rise to the “standard” W-algebras.

Denote SL(2) generators by $\{\Lambda^-, \Lambda^0, \Lambda^+\}$

$$[\Lambda^0, \Lambda^\pm] = \pm\Lambda^\pm, \qquad [\Lambda^+, \Lambda^-] = 2\Lambda^0.$$

Then

$$a = (\Lambda^+ + \mathcal{W})dx^+ + (\mathcal{M} + \ldots)dx^-$$

$[\Lambda^-, \mathcal{W}] = 0, \quad \mathcal{W}:$ normalizable modes

$[\Lambda^+, \mathcal{M}] = 0, \quad \mathcal{M}:$ non-normalizable modes

Example for SL(3) principal embedding:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ T & 0 & 1 \\ W & T & 0 \end{pmatrix} dx^+ + \begin{pmatrix} * & \mu + * & \nu \\ * & * & \mu - * \\ * & * & * \end{pmatrix} dx^-$$

Flatness: Conservation of currents in the presence of sources (aka Ward identities) $\int d^2x \sum_i \mu_i(x) W_i(x)$

Gauge transformations that preserve the form of a: non-linear classical W-algebra. These form the asymptotic symmetry group of the system.

Campoleoni et al; Henneaux, Rey; work in 90's

So SL(N)xSL(N) Chern-Simons theory with a suitable boundary term and with the above boundary conditions describes the universal sector of CFTs with higher spin symmetries.

$$c = 12k\mathrm{Tr}(\Lambda^0\Lambda^0)$$

The relevant boundary term to add is

$$S_{\rm bdy} \sim \frac{k}{2\pi} \int d^2x \mathrm{Tr} \left[(a_+ - 2\Lambda^+) a_- \right]$$

With this boundary term the variation becomes

$$\delta(S_{\rm CS} + S_{\rm bdy}) \sim \frac{k}{\pi} \int d^2x \mathrm{Tr} \left[(a_+ - \Lambda^+) \delta a_- \right]$$

which indeed has the right form since

$$a = (\Lambda^+ + \mathcal{W}) dx^+ + (\mathcal{M} + \ldots) dx^-$$

Important subtlety:

The parameter μ corresponds to turning on a non-trivial metric in the boundary theory. (“Beltrami differential”)

Instead of putting it in the gauge field one can also put it in the choice of modular parameter of the boundary torus in the Euclidean case. Then connection to temperature is manifest.

One can use either formulation but there are technical differences in choices of boundary terms etc. The standard choice is to not include μ but to use the modular parameter of the torus instead.

We want to consider systems at finite temperature/finite chemical potentials for the higher spin fields.

Idea: Euclidean signature, impose regularity for the gauge field along the contractible time circle: trivial monodromy.

$$\mathrm{spec}(2\pi(\tau a_+ - \bar{\tau} a_-)) = \mathrm{spec}(2\pi i \Lambda^0)$$

Gutperle, Kraus

The rhs follows by insisting that when we turn of all charges we recover BTZ.

There may be other branches but will ignore this.

David, Ferlaino, Prem Kumar

For example, for SL(2):

$$\text{spec } 2\pi\tau \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix} = \text{spec } 2\pi i \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\Longrightarrow T \sim L_0 \sim \tfrac{1}{\tau^2}$$

The entropy follows from the on-shell action through a Legendre transform.

cf Banados, Canto, Theisen

We have to be careful to identify the variable conjugate to the modular parameter of the torus.

With our boundary term we obtain

$$\delta I_{os}^{(E)} = -\frac{ik_{cs}}{2\pi}\int_{\partial M}\frac{d^2 z}{\mathrm{Im}(\tau)}\mathrm{Tr}\left[\left(\frac{a_z^2}{2}+a_z a_{\bar z}-\frac{\bar a_z^2}{2}\right)\delta\tau + \left(a_z-\Lambda^+\right)\delta\Big(\left(\bar\tau-\tau\right)a_{\bar z}\Big)\right.$$
$$\left. -\left(-\bar a_{\bar z}+\Lambda^-\right)\delta\Big(\left(\bar\tau-\tau\right)\bar a_z\Big) - \left(\frac{\bar a_{\bar z}^2}{2}+\bar a_{\bar z}\bar a_z-\frac{a_{\bar z}^2}{2}\right)\delta\bar\tau\right]$$

stress tensor

higher spin charges

higher spin sources

Perez, Tempo, Troncoso

After a Legendre transform we finally obtain for the entropy

$$\boxed{S = -2\pi i k \mathrm{Tr}\left[(a_z + a_{\bar{z}})(\tau a_z + \bar{\tau} a_{\bar{z}}) - (\bar{a}_z + \bar{a}_{\bar{z}})(\tau \bar{z}_z + \bar{\tau} \bar{a}_{\bar{z}})\right]}$$

One can simultaneously diagonalize all connections and the entropy then becomes a very simple fixed linear combination of the eigenvalues of $a_z + a_{\bar{z}}$ and $\bar{a}_z + \bar{a}_{\bar{z}}$

This formalism plus choices of conjugate variables (and no obvious holomorphic factorization) is sometimes referred to as the canonical formalism.

One can also start with a square torus and work with the parameter μ instead. In this case we think we understand the connection to CFT partition functions at every step and we find that the entropy becomes

JdB, Goeree

$$S = -2\pi i k \mathrm{Tr}\left[a_z(\tau a_z + \bar{\tau} a_{\bar{z}}) - \bar{a}_{\bar{z}}(\tau \bar{z}_z + \bar{\tau}\bar{a}_{\bar{z}})\right]$$

This expression (not in this form) was for spin-three black holes compared to explicit CFT computations and one finds perfect agreement.

Kraus, Perlmutter;
Gaberdiel, Hartman, Yin

By diagonalizing, for spin-three, the entropy is e.g. the difference between the largest and smallest eigenvalue of:

$$\begin{pmatrix} 0 & 1 & 0 \\ T & 0 & 1 \\ W & T & 0 \end{pmatrix}$$

$$S = -2\pi i k \mathrm{Tr}\left[(a_z + a_{\bar z})(\tau a_z + \bar\tau a_{\bar z}) - (\bar a_z + \bar a_{\bar z})(\tau \bar z_z + \bar\tau \bar a_{\bar z})\right]$$

CFT computation unclear, but natural from CS point of view

summary

$$Z = \left\langle e^{2\pi i \tau L_0 - \sum_i \mu_i W_i - 2\pi i \bar L_0 - \sum_i \bar\mu_i \bar W_i} \right\rangle$$

$$S = -2\pi i k \mathrm{Tr}\left[a_z(\tau a_z + \bar\tau a_{\bar z}) - \bar a_{\bar z}(\tau \bar z_z + \bar\tau \bar a_{\bar z})\right]$$

Entanglement entropy?

Starting observation: geodesic distance in AdS3 can be written as

$$\cosh d(P,Q) = \frac{1}{2}\mathrm{Tr}\left[P\exp\int_P^Q \bar{A}\ \ P\exp\int_Q^P A\right]$$

Not gauge invariant?? Not a problem.

Since entanglement entropy in AdS3 is related to the geodesic distance (Ryu Takayangi) this motivates us to look for an expression in terms of Wilson lines.

Further motivation:

1) Bulk theory is topological so it is reasonable to look for topological quantities

2) Entanglement entropy related to two-point function of twist fields. In first quantized form such a two-point function involves the action of a point particle coupled to the gauge field. Except that there is no propagating point particle – all that is left is the coupling to a gauge field i.e. a Wilson line. (one might expect infinite dimensional representations, cf Castro-Iqbal)

3) In higher spin theory equation for the master field

$$dC + AC - C\bar{A} = 0$$

Kraus, Perlmutter

Proposal:

$$S \equiv k \log \left[\lim_{\rho_0 \to \infty} W(P,Q)\big|_{\rho_P=\rho_Q=\rho_0} \right]$$

$$W(P,Q) = \mathrm{Tr}_{R_\Lambda} \left[P \exp \int_P^Q \bar{A} \;\; P \exp \int_Q^P A \right]$$

A special representation is used: in the Weyl orbit of Λ^0 a unique principal highest weight appears. R is the corresponding highest weight representation. If half-integer spins are present, we need to look at the Weyl orbit of $2\Lambda^0$ instead.

Result agrees with that of Ammon-Castro-Iqbal in cases where one can check this explicitly.

Test 1: reproduce standard AdS3 results

black hole: $$S_{BTZ} = \frac{c}{6} \log \left\{ \frac{\beta_+ \beta_-}{\pi^2 a^2} \sinh\left[\frac{\ell}{\beta_+} \pi \left(\varphi_P - \varphi_Q \right) \right] \sinh\left[\frac{\ell}{\beta_-} \pi \left(\varphi_P - \varphi_Q \right) \right] \right\}$$

global: $$S_{AdS_3} = \frac{c}{3} \log \left[\frac{\ell}{a} \sin \left(\frac{\varphi_P - \varphi_Q}{2} \right) \right]$$

Poincaré: $$S_{AdS_3} = \frac{c}{3} \log \left[\frac{x_P - x_Q}{a} \right]$$

works OK! The special representation R is precisely such that the prefactors c/6 and c/3 automatically appear.

Test 2: reproduce thermal entropy of higher spin black holes

To do this, we can do two things. We can either take the boundary to be non-compact and find the extensive contribution to the entanglement entropy in the limit where P and Q become very widely separated from each other.
Result: reproduce exactly canonical form of thermal entropy.
One particular weight in R dominates in this limit.

Alternatively, can take the boundary to be a circle and let Q go around the circle while keeping P fixed. The entanglement entropy is then given by a closed loop around the horizon. It is not immediately obvious how to generalize our ansatz (with P and Q on the boundary) to this case.

Test 3: strong subadditivity

$$S_{EE}(A) + S_{EE}(B) \geq S_{EE}(A \cup B) + S_{EE}(A \cap B)$$

We have verified numerically that our result for the entanglement entropy for W3

$$S = \frac{2k}{m_{Adj}} \log\left[\frac{8e^{4\rho_0}}{\lambda_1\lambda_2\lambda_3}\right] + \frac{2k}{m_{Adj}} \log\left[\frac{\lambda_1^2 - \lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} + \frac{\cosh(\lambda_1\Delta\varphi)}{\lambda_1} - \frac{\cosh(\lambda_2\Delta\varphi)}{\lambda_2} - \frac{\cosh(\lambda_3\Delta\varphi)}{\lambda_3}\right]$$

indeed obeys strong subadditivity. We do not yet have a general proof. Caveat: for certain values of the chemical potentials there are discontinuities in the entanglement entropy?

There are also some issues when we apply our formalism to RG flows.

For a simple RG flow, we e.g. obtained

$$S = \frac{c_{\mathrm{UV}}}{3} \log \left[\frac{\Delta x}{a_{\mathrm{UV}}} \sqrt{|1 - \lambda^2 \Delta x^4|} \right]$$

which breaks down at some point. Perhaps we need to redefine the cutoff to get the right IR answer?

Take home message:

There appear to be generalizations of the Cardy formula and the expression for entanglement entropy for theories with higher spin charges.

It is not clear whether they are universal for all CFTs (what replaces modular invariance?) or only for CFTs with a gravitational dual.

Everything depends in a relatively simple way on the eigenvalues of the DS matrix (which contains all gauge invariant information about the higher spin black hole)

$$\begin{pmatrix} 0 & 1 & 0 \\ T & 0 & 1 \\ W & T & 0 \end{pmatrix}$$

Conclusions/outlook

- Understand relation between canonical formalism and the dual CFT
- Explore other solutions where monodromy is in the center of the gauge group, connection to conical defects.
- Look at supersymmetric extensions – presumably everything remains the same with groups replaced by supergroups
- Prove strong subadditivity in general and fix caveat.
- Find correct expression for a closed loop in the bulk.
- Any first principles derivation that does not use AdS/CFT?
- Give a direct proof of our expression for the entanglement entropy using the replica trick.
- Consider modifications of the form

$$W(P,Q) = \mathrm{Tr}_{R_\Lambda}\left[M_Q \, P \exp \int_P^Q \bar{A} \;\; M_P \;\; P \exp \int_Q^P A \right]$$

- Is there a generalization of modular invariance to higher spin systems?
- Is there an interesting underlying geometry?
- Are there universal results in higher spin CFTs which can somehow be explained? Can one e.g. show that a single conformal block dominates computations? (cf Hartman)