SWANSEA
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Integrability and DLCQ

Work in progress
(with P. Zhao)

Integrability of planar
$\mathcal{N}=4$ SUSY Yang-Mills

"bootstrap"

exact results for generic observables: ie non-BPS

operator dimensions
Wilson loops
Gluon "S-matrix"
Correlation functions

Origin/extent of integrability still mysterious .....

This talk;

- Brief review
- New "first-principles" approach based on,

Discrete

Light

Cone

Quantization

a.k.a. M(atrix) theory

BFSS

Aharony, Berkooz, Seiberg

Kapustin, Sethi

Hamiltonian system:

Integrability $\equiv$ Large hidden symmetry

$\Big\downarrow$ Bethe Ansatz

Exact spectrum

<u>Example</u>

1d particle mechaniss

$x_\ell, p_\ell \quad \ell = 1, \dots, k$

positions, momenta

$$H = \sum_\ell p_\ell^2/2 + \sum_{\ell > m} V(x_\ell - x_m)$$

integrable for <u>special</u> choice,

$V(x) = \quad 1/x^2$ rational

$\frac{1}{\sin^2 x} \quad 1/\sinh^2 x$ trig / hyperbolic

Wieirstraß function $\rightarrow \wp_2(x)$ elliptic

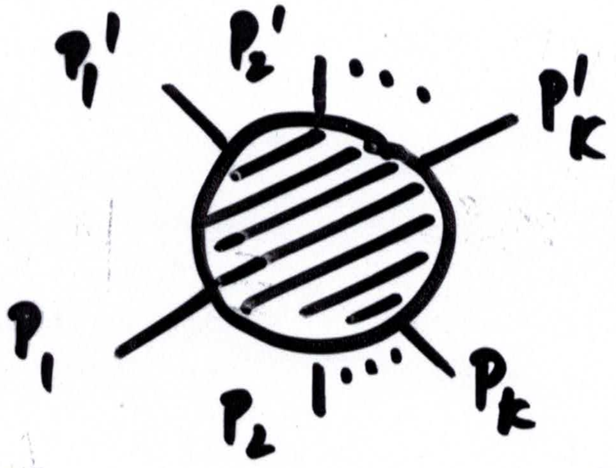
Non-compact case,

$$\hat{H} = \sum_{\ell} \hat{p}_{\ell}^{2}/2 + \sum_{\ell > m} \frac{\mu(\mu-1)}{\sinh^2(x_\ell - x_m)}$$

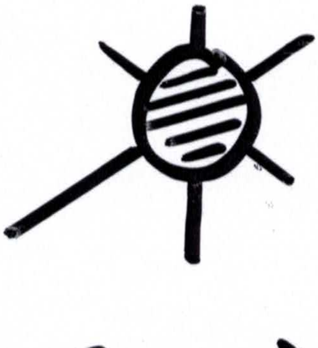
$$\hat{p}_\ell = -i\hbar\, \partial/\partial x_\ell$$

K-particle elastic scattering,

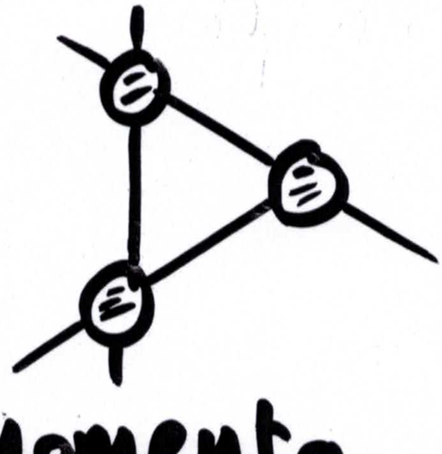
$S^{(K)} \simeq$



Integrability $\Leftrightarrow$ Factorization



$\equiv$



$\Leftrightarrow$ <u>Individual</u> momenta conserved

$$\{p_1', p_2', \ldots, p_K'\} = \{p_1, p_2, \ldots, p_K\}$$

Compact case

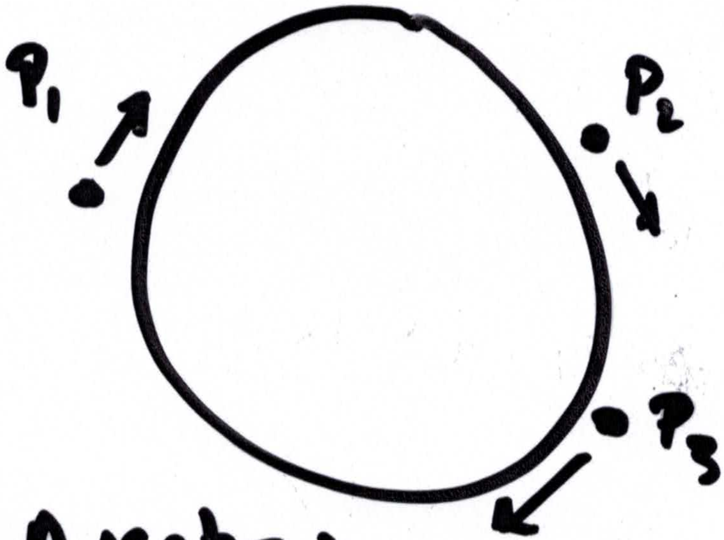
$$\hat{H} = \sum_{l} \hat{p}_l^2/2 + \sum_{l>m} \mu(\mu-1)\wp_2(X_l - X_m)$$

Large volume limit $\Im\tau \gg 1$

$$\wp_2(X) \simeq \frac{1}{\sinh^2(X)} + \cdots$$

$X \sim X + 2\pi L$

$L \sim m\tau$



Bethe Ansatz:

$$E = \sum_{l} p_l^2/2$$

p_l for $l = 1, \ldots, k$ solve,

$$e^{ip_l L} = \prod_{m \neq l} S^{(2)}(p_l - p_m)$$

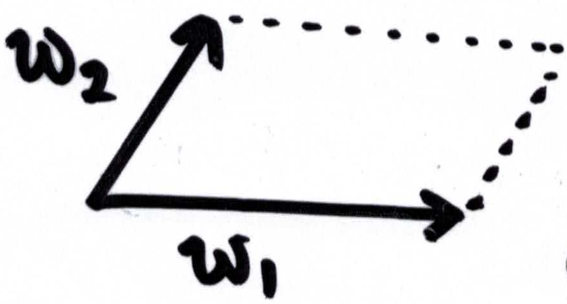
starting point

- (2,0) theory in $D=6$

  Type A_{N-1}

  - worldvolume theory on N M5 branes
  - strongly coupled SCFT

- Compactify on $T^2 \times \mathbb{R}^{3,1}$



complex structure: $\tau = w_2/w_1$

area: $A \sim \mathrm{Im}[\bar{w}_2 w_1]$

- Send $A \to 0$, τ fixed to get,

$$\boxed{\mathcal{N}=4 \text{ SUSY Yang-Mills on } \mathbb{R}^{3,1}}$$

$$G = SU(N), \quad \tau = \frac{4\pi i}{g^2_{YM}} + \theta/2\pi$$

DLCQ of (2,0) – Aharony, Berkooz, Seiberg

(2,0) on $\mathbb{R}^{5,1}$

- Compactify <u>null</u> direction

$$x_+ = x_0 + x_1 \sim x_+ + 2\pi R_+$$

- Conjugate momentum

– positive $p^- = E - p \geqslant 0$

– quantised $p^- = k/R^+ \quad k \in \mathbb{Z}^+$

- Sector of fixed $p^- = k/R^+$

– finite dimensional QM

– recover full theory in $k \to \infty$ limit.

ABS proposal

$(2,0)$ on $\mathbb{R}^4 \times \mathbb{R}_- \times S^1_+$ ← Null circle
with $P^- = k/R_+$ A_{N-1} theory

$\downarrow$

5d super-Yang-Mills $G = SU(N)$
with k instantons $g_5^2 \sim R_+$

$\downarrow$ low energy

SUSY QM on $\mathcal{M}_{k,N}$

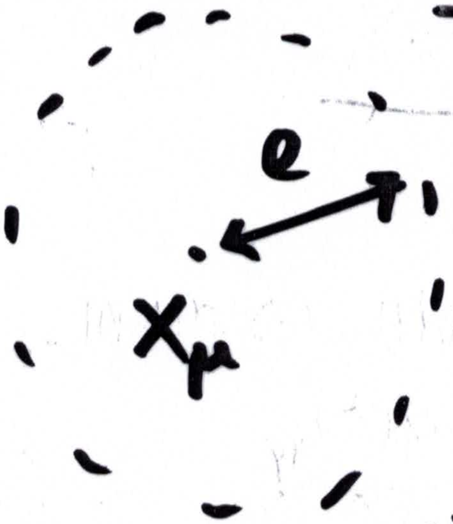
moduli space of k $SU(N)$ instantons → $\mathcal{M}_{k,N}$

Hamiltonian $P^+ = \Delta$

Laplacian on $\mathcal{M}_{k,N}$ → Δ

Superconformal QM

$K=1$



Collective Coordinate		#
position	$x_\mu \in \mathbb{R}^4$	4
size	$\rho \in \mathbb{R}^+$	1
orientation	$\hat{U} \in \dfrac{SU(N)}{SU(N-2)\times U(1)}$	4N-5
		4N

$K>1$

$$\dim[\mathcal{M}_{K,N}] = 4KN$$

ADHM construction

Realize $\mathcal{M}_{K,N}$ as Higgs branch of $U(K)$ gauge theory in $D=0+1$

D0-D4 system

matter content;

	$U(K)$	$U(N)$ (global)
$X, \tilde{X}$	adj	1
Q	K	N
$\tilde{Q}$	$\bar{K}$	$\bar{N}$

- Impose F-term constraint

$$[X, \tilde{X}] + Q\tilde{Q} = 0$$

- Mod out complexified gauge symmetry $G_{\mathbb{C}} = GL(K, \mathbb{C})$

$$\dim[\mathcal{M}_{K,N}] = [4K^2 + 4KN] - [2K^2 + 2K^2]$$
$$= 4KN \checkmark$$

Integrability and Instantons

$\mathcal{M}_{K,N}$ as complex, symplectic manifold

- **Big space**

$$\{ X, \tilde{X}, Q, \tilde{Q} \} \in \mathbb{C}^{2K^2 + 2KN}$$

symplectic form,

$$\omega = dX \wedge d\tilde{X} + dQ \wedge d\tilde{Q}$$

$\Leftrightarrow$ Poisson brackets

$$\{ X, \tilde{X} \} = 1, \quad \{ Q, \tilde{Q} \} = 1$$

Commuting "Hamiltonians"

$$H_\ell = \mathrm{Tr}[\tilde{X}^\ell] \qquad \ell = 1, \dots, K$$

$$\qquad i,j = 1, \dots, N$$

$$\tilde{H}_\ell = Q_i \tilde{X}^\ell \tilde{Q}_j$$

Realize $\mathcal{M}_{K,N}$ as symplectic quotient

- Solve F-term constraint

$$[X,\tilde{X}]_{\ell m} + \sum_{i=1}^{N} Q^i_\ell \tilde{Q}^i_m = 0 \quad (*)$$

- mod out by $GL(K,\mathbb{C})$

I) Diagonalize;

$$X = \mathrm{diag}(x_1, \dots, x_K)$$

II) Solve $(*)$ for $\tilde{X}$

$$\tilde{X}_{\ell m} = \delta_{\ell m} P_\ell + (1-\delta_{\ell m}) \sum_{i=1}^{N} \frac{Q^\ell_i \tilde{Q}^m_i}{x_\ell - x_m}$$

diagonal components unconstrained

III) Form "spins"

$$S^\ell_{ij} = Q^\ell_i \tilde{Q}^\ell_j$$

$M_{K,N}$ symplectic with Poisson brackets, $\{X_l, P_m\} = \delta_{lm}$

$$\{S^l_{ij}, S^m_{i'j'}\} = \delta_{lm}\left[\delta_{ij'} S^l_{i'j} - \delta_{ji'} S^l_{ij'}\right]$$

$$H_2 = \mathrm{Tr}_K[\tilde{X}^2]$$

$$= \sum_l P_l^2/2 + \sum_{l>m} \frac{\tilde{S}^l_{ij}\,\tilde{S}^m_{ji}}{(X_l - X_m)^2}$$

Hamiltonian of <u>rational spin Calogero-Moser model</u>

$M_{K,N}$ is phase space of complex integrable system

Hamiltonians generate commuting flows on $\mathcal{M}_{K,N}$

$$\delta^{(\ell)} Z = \{ H_\ell , Z \}$$

- preserve complex structure
  symplectic form

but <u>not</u> metric

Kapustin + Sethi

DLCQ of $\mathcal{N}=4$ SUSY Yang-Mills

$\mathcal{N}=4$ SUSY YM on $\mathbb{R}^2 \times \mathbb{R}_- \times S^1_+$

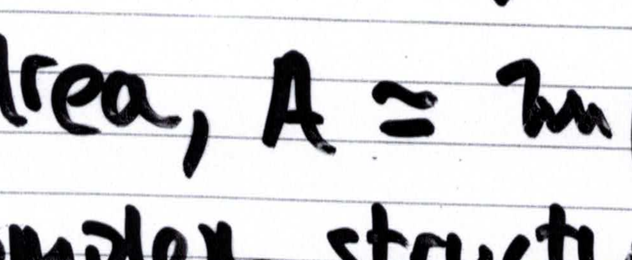
with $p^- = K/R_+$, $G = SU(N)$

Null circle

$$\downarrow$$

SUSY QM on $\hat{\mathcal{M}}^{\tau}_{K,N}$

moduli space of K SU(N) instantons on $\mathbb{R}^2 \times T^2$



Area, $A \simeq \mathrm{Im}(\bar{w}_2 w_1) \to 0$

Complex structure,

$$\tau = \frac{4\pi i}{g^2_{YM}} + \frac{\theta_{YM}}{2\pi}$$ held fixed

Instantons on $\mathbb{R}^2 \times T^2$

Dual descriptions related by,

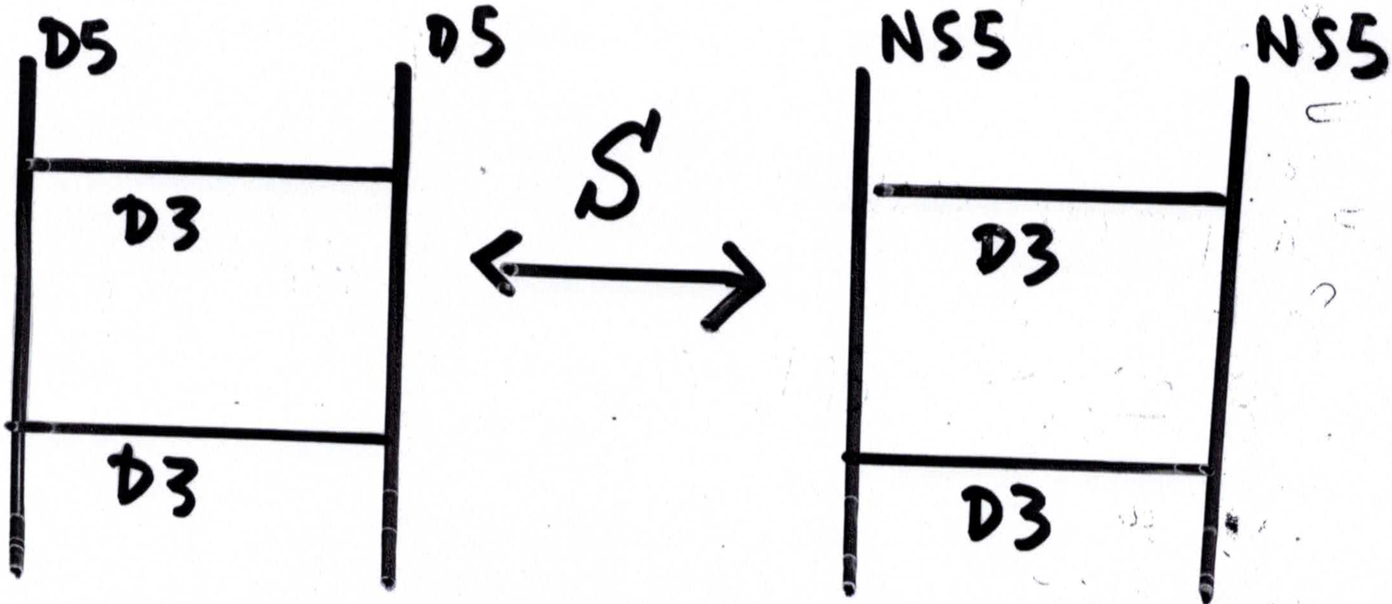
D=3 Mirror Symmetry

Intriligator + Seiberg

Higgs Branch of Theory A $\equiv$ Coulomb Branch of Theory B

Classically exact

quantum corrections



Higgs

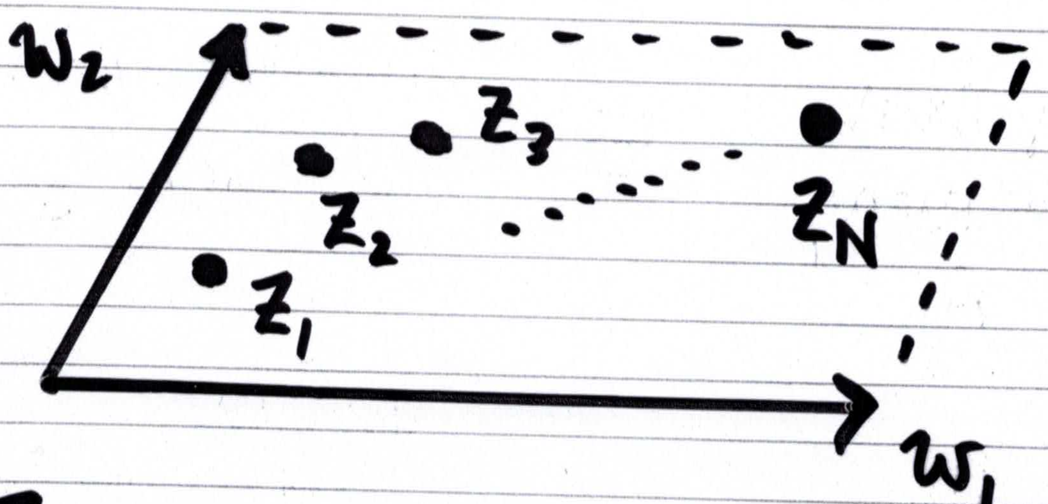
Coulomb

Higgs Branch Description

Kapustin

ADHMN transform

$U(K)$ gauge theory on $\mathbb{R}^{2,1} \times T^2$ with impurities



$\Phi(z)$, $A_{\bar{z}}(z)$, $Q_i, \tilde{Q}_i$

adjoint scalar

gauge field

N impurities in fundamental

F-term constraint,

$$\partial_{\bar{z}} \Phi - [A_{\bar{z}}, \Phi] = \sum_{i=1}^{N} Q_i \tilde{Q}_i \, \delta^{(2)}(z - z_i)$$

Hitchin's eqn

KN Commuting "Hamiltonians"

Nekrasov

ND, Kumar, Hollowood

$$\mu_\ell^i = \underset{z=i}{\mathrm{Res}}\, \mathrm{Tr}_K \left[\Phi^\ell(z) \right]$$

$$\bar{\mu}_2 = \sum_{\ell=1}^{K} P_\ell^2 / 2 \qquad S_{ij}^\ell = Q_i^\ell \tilde{Q}_j^\ell$$

$$+ \sum_{\ell > m} S_{ij}^\ell S_{ji}^m \wp_\tau (X_\ell - X_m)$$

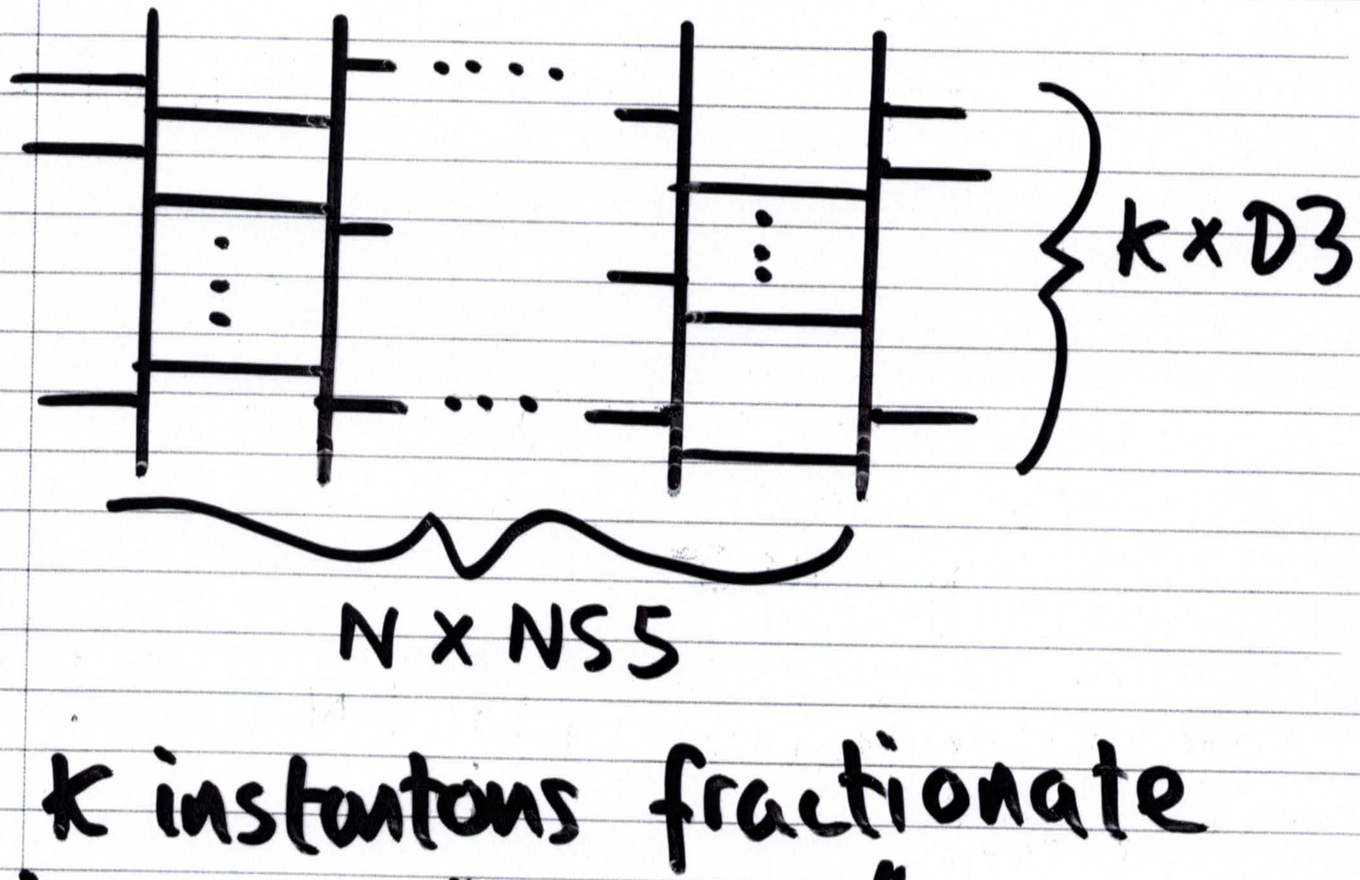
$\hat{\mathcal{M}}_{K,N}^\tau$ is <u>phase space</u> of elliptic spin Calogero - Moser model

Coulomb Branch Description

$\mathcal{N}=2$ SUSY Yang-Mills on

$\mathbb{R}^{2,1} \times S^1$ radius $\tilde{R}$

$\hat{A}_{N-1}$ quiver $\tilde{G} = [U(k)]^N$



k instantons fractionate into kN "partons"

Classical Coulomb Branch

$$\tilde{G} = [U(k)]^N \longrightarrow [U(1)]^{kN}$$

massless fields on $\mathbb{R}^{2,1} \times S^1$

- 4d adjoint scalar Φ_I
- Wilson lines

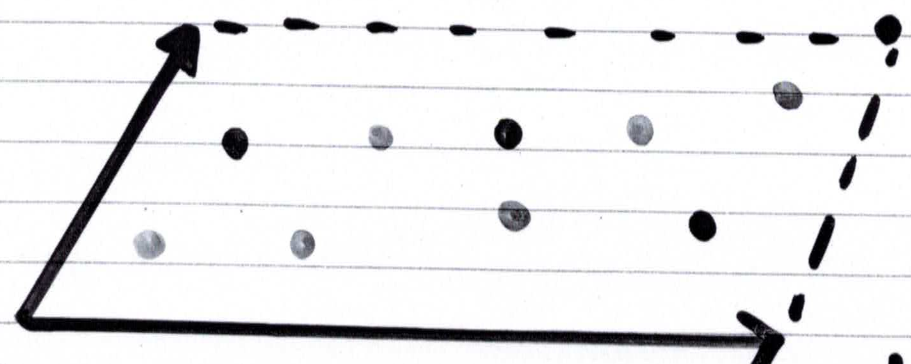
$$w_I = \oint A_I \cdot dx \sim w_I + 2\pi$$

- Dual photons

$I = 1, \ldots, kN$

$$\sigma_I \sim \sigma_I + 2\pi$$

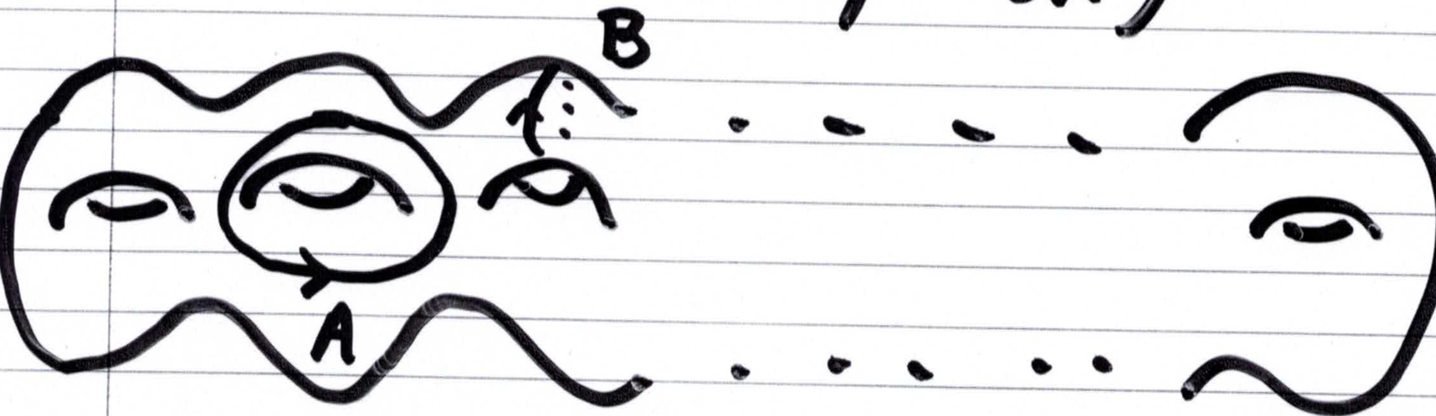
$$\mathcal{M}_{cl} \approx \left[\mathrm{Sym}_k \left(\mathbb{R}^2 \times T^2 \right) \right]^N$$



k particles of each "colour" on T^2

Quantum Coulomb Branch

SW curve (Σ, λ_{SW})



massless fields   genus NK

- Periods $a_I = \oint_{A_I} \lambda_{SW}$

$\tau_{eff} = \text{Im}\,\frac{\partial a_D}{\partial a}$ $a_I^D = \oint_{B_I} \lambda_{SW}$

moduli space of Σ

parametrise $\mathcal{M}(\Sigma)$

- Wilson lines / dual photons

$$z^I = w^I + \tau_{eff}^{IJ} \sigma_J \in \mathcal{J}(\Sigma)$$

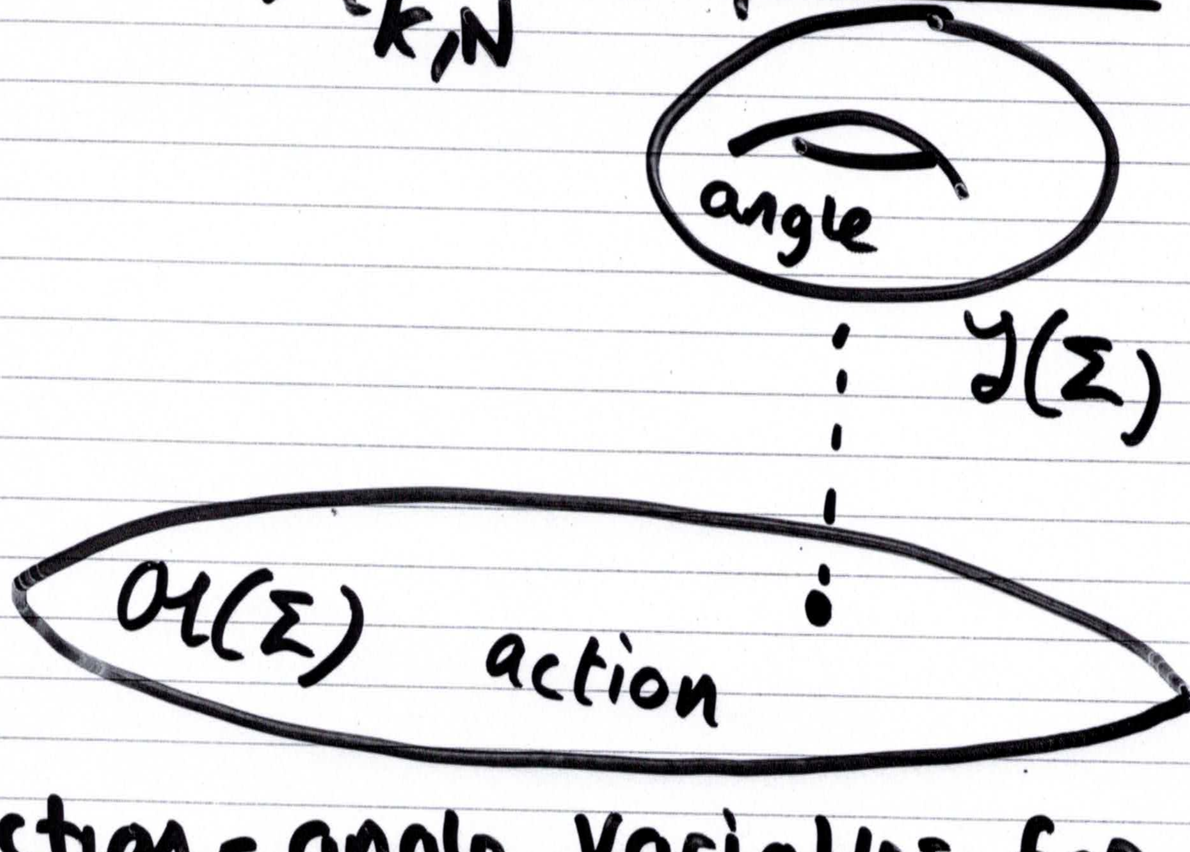
Jacobian

$\underline{R \to \infty}$ Gaiotto-Moore-Nietzke

"semiflat" metric,

$$ds^2 = \widetilde{R}\,(\mathrm{Im}\,\tau_{eff})\,|da|^2 + \frac{1}{4\pi^2 \widetilde{R}}\,(\mathrm{Im}\,\tau_{eff})^{-1}\,|dz|^2$$

exhibits $\mathcal{M}^{\tau}_{k,N}$ as <u>fibration</u>



$\equiv$ Action-angle variables for elliptic spin Calogero-Moser

Main point

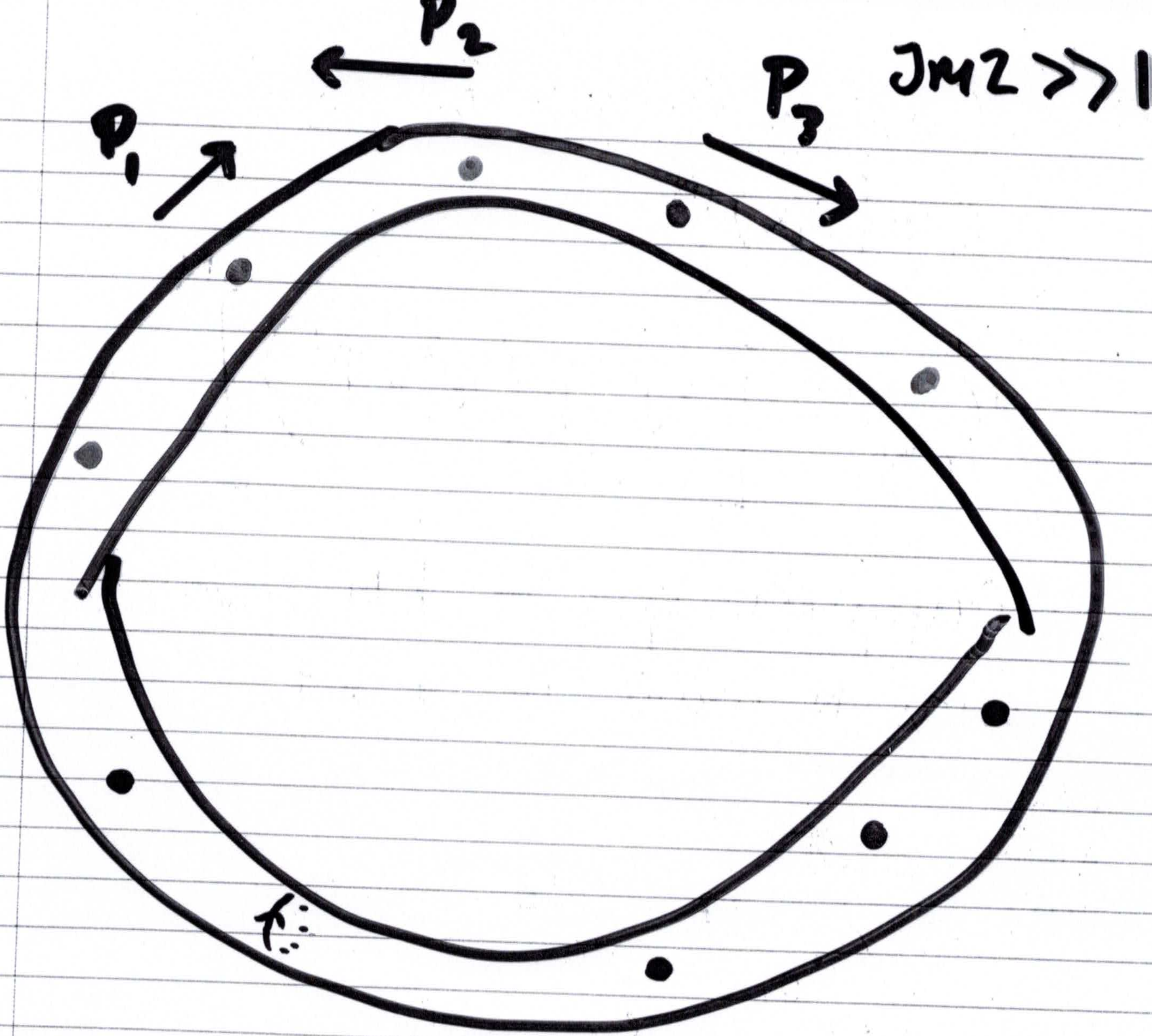
- In this limit Hamiltonian flows become isometries

$\Downarrow$

- Large new symmetry of DLCQ description of (2,0) on $\mathbb{R}^{3,1} \times T^2$ emerges as $A' \to 0$ ie $\mathcal{N}=4$ SUSY YM limit

- In context of diagonalising dilation operator in 't Hooft limit $N \to \infty$, $\operatorname{Im} \tau \to \infty$

This is enough for 'integrability!'



- individual momenta of partons conserved
- Work in progress with P. Zhao

Solution via Bethe Asatz