The return of the analytic S-matrix

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(Partly) based on work with

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Strong fields, strings, and holography   Swansea, 17 July 2013

Why amplitudes ?

- Because they are simple
  - calculation with Feynman diagrams cumbersome, however final results often strikingly simple
- Gluon scattering is an important background for LHC
  - at tree level, gluon scattering can be equivalently calculated in any supersymmetric theory
  - one loop supersymmetric decomposition (Bern, Dixon, Dunbar, Kosower)

$$\boxed{\mathcal{A}_g = (\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s) - 4(\mathcal{A}_f + \mathcal{A}_s) + \mathcal{A}_s}$$

$\mathcal{N}=4$ $\mathcal{N}=1$ $\mathcal{N}=0$

one-loop amplitude in pure YM with a gluon running in the loop

gluon
4 Weyl fermions
6 real scalar fields

the most difficult piece, but simpler than $\mathcal{A}_g$

Textbook approach to amplitudes:

Calculate Feynman diagrams !

A typical Feynman diagram contains:

Propagators

Vertices

Gauge-dependent, off-shell internal states

- symmetries of the problem not preserved by our calculational approach
  - Feynman diagrams are not separately gauge invariant
  - Unphysical, off-shell internal states (vertices & propagators)

- vast redundancy from field redefinitions
  - S-matrix equivalence theorem $\phi \to \left(\frac{\Box}{m^2}\right)^6 \phi + 5\frac{(\Box + m^2)^7}{m^{14}}\phi$
  - in a sense, Lagrangian is not unique !

- locality & unitarity as derived concepts
  - non manifestly local/unitarity descriptions (Arkani-Hamed, Cachazo et al)

Unwanted complexity (I)

Number of Feynman diagrams for $gg \longrightarrow n\, g$ scattering: (tree level)

☞

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Gluon scattering

☞ Result is: $\mathcal{A}(1^{\pm}, 2^{+}, \ldots, n^{+}) =$ 

`FullSimplify[`

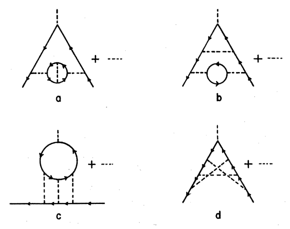
+ many more pages like this...]

$= 0$!!

Why so simple? Why zero?

Unwanted complexity (II)

- Three-loop correction to electron g–2

72 diagrams like  $= (1.181241456\ldots)\,(\alpha_{\mathrm{e.m.}}/\pi)^3$

(Cvitanovic & Kinoshita ’74)

(Laporta & Remiddi ’96)

- wild oscillations between the values of each diagram/integral
- final result is $O(1)$
- another example of unexplained simplicity...

Form factors

- Partially **off-shell** quantities

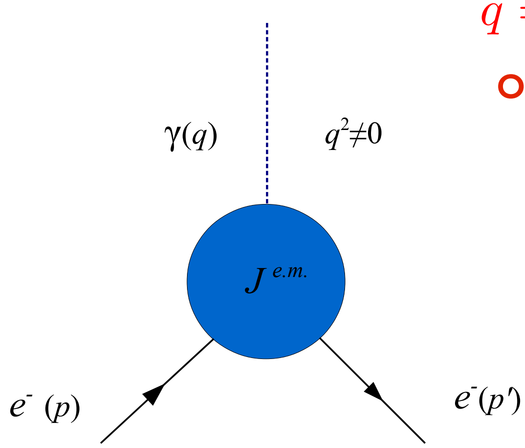
$$F = \int d^4x\, e^{-iqx} \langle state|\mathcal{O}(x)|0\rangle = \delta^{(4)}(q - p_{state}) \langle state|\mathcal{O}(0)|0\rangle$$

- g–2 : electromagnetic form factor

$q = p - p'$

off shell

$$\langle e^-(p')|\, J_\mu^{e.m.}(0)\, |e^-(p)\rangle =$$

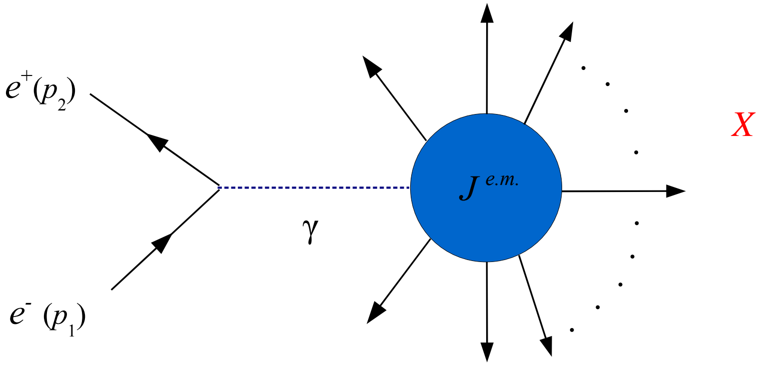


$$J_\mu^{e.m.} = \bar{\psi}\gamma_\mu\psi$$

on shell

on shell

- Form factors appear in several interesting contexts:
  - deep inelastic scattering $(e^- + p \rightarrow e^- + \text{hadrons})$
  - $e^+ e^- \rightarrow$ hadrons :

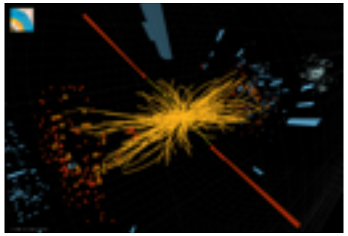


$$= e\,\bar{v}(p_2)\gamma_\mu u(p_1)\,\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}\,(-e)\langle X|\,J_\nu^{e.m.}(0)\,|0\rangle$$

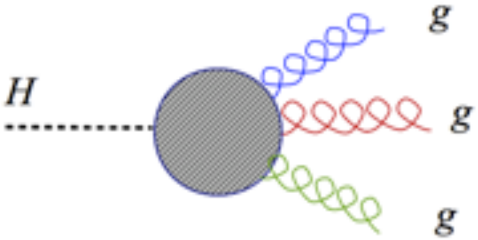
$e^+ e^- \rightarrow$ hadrons (X)

all orders in α_{strong}, first order in $\alpha_{\text{e.m.}}$

hadronic electromagnetic current



- Higgs + multi-gluon amplitudes in QCD
  - at low M_H : dominant Higgs production at the LHC through gluon-gluon fusion
  - coupling to gluons through a fermion loop
    - proportional to the quark mass $\Rightarrow$ top quark dominates
  - for $M_H < 2\, m_{top}$ integrate out the top quark



- Effective Lagrangian description $\mathcal{L}_{\text{eff}} \sim H\, \mathrm{Tr}\, F_{\text{SD}}^2$
  - coupling is independent of m_{top}
  - efficient MHV rules (Dixon, Glover & Khoze; Badger, Glover & Risager; Boels & Schwinn)

- Higgs + multi-gluon scattering is a form factor!
  - form factor of $\mathrm{Tr}\,(F_{\mathrm{SD}})^2$ (= amplitude of a different theory!)

$$F_{\mathrm{Tr}F^2_{\mathrm{SD}}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state|\,\mathrm{Tr}\,F^2_{\mathrm{SD}}(x)\,|0\rangle$$

  - in N=4 SYM, this is related to the form factor of $\mathrm{Tr}\,(\phi_{12})^2$

$$F_{\mathrm{Tr}\phi^2_{12}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state'|\,\mathrm{Tr}\,\phi^2_{12}(x)\,|0\rangle$$

    - $\mathrm{Tr}\ \phi^2{}_{12}$ and $\mathrm{Tr}\ F_{\mathrm{SD}}{}^2$ part of the same 1/2 BPS supermultiplet
    - supersymmetric form factor of the chiral part of the stress tensor multiplet (Brandhuber, Gurdogan, Mooney, Yang, GT)

- Recent QCD calculation of Gehrmann, Glover, Jacquier & Koukoutsakis:
  - $H\, g^+\, g^-\, g^-$ MHV
  - $H\, g^+\, g^+\, g^+$ maximally non-MHV
  - $H\, q\ \ \bar{q}\ g$ fundamental quarks
- We will compare our result in N=4 super Yang-Mills to theirs later
  - surprising result: maximally transcendental parts in perfect agreement!

Unwanted complexity (III)
General Relativity

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma''\sigma'')\beta'} \to -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma}+p''^{\sigma}). \quad (2.3)$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \to \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \quad (2.4)$$

$$\hat{G} \to \gamma^{\alpha\beta}/p^2, \quad (2.5)$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational field are much more complicated. In this case we shall employ the momentum-index combinations $p\mu\nu$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''\iota'''\kappa'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms[2] and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho})\\&-\tfrac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu})\\&\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big], \quad (2.6)\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\&-P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_{\sigma}p'^{\rho}\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^{\rho}p'^{\iota}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\&\qquad-P_6(p^{\rho}p'^{\iota}\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big]. \quad (2.7)\end{aligned}$$

← 3-point vertex: 171 terms

← 4-point vertex: 2850 terms

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once $S_2{}^0$ is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$\begin{aligned}&V_{(\mu}{}^{\sigma''\tau''}{}_{)\nu'} \to\\&\quad\tfrac{1}{2}\mathrm{Sym}\big[2p''_{\mu}p'^{\sigma}\delta_{\nu}{}^{\tau}-p''_{\mu}p'_{\nu}\eta^{\sigma\tau}\\&\qquad+(p_{\nu}p'^{\sigma}-p'_{\nu}p^{\sigma})\delta_{\mu}{}^{\tau}+p\cdot p'\delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\tau}\big], \quad (2.8)\end{aligned}$$

where the momentum-index combinations are $p\mu$, $p'\nu'$, $p''\sigma''\tau''$, and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \to (\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \quad (2.9)$$

$$\hat{G} \to \eta^{\mu\nu}/p^2. \quad (2.10)$$

[2] The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

Einstein-Hilbert Lagrangian and Yang-Mills Lagrangian give rise to very different-looking Feynman rules...

Bryce S. DeWitt , Phys. Rev. 162:1239-1256,1967.

- ....however:

$$\boxed{\begin{aligned} \mathcal{A}_{\mathrm{GR}}(1^+2^+3^-) &= [\mathcal{A}_{\mathrm{YM}}(1^+2^+3^-)]^2 \\ \mathcal{A}_{\mathrm{GR}}(1^-2^-3^+) &= [\mathcal{A}_{\mathrm{YM}}(1^-2^-3^+)]^2 \end{aligned}}$$

  - KLT relations
  - hint at further secret similarities between GR and YM amplitudes...

- three-point amplitudes are the smallest amplitudes
  - entirely determined by helicities + Lorentz invariance
  - appear only in complexified Minkowski

- EH Lagrangian (and Feynman rules) not needed !

Unexplained simplicity hints at...

- ...hidden structures in perturbative quantum field theory...
- ...which are not captured by Feynman diagrams
- Need new framework to calculate S-matrix directly

(Cambridge, 1966)

"Strings, gauge fields and duality",
a conference to mark the retirement of David Olive
Swansea 24-27 March 2004

Key ideas in

- On-shellness
  - "The fields themselves are of little interest. They are merely used to calculate transition amplitudes for interactions. These amplitudes are the elements of the S-matrix"
  - "One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, whether or not some underlying Lagrangian theory exists"
- Complexify
  - "One of the most remarkable discoveries in elementary particle physics has been that of the complex plane"

What was “missing” in 1966

- Massless particles
  - most of the beautiful structure uncovered so far is in theories of massless particles
- New symmetries/concepts
  - large-N limit
  - supersymmetry
  - string theory, AdS/CFT correspondence
  - conformal symmetry, new hidden symmetries
  - simplest S-matrix: N=4 SYM & N=8 supergravity (maximal supersymmetry)

Plan

- Look at some incarnations of these ideas

- Hidden structures in scattering amplitudes & form factors
  - MHV amplitude and recursion relations
  - amplitude/ Wilson loop duality at strong and weak coupling
  - dual conformal symmetry
  - maximal transcendentality & symbols of "finite remainder functions"

MHV amplitude

- First non-vanishing amplitude: Maximally Helicity Violating

$$\mathcal{A}_{\mathrm{MHV}}(1^+ \ldots i^- \ldots j^- \ldots n^+) = \frac{\langle i j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

helicities are a permutation of −−++....+

(Parke & Taylor, 1986; Berends, Giele 1987; Mangano, Parke, Xu 1988)

- Simple geometry in Penrose’s twistor space
(Witten, 2003)

  - localised on a line in twistor space

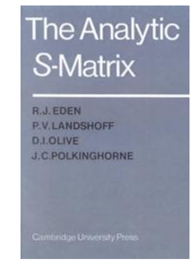
  - holomorphic (only < > spinor products)

  - generic amplitudes (with more negative helicities) localise on unions of lines

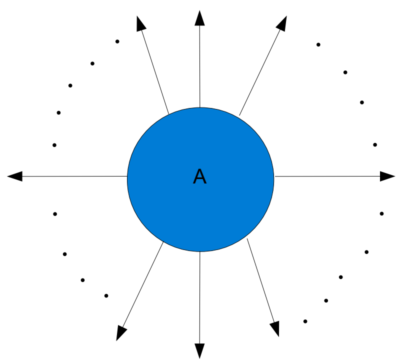
  - first example of hidden structure

On-shell (BCF) recursion relations

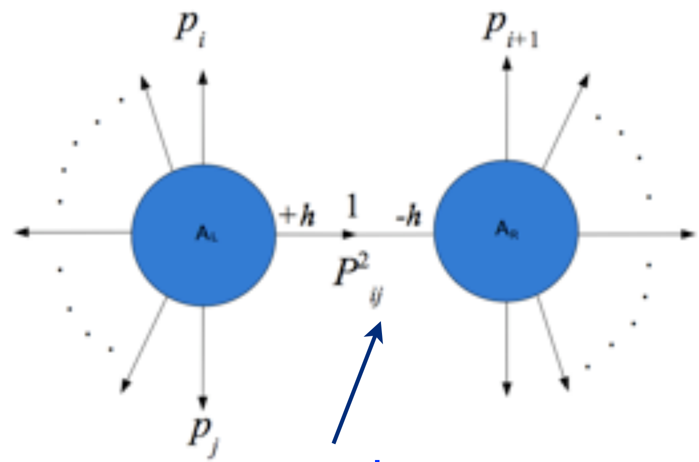
(Britto, Cachazo, Feng; BCF + Witten, 2005)



- Exploit analytic structure of amplitudes
  - Singularities of tree amplitudes:



$$\xrightarrow[P_{ij}^2 \to 0]{} \quad \sum_h$$



- Factorisation on multi-particle poles (simple poles, tree level)

simple pole

h = internal particles helicities

  - <u>idea</u>: physical singularities → poles in a single complex variable z

- Shift momenta: $\hat{p}_1(z) = p_1 + z\,\eta\,,\quad \hat{p}_2(z) = p_2 - z\,\eta$
  with $\hat{p}_1^2 = \hat{p}_2^2 = 0$ for all z and $\eta^2 = 0$

  - shifted momenta are complex!

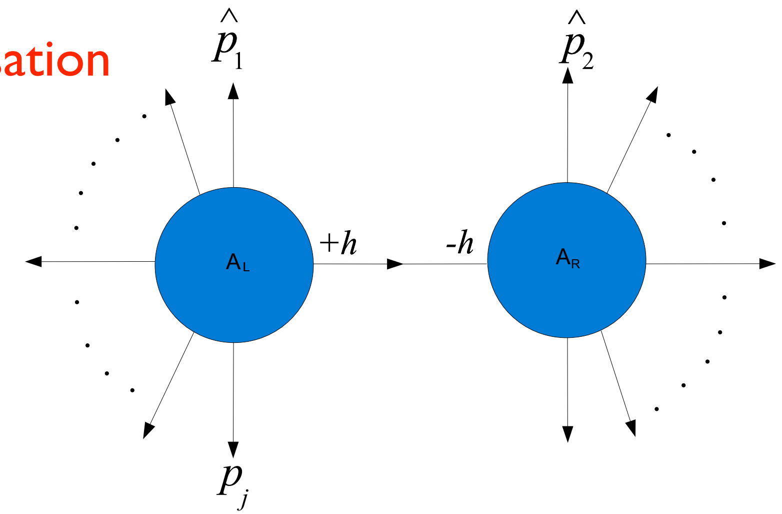
- $\mathcal{A}(z) \;:=\; \mathcal{A}(\hat{p}_1, \hat{p}_2, p_3, \ldots, p_n)$ $\quad$ $\mathcal{A}(0)$ is the amplitude

- $$\mathcal{A}(z) \;=\; \sum_P \frac{c_P}{z - z_P}$$ only simple poles

  - assume $\mathcal{A}(z) \to 0$ as $z \to \infty$ (depends on theory)

  - residues c_P from factorisation

<u>Final result:</u> $$\mathcal{A} \;=\; \sum_{j,h}$$



- Results very simple!
  - 3-pt amplitudes “seed” the recursion, everything on shell
- Wide applicability:
  - General Relativity (Bedford, Brandhuber, Spence, GT ‘05; Cachazo, Svrcek’05; Benincasa, Boucher-Veronneau, Cachazo ‘07; Arkani-Hamed, Kaplan ‘08)
  - rational part of QCD (Bern, Dixon, Kosower; “BLACKHAT” collaboration) and gravity amplitudes (Brandhuber, McNamara, Spence, GT; Alston, Dunbar, Perkins)
  - massive particles (Badger, Glover, Khoze, Svrcek)
  - N=4/N=8 manifestly supersymmetric recursion relations (Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan; Drummond, Henn)
  - ABJM theory (Gang, Huang, Koh, Lee, Lipstein)

Hidden structures in planar N=4 SYM

i. Iterative structure at weak coupling

(Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)

- $\mathcal{A}_{n,\mathrm{MHV}} = \mathcal{A}^{\mathrm{tree}}_{n,\mathrm{MHV}} \mathcal{M}_n$ $\mathcal{M}_n$ is "helicity-blind"

- All-loop MHV amplitude:

$$\boxed{\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} \lambda^L \mathcal{M}_n^{(L)} \sim e^{\mathrm{BDS} + \mathcal{R}}}$$

$\lambda \sim g^2 N/(8\pi^2)$

- $\mathrm{BDS} \sim \mathrm{div} + \gamma_K \mathrm{Finite}^{(1)}(p_1, \ldots, p_n)$ BDS ansatz
- div = universal infrared-divergent part
- γ_K = cusp anomalous dimension
- $\mathrm{Finite}^{(1)}(p_1, \ldots, p_n)$ = finite part of one-loop amplitude
- $\mathcal{R}$ is the Remainder Function, $\mathcal{R} = 0$ for $n = 4, 5$ $\mathcal{R} \neq 0$ for $n \geq 6$

- BDS:
  - contains infrared divergences, which are known to exponentiate
    (Giele, Glover; Kunszt, Signer, Trocsany; Sterman, Teyeda-Yeomans; Catani; Magnea, Sterman)
  - exponentiation of finite parts: new and unexpected
  - modern explanation: hidden dual conformal symmetry

- Remainder:
  - $\mathcal{R} = 0$ for $n = 4, 5$ and any loop; $\mathcal{R} \neq 0$ for $n \geqslant 6$ starting at 2 loops
  - hard to calculate, even numerically (one data point takes one week)
  - will approach from the Wilson loop side......

ii. Wilson loop/amplitude duality

(Alday, Maldacena; Drummond, Korchemsky, Sokatchev + Henn; Brandhuber, Heslop, GT)

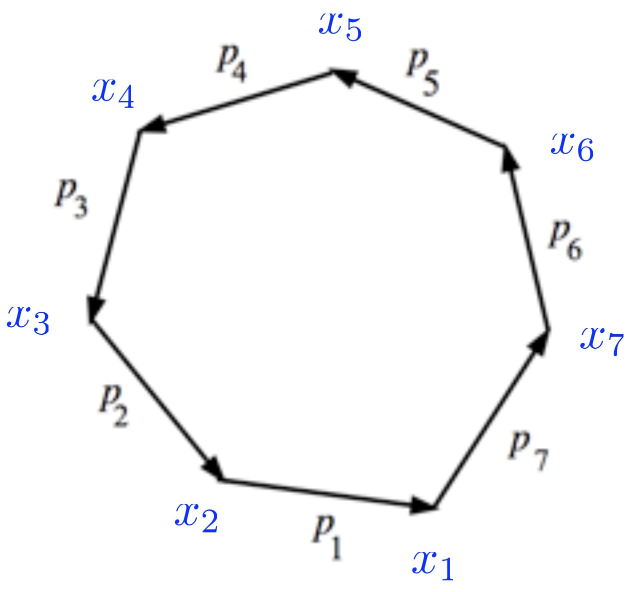
- MHV amplitudes in planar N=4 super Yang-Mills calculated by a Wilson loop

$$\langle W[C]\rangle \;:=\; \mathrm{Tr}\, P \exp\left[ig \oint_C d\tau \Big(\dot{x}_\mu(\tau) A^\mu\big(x(\tau)\big)\Big)\right]$$

  - Strong coupling (Alday & Maldacena)
  - Weak coupling (Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, GT)

- C determined by the momenta of the scattered particles

- The contour of the Wilson loop:
  - A particular polygonal contour, made of lightlike segments:
    - colour ordering $\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$
  - $\sum_{i=1}^{n} p_i = 0$ momentum conservation
    closed contour



    - $p_i = x_i - x_{i+1},$ lightlike
    - x are T-dual (region) momenta

All-loop conjecture

(Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, GT)

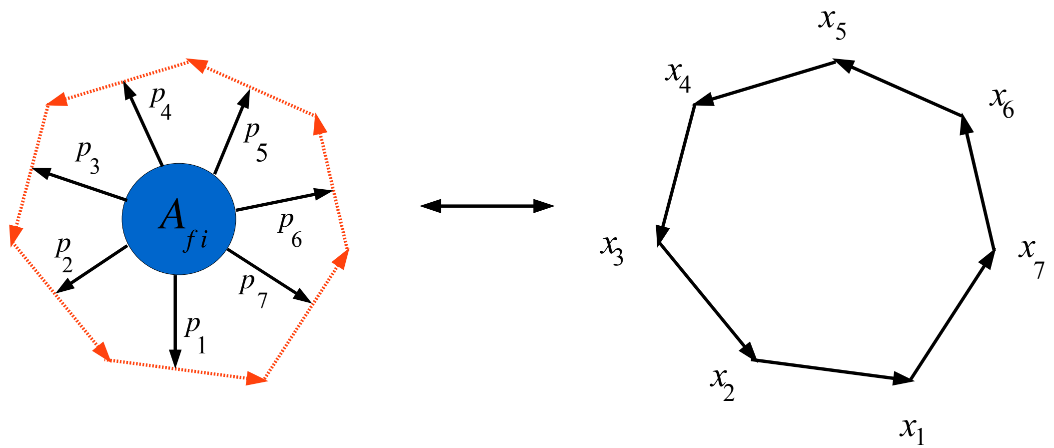
- MHV Amplitude "=" Wilson loop
  - more precisely: Wilson loop calculates $\mathcal{M}$
    - $\mathcal{M}$ is the helicity-blind function in $\mathcal{A}_{\rm MHV}^{(L)} = \mathcal{A}_{\rm MHV}^{\rm tree}\,\mathcal{M}^{(L)}$
    - Subtlety in the infrared-divergent part

- <u>Conjecture:</u> (Log) $< W[C] >$ = (Log) $\mathcal{M}$ to all loops

In terms of the remainders: $$\boxed{\mathcal{R}_{n,\rm WL} = \mathcal{R}_n}$$

Why is this interesting/useful ?

- New duality



- Remainder function is easier to compute

$$< W[C] > \; = \mathrm{Exp}\,(\;\; \mathrm{BDS} + \mathcal{R} \;\;)$$

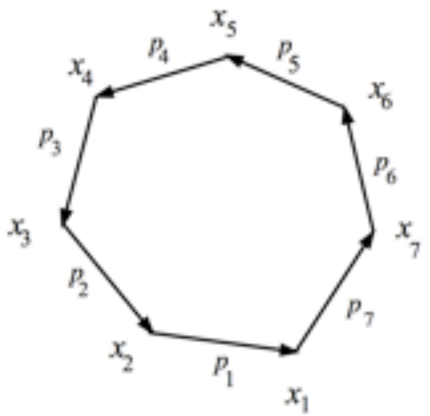
- Wilson loop: one hour. Amplitude: one week
  - (dimensionally regularised) Wilson loop integral functions much simpler to evaluate than corresponding amplitude integral functions
- Functional dependence of $\mathcal{R}$ constrained by dual conformal symmetry

iii. Dual conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev)

- Natural symmetry from Wilson loop perspective:

  - it is the standard conformal group acting on dual momenta x's

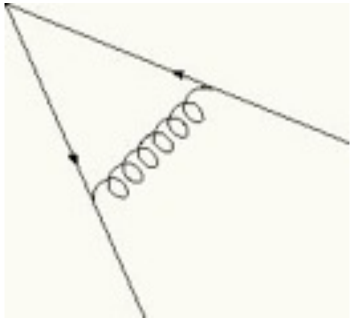


$$p_i = x_i - x_{i+1}$$

$$x_{n+1} = x_1$$

  - symmetry is anomalous

    - UV divergences from cusps in the contour
      (UV for the Wilson loop = IR for the amplitude)



- BDS Ansatz explained by dual conformal symmetry
  - a solution to the associated anomalous Ward identity
  - remainder $\mathcal{R}$ is a function of cross-ratios
    - $\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$ invariant under $x_i \to \frac{x_i}{x_i^2}$
  - solution is unique at four and five points (modulo constants)
    - lightlike condition forbids nontrivial cross ratios for $n < 6$
- For $n \geqslant 6$ points, cross ratios open up and $\mathcal{R} \neq 0$
  - e.g. at $n = 6$: $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2}\,, \quad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}\,, \quad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$
  - $\mathcal{R}_6 = \mathcal{R}_6(u_1, u_2, u_3)$ non-vanishing starting at 2 loops

- Remarkable series of recent strong-coupling calculations
(Alday, Maldacena; Alday, Gaiotto Maldacena; Alday, Maldacena, Sever, Vieira)
  - integrability of worldsheet theory, Y-systems...
- Weak-coupling side:
  - n-point remainder integrals (Anastasiou, Brandhuber, Heslop, Khoze, Spence, GT)
  - 6-point integrals calculated by Del Duca, Duhr, Smirnov.
    17-pages result, contains Goncharov polylogs

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

  - Goncharov, Spradlin, Vergu and Volovich introduced the concept of "symbol of a transcendental function" and rewrote this as 2 lines of classical polylogs $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$

Comments:

1. Conjecture: dual (super)conformal symmetry lifted from Wilson loops to amplitudes
(Drummond, Henn, Korchemsky, Sokatchev)

- new hidden symmetry of planar N=4 amplitudes!
  - on-shellness, large-N limit, N=4 symmetry
- tree-level S-matrix of N=4 SYM is dual superconformal covariant
(Brandhuber, Heslop, GT)

2. Weak coupling: Yangian symmetry of tree-level scattering amplitudes (Drummond, Henn, Plefka)

- commute the generators of the two superconformal algebras
- it is still a matter of debate whether the predictive power of the Yangian symmetry exceeds that of the two superconformal symmetries

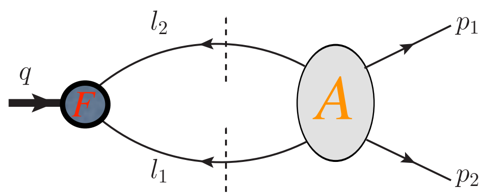
The form factor remainder function

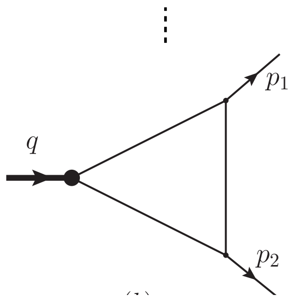
One loop

(Brandhuber, Spence, GT, Yang; + Gurdogan & Mooney)

- Form factors from unitarity
- Simplest application: Sudakov form factor (= two points) of a half-BPS operator

$$F(q^2) \;:=\; \langle \phi_{12}(p_1)\phi_{12}(p_2)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\rangle \qquad q \;:=\; p_1+p_2$$





$$\big[F(q^2)\big]^{1\,\mathrm{loop}} \;=\; 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big] \qquad D=4-2\epsilon \;☜$$

regulates infrared divergences

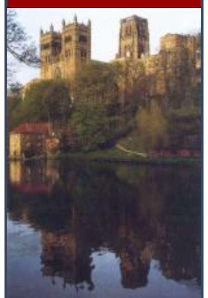
- each term has fixed degree of "transcendentality"

Transcendentality

- constants have transcendentality 0
- π, log transcendentality 1
- π^2 , $\log^2$, Li_2 transcendentality 2
- ... ζ_n , Li_n , $\log \times \mathrm{Li}_{n-1}$ ... transcendentality n
- At L loops, term in ϵ^p has transcendentality $2L + p$

- Principle of maximal transcendentality
  - observed by Gracey in supersymmetric non-linear sigma models
  - Kotikov, Lipatov + Onischchenko, Velizhanin introduced it in N=4 SYM for anomalous dimensions of twist-2 operators
- connections to number theory!

Participants
Schedule
Talks



London Mathematical Society - EPSRC Durham Symposium

Polylogarithms as a Bridge between Number Theory and Particle Physics

School: 3 July (from 9:30) - 6 July 2013 (finishing 13:30)
Workshop: 8 July (from 9:30) - 12 July 2013 (finishing 17:30)

The LMS Durham Research Symposia began in 1974, and form an established series of international research meetings, with over 90 symposia to date. They provide an excellent opportunity to explore an area of research in depth, to learn of new developments, and to instigate links between different branches. The format is designed to allow substantial time for interaction and research. The meetings are held in July and August, usually lasting for 10 days, with up to 70 participants, roughly half of whom will come from the UK. Lectures and seminars take place in the Department of Mathematical Sciences, Durham University. The school and conference are supported in part by the EU network GATIS.

Participants include:

School lecturers:

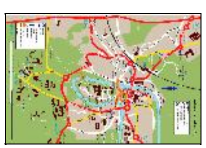
Jacob Bourjaily: The On-Shell Analytic S-Matrix
Johannes Henn: Introductory lectures on amplitudes, Wilson loops and symmetries
Gregory Korchemsky: Correlators and integrability
Cristian Vergu: Multiple polylogarithms/symbols and physical applications
Jianqiang Zhao: Multiple Polylogarithms, Multiple Harmonic Sums and Multiple Zeta Values

Workshop speakers:

Nima Arkani-Hamed
Christian Bogner
Andreas Brandhuber
David Broadhurst
Ozgur Ceyhan
Lance Dixon
Dzmitry Doryn
James Drummond
Claude Duhr
Burkhard Eden
Michael Green
Matt Kerr
Dirk Kreimer
Kasper Larsen

Lionel Mason
Jan Plefka
Radu Roiban
Oliver Schnetz
Emery Sokatchev
Mark Spradlin
Stephan Stieberger
Congkao Wen

Outline
Over the last decade, there have been numerous interactions between Number Theory and Particle Physics, often involving polylogarithms and associated structures. The first week of the symposium will consist of a four-day school covering the following topics: Introductory lectures on amplitudes, Wilson loops and symmetries; Symbols/mixed Hodge structures; multiple polylogarithms/symbols and physical applications; Grassmannian approach to amplitudes; correlators and integrability. In the second week there will be a workshop with leading researchers on both the Physics and Number Theory side. The scientific goals of the workshop include: to review recent progress, highlight the remarkable connections between Number Theory and Particle Physics, stimulate interaction and collaboration among participants, and inspire further outstanding developments in the field.



Travel Information

Useful information about travelling to Durham can be found on the Department of Mathematical Sciences webpages and on the Durham University's pages.

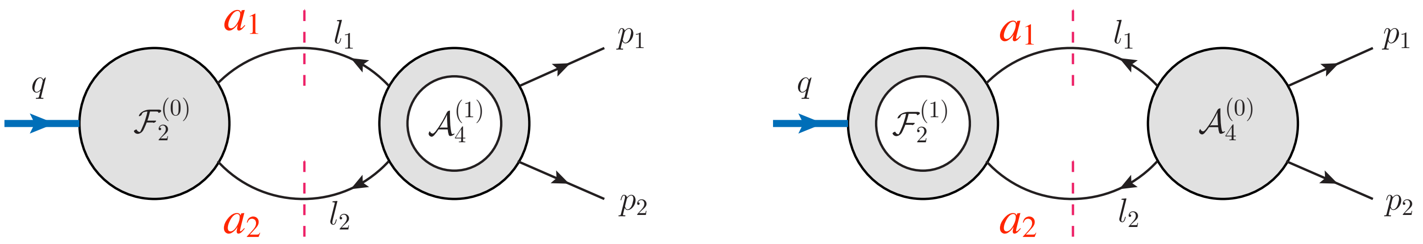
Accommodation

Accommodation for most participants will be in Holgate House, Grey College. This is conveniently located near to the lecture rooms in the Department of Mathematical Sciences. Guest rooms offer en-suite and internet-connection facilities. Attendance is by invitation only and fees for self-supporting participants are payable by cash, credit card, sterling cheques or sterling travellers cheques at registration.

Organising Committee:
Herbert Gangl (Durham), Paul Heslop (Durham), Gabriele Travaglini (QMUL).

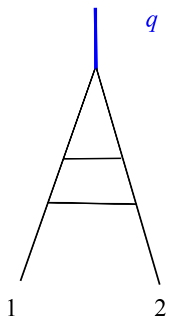
Two loops

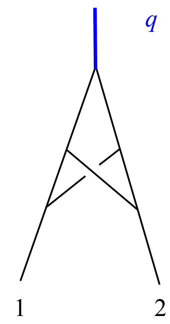
- Result derived from various cuts:



- F proportional to $\delta^{a_1 a_2}$
- non-planar one-loop amplitude are also relevant in the cuts!

Sudakov at two loops:

$F^{(2)}(q^2) =$ 4 

$+$ 

- first obtained by van Neerven in a pioneering paper in 1986!

- two-loop result exponentiates as expected:

$$\left[F(q^2)\right]^{1\,\text{loop}} = 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2} + \mathcal{O}(\epsilon)\Big]$$

$$\left[\text{Log } F(q^2)\right]^{2\,\text{loop}} = (-q^2)^{-2\epsilon}\Big[\frac{\zeta_2}{\epsilon^2} + \frac{\zeta_3}{\epsilon} + \mathcal{O}(\epsilon)\Big]$$

- result is transcendental (non-planar integral topology)

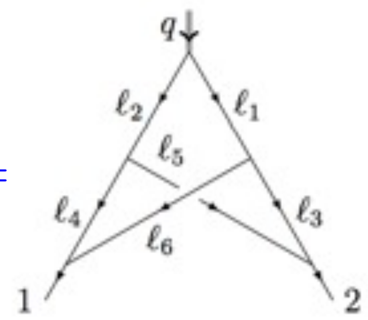
- recent nice three-loop calculation confirms principle of maximal transcendentality (Gehrmann, Henn, Huber)

- Two-loop Sudakov in ABJM (Brandhuber, Gurdogan, Korres, Mooney, GT; Young)

$$F^{(2)}(q^2) \;=\; \left(\frac{N}{k}\right)^2 \mathbf{XT}(q^2)$$

note particular numerator

$$\mathbf{XT}(q^2) \;=$$



$$\times\, q^2\big[-\mathrm{Tr}(p_1p_2l_3l_1) + q^2 l_3^2\big]$$

- $$\boxed{F^{(2)}(q^2) \;=\; \frac{1}{64\pi^2}\left(\frac{N}{k}\right)^2\left(-\frac{q^2}{\mu'^2}\right)^{-2\epsilon}\left[-\frac{1}{\epsilon^2} + 6\log^2 2 + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)\right]}$$

$$\mu'^2 \;:=\; 8\pi e^{-\gamma_E}\mu^2$$

π, $\log 2$

- agreement with the IR divergences of the known two-loop amplitudes, result has maximal degree of transcendentality...same as N=4 SYM!

3-point form factor at 2 loops

(Brandhuber, GT, Yang)

- MHV $\quad F_3(1,2,3) \;=\; \langle \phi_{12}(p_1)\,\phi_{12}(p_2) g^{+}(p_3)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\rangle$

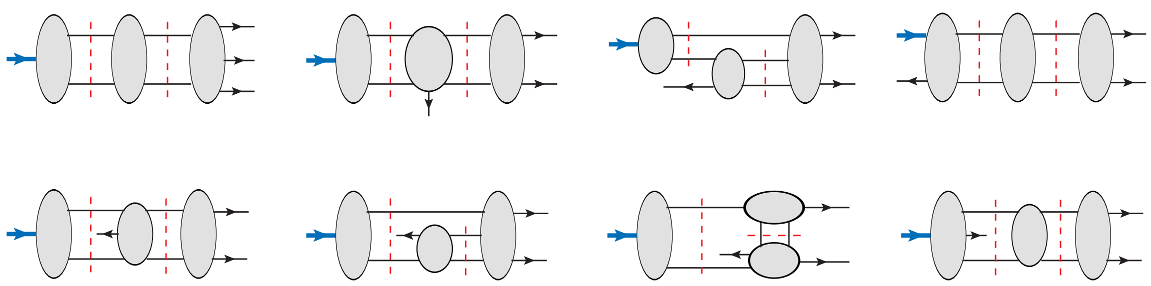
  - Tree: $\quad F_3^{\mathrm{tree}} \;=\; \dfrac{\langle 1\,2\rangle}{\langle 2\,3\rangle\langle 3\,1\rangle}$

  - Loops: $\quad F_3^{(L)} \;=\; F_3^{\mathrm{tree}}\;\mathcal{G}_3^{(L)}(1,2,3)$
    - $\mathcal{G}_3^{(L)}$ helicity-blind function
    - totally symmetric under legs exchange
    - one loop: IR divergences + sum of finite 2me box
    - two loops: nontrivial remainder function?

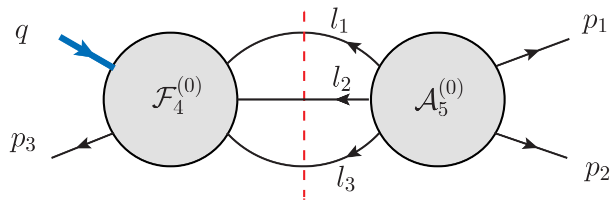
The traditional way

- Do a 2-loop calculation, use generalised unitarity

  1. detect all possible integrals and coefficients with iterated two-particle cuts



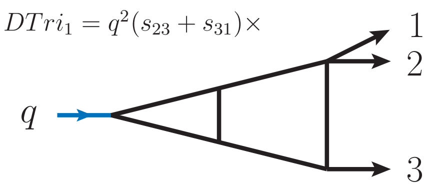
  2. next, fix all remaining ambiguities using three-particle cuts, such as



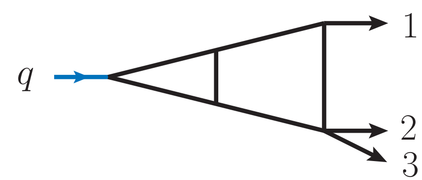
- **Final result:**

$$\boxed{\frac{F_3^{(2)}}{F_3^{\mathrm{tree}}} = \sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}}$$

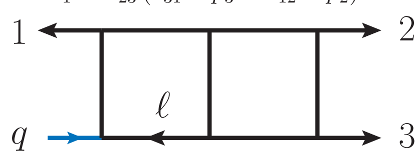
$DTri_1 = q^2(s_{23}+s_{31})\times$



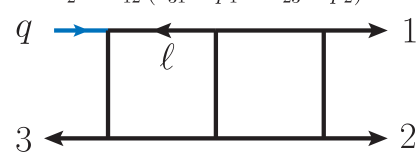
$DTri_2 = q^2(s_{12}+s_{31})\times$



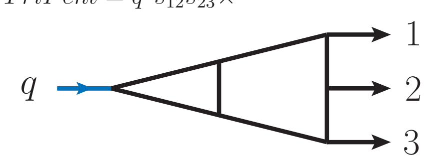
$DBox_1 = s_{23}\left(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2\right)\times$



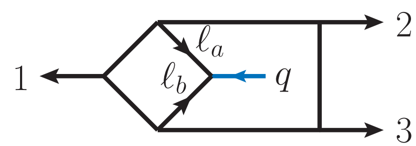
$DBox_2 = s_{12}\left(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2\right)\times$



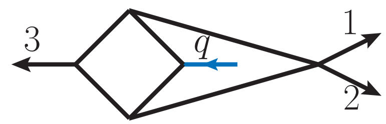
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



- result expressed in terms of two-loop planar and non-planar integrals

- Several analytic results (Gehrmann & Remiddi)
  - variables: $u := \frac{s_{12}}{q^2}, \ v := \frac{s_{23}}{q^2}, \ w := \frac{s_{31}}{q^2}$, with $q = p_1 + p_2 + p_3$
  - $u + v + w = 1$
  - all known integrals appearing in our answer are transcendental
  - unknown integrals can be re-expressed in terms of master integrals which are transcendental (Gehrmann & Remiddi)
- Evaluate integrals with sophisticated technologies:
  - AMBRE (Gluza, Kajda, Riemann, Yundin) (only for planar or non-planar with 1 scale)
  - MB.m (Czakon)
  - MBresolve.m (Smirnov & Smirnov)

- Some features of the final result:
  - has fixed degree of transcendentality (at each loop order and power of the dimensional regularisation parameter ϵ)
  - can be expressed in terms of Goncharov multiple polylogarithms...
  - ...which disappear in our final expression for the remainder
  - cancellations impossible to find without resorting the symbols

The fast way: go straight to the answer!

- Compute directly the finite remainder using symbols, then lift the symbol to a function
  - define an appropriate remainder function:
    - finite
    - trivial/understood collinear limits
  - determine its symbol (Goncharov, Spradlin, Vergu, Volovich)
    - remainder is a transcendentality-four function (two loops)
    - impose symmetries and physical constraints
  - fix "beyond-the-symbol" terms
  - lift symbols to functions

Examples of this strategy so far:

- Six-point MHV remainder (Goncharov, Spradlin, Vergu, Volovich)
- MHV remainder in $(1{+}1)$-dim kinematics (Heslop & Khoze)
  - 2 loops, all n
  - 3 loops, all n (7 undetermined constants)
- MHV remainder, any n (Caron-Huot)
- Six-point NMHV remainder at 2 loops (Dixon, Drummond, Henn)
- Six-point, MHV remainder at 3 and 4 loops
  (Dixon, Drummond, Henn; Caron-Huot, He; Dixon, Drummond, Duhr, Pennington)
- Our example: three-point (1 leg off shell, 3 on shell) form factor remainder at 2 loops

Step I: define form factor remainder

- Define ABDK/BDS remainder, $\mathcal{R}$

$$\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)} - \mathrm{BDS}_n^{(2)}$$

- Ingredients:
  - two-loop form factor $\mathcal{G}_n^{(2)}$
  - BDS part, contains all infrared divergences
  - first nontrivial remainder appears for n=3

- Properties of the remainder:
  - finite
  - trivial collinear limits $\mathcal{R}_n^{(2)} \to \mathcal{R}_{n-1}^{(2)}$
  - in particular: $\mathcal{R}_3^{(2)} \to 0$ (there is no Sudakov remainder $\mathcal{R}_2^{(2)}$!)

Crash review of symbols

- The symbol of a transcendentality-k function is an element of the k-fold tensor product of rationals
(Goncharov, Spradlin, Vergu, Volovich)

  - $f^{(k)} \longrightarrow \mathcal{S}[f^{(k)}] \;=\; R_1 \otimes \cdots \otimes R_k$

"fingerprints" of the function
(a linear combination of
elementary tensors)

- Recursive definition:

  - $df^{(k)} \;=\; \sum_a f_a^{(k-1)}\, d\log R_a \qquad \Longrightarrow \qquad \mathcal{S}[f^{(k)}] \;=\; \sum_a \mathcal{S}[f_a^{(k-1)}] \otimes R_a$

- Two key properties:

  - $\cdots \otimes R_a\, R_b \otimes \cdots \;=\; \cdots \otimes R_a \otimes \cdots + \cdots \otimes R_b \otimes \cdots$
  - $\cdots \otimes c\, R_a \otimes \cdots \;=\; \cdots \otimes R_a \otimes \cdots$ where c = constant

- Examples:

  - $\mathcal{S}\,[\log x] = x\,,\quad \mathcal{S}\,[\mathrm{Li}_2\,(x)] = -\,((1-x)\otimes x\,),\quad \mathcal{S}\,[\mathrm{Li}_3\,(x)] = -\,((1-x)\otimes x\otimes x)$

  - $\mathcal{S}\,[\log x\,\log y] = x\otimes y + y\otimes x$ (<u>note</u>: $x\otimes y$ is not the symbol of a function)

- The symbol transforms complicated polylogarithmic identities into algebraic ones, e.g.

  - $\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)+\log(z)\log(1-z)-\frac{\pi^2}{6} = 0$ (Euler) translated into

$$-((1-z)\otimes z) - (z\otimes(1-z)) + (1-z)\otimes z + z\otimes(1-z) = 0$$

  - loss of information on π's (beyond-the-symbol terms) and branch cuts where the function has to be evaluated

Step II: constructing the symbol of $\mathcal{R}$

- **Entries:** $(u, v, w, 1\text{-}u, 1\text{-}v, 1\text{-}w)$ $u = s_{12} / q^2$, $v = s_{23} / q^2$, $w = s_{31} / q^2$
  - from inspecting the relevant integrals in Gehrmann & Remiddi (GR)
- **First entry:** (u, v, w) for correct branch cuts
  (Gaiotto, Maldacena, Sever, Vieira)
  - $\mathcal{S}[\mathcal{R}^{(2)}] = \sum_{i,j} P_{i,j}^2 \otimes \mathcal{S}[\mathrm{disc}_{i,j}\mathcal{R}^{(2)}]$ with $P_{ij} := p_i + \ldots + p_j$
  - also satisfied at the GR integral function level
- **Further constraints on entries**
  (Gaiotto, Maldacena, Sever, Vieira; Caron-Huot; Dixon, Drummond, Henn)
  - second & last entries

- The unique symbol satisfying these requirements:

☞

$$
\begin{aligned}
\mathcal{S}^{(2)} \quad = \quad & -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\
& + \text{cyclic permutations}\,.
\end{aligned}
$$

- overall coefficient fixed from numerics for $n=3$
  (from collinear limits for $n>3$)

- coefficients ±1, ±2 (well... -2)

- can we determine uniquely the function with this symbol?

- Yes!

  - $\mathcal{S}^{(2)}$ satisfies a particular relation of Goncharov, Spradlin, Vergu & Volovich:

$$\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$$

  - $\Rightarrow$ can re-express as a linear combination of classical polylogarithms only

$$\log x_1 \log x_2 \log x_3 \log x_4\ ,\ \mathrm{Li}_2(x_1) \log x_2 \log x_3\ ,\ \mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\ ,\ \mathrm{Li}_3(x_1) \log x_2 \text{ and } \mathrm{Li}_4(x_i)$$

  - we find the following arguments:

$$\left(u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right)$$

- Final answer fits on one line (for appropriately chosen fonts):

- Final answer: (Brandhuber, GT, Yang)

$$\mathcal{R}_3^{(2)} = -2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i^{-1})\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4$$

  - $u_1 = u\,,\ u_2 = v\ ,\ u_3 = w$
  - $J_4(z) := \mathrm{Li}_4(z) - \log(-z)\mathrm{Li}_3(z) + \frac{\log^2(-z)}{2!}\mathrm{Li}_2(z) - \frac{\log^3(-z)}{3!}\mathrm{Li}_1(z) - \frac{\log^4(-z)}{48}\ .$
  - Block-Wigner-Ramakrishnan(-Zagier) polylogarithmic function
  - no Goncharov polylogarithms!

- Next: QCD

Higgs amplitudes in QCD

- Higgs + 3 partons (Koukoutsakis 2003; Gehrmann, Glover, Jaquier & Koukoutsakis 2011)
  - $H\, g^+\, g^-\, g^-$ MHV
  - $H\, g^+\, g^+\, g^+$ maximally non-MHV
  - $H\, q\ \ \bar{q}\ \ g$ fundamental quarks

$$F^{\rm tree}(H, g_1^-, g_2^-, g_3^+) = \frac{\langle 1\, 2\rangle^2}{\langle 2\, 3\rangle\, \langle 3\, 1\rangle}$$

$$F^{\rm tree}(H, g_1^+, g_2^+, g_3^+) = \frac{q^4}{[1\, 2]\, [2\, 3]\, [3\, 1]}$$

$$q^2 = M_H^2$$

- In N=4 SYM:
  - $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$ both derived from super form factor
  - from supersymmetric Ward identities: (Brandhuber, GT, Yang)

$$\frac{F^{(L)}(g_1^-, g_2^-, g_3^+)}{F^{\rm tree}(g_1^-, g_2^-, g_3^+)} = \frac{F^{(L)}(g_1^+, g_2^+, g_3^+)}{F^{\rm tree}(g_1^+, g_2^+, g_3^+)} = \mathcal{G}^{(L)}(u, v, w) \quad \leftarrow \text{what we computed}$$

- QCD answer from Gehrmann, Glover, Jaquier & Koukoutsakis :
  - expressed in terms of a few pages of Goncharov polylogarithms
  - entirely expected because of expansion as $\sum$ (coefficient x integral) !
    - e.g. scalar non-planar double box does not satisfy the Goncharov et al criterion
- Next, relate N=4 form factors to Higgs amplitudes:
  - take maximally transcendental piece of $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$

- We find a surprising relation...

$$\boxed{\mathcal{R}^{(2)}_{H\,g^-g^-g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{H\,g^+g^+g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{\mathcal{N}=4\,\text{SYM}}}$$

!

  - from symbols and numerics
  - all Goncharov polylogarithms in QCD results can be eliminated in favour of classical polylogarithms
  - we don't know why!

- Nothing similar seems to hold for the $(H,q,\bar{q},g)$ form factor

  - maximally transcendental part does not satisfy Goncharov et al criterion

- Final surprise: amplitude vs form factor remainders

  - the six-point MHV amplitude remainder is built out of six variables $(u, v, w; y_u, y_v, y_w)$:

    - cross ratios: $u := \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,, \quad v := \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}\,, \quad w := \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$

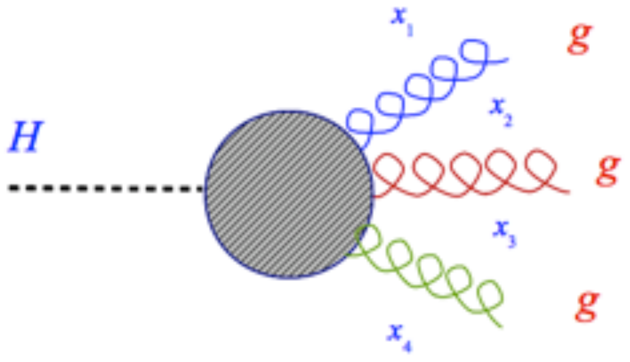
    - y variables: $y_u := \frac{u - z_+}{u - z_-}\,, \quad y_v := \frac{v - z_+}{v - z_-}\,, \quad y_w := \frac{w - z_+}{w - z_-}$

$$z_\pm := \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]\,, \quad \Delta := (1 - u - v - w)^2 - 4\,uvw$$

  - three-point form factor variables:

$$u := \frac{x_{13}^2}{x_{14}^2}\,, \quad v := \frac{x_{24}^2}{x_{14}^2}\,, \quad w := \frac{x_{34}^2}{x_{14}^2}$$

$$u + v + w = 1$$



- Symbol of 6-pt MHV amplitude remainder has two parts:

$$\mathcal{S}^{(2)}_{6,\,\mathrm{ampl}} = \hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w) + \tilde{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w;y_u,y_v,y_w)$$

  - both $\hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w)$ and $\tilde{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w;y_u,y_v,y_w)$ have trivial collinear limits (independently)

- We find: $\boxed{\mathcal{S}^{(2)}_{3,\,\mathrm{form\ factor}}(u,v,w) = \hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w)}$

  - identify the (independent) cross ratios (u,v,w) with the (dependent) form factor ratios (u,v,w)
  - In general, form factor remainder depends on $3n$ - 7 ratios, amplitude remainder depends on $3n$ - 15 cross ratios
  - Furthermore (unpublished observation, Dixon & Duhr)

$$\tilde{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w;y_u,y_v,y_w)\Big|_{u+v+w=1} = 0 \qquad !$$

Summary

- Hidden structures in (amplitudes &) form factors

- Form factors in N=4 super Yang-Mills

- Three-point form factor in N=4 super Yang-Mills & QCD
  - remainder function from symbols and explicit calculations
  - relation to Higgs + multi-gluon QCD remainder...
  - ...and to the N=4 six-point MHV remainder

Open questions

- Further relations between amplitude and form factor remainders ? is this just an accident ?
- More loops, more legs
- Further applications of symbol to QCD?
- Connection to correlation functions
- Recursion relations for form factors integrands? Grassmannians?
- Symmetries of form factors?
- Go beyond symbols...
- arguments of polylog functions?
- applications to other superconformal theories, e.g. ABJM