Making Up for Lost Time: Emergent Time and the M5-Brane

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Strong Fields, Strings and Holography

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C. Hull and NL., to appear

Outline

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Introduction

Our beloved M5-brane remains very mysterious. Low-energy, decoupled, dynamics governed by a 6D theory with:

- $(2,0)$ supersymmetry
- conformal invariance
- $\mathrm{SO}(5)$ R-symmetry

Multiplet contains 5 scalars and a selfdual antisymmetric 3-form field strength + fermions

Very rich and novel 6D CFT dual to $AdS_7 \times S^4$

Strong Coupling, UV completion of 5D SYM

Various related conjectures based on lower dimensions:

- DLCQ of QM on instanton moduli space [Aharony, Berkooz, Kachru, Seiberg, Silverstein]
- Deconstruction from D=4 SCFT [Arkani-Hamed,Cohen,Karch,Motl]
- Strong coupling limit of 5D SYM [Douglas],[NL,Papageorgakis,Schmidt-Sommerfeld]
- 5D SYM on $\mathbb{R} \times \mathbb{C}P^2$ [Kim,Lee]

Also 6D approaches

- [Chu],[Ho, Huang, Matsuo],[Saemann, Wolf],[Samtleben, Sezgin, Wulf][Bonetti, Grimm, Hohenegger],[Bandos, Samtleben,Sorokin].

Here we will continue looking at the M5 from lower-dimensional field theories. This time using 5D Euclidean SYM

Can be obtained from dimensional reduction of 5+5D MSYM

- 16 supersymmetries
- $SO(5)$ rotational symmetry (and translations)
- $SO(5)$ R-symmetry

Conjectured by [Hull],[Hull,Khuri] to arise as the theory of E5-branes arising from timelike reduction of M-theory.

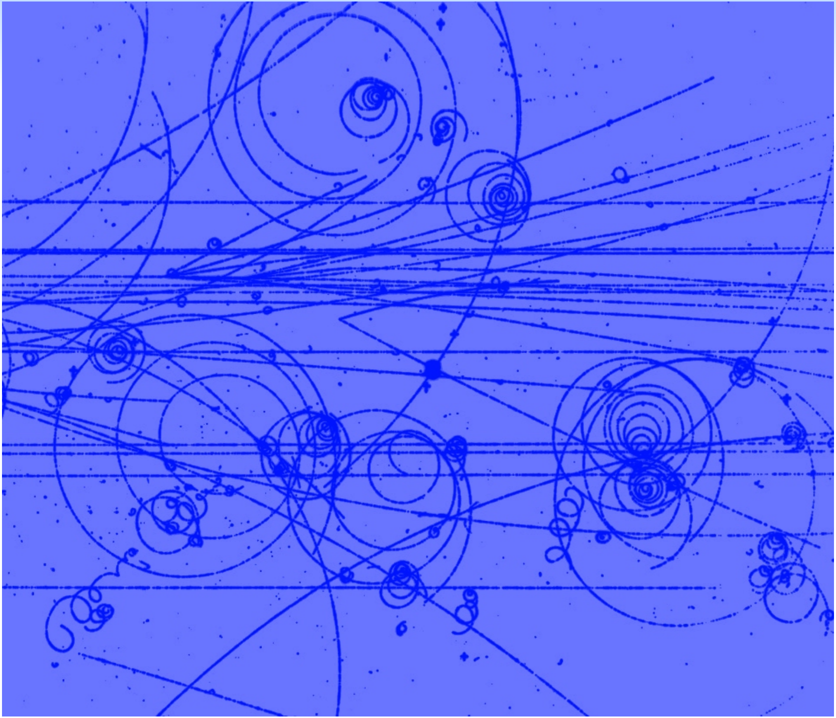
We will also construct it from an explicit relalization of the $(2,0)$ system given by [NL, Papageogakis]

We will show that it has a hidden $SO(5,1)$ symmetry acting on it string soliton states.

So this Euclidean theory knows about time and dynamics.

- Sees entire worldline/worldsheet
- can define energy and momentum which respect $SO(5,1)$.
- Analogous to time-independent Schrödinger equation

We should view it like one views bubble chamber tracks.



A Little Review

At linearized level the free susy variations of the (2,0) theory are

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon \\ \delta H_{\mu\nu\lambda} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi\ , \end{aligned}$$

and the equations of motion are those of free fields with $dH = 0$ (and hence $dH = d\star H = 0$).

Reduction to the D4-brane theory sets $\partial_5 = 0$ and

$$F_{\mu\nu} = H_{\mu\nu5}$$

We wish to generalise this algebra to nonabelian fields with

$$D_\mu X^I_a = \partial_\mu X^I_a - A_{\mu}{}^{b}{}_{a} X^I_b$$

Thus we need a term in $\delta\Psi$ that is quadratic in X^I and which has a single Γ_μ:

- need to invent a field C^μ

N.B. In the original formulation $C^\mu \to C^\mu_a$

- Lie algebra structure constants $f^{ab}{}_c \to f^{abc}{}_d$ - 3-algebra structure constants.

After starting with a suitably general anstaz we find closure of the susy algebra implies [NL, Papageorgakis]

$$\begin{aligned}
\delta X^I_a &= i\bar{\epsilon}\Gamma^I\psi_a \\
\delta\psi_a &= \Gamma^\mu\Gamma^I\epsilon D_\mu X^I_a + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}\epsilon H^{\mu\nu\lambda}_a - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}\epsilon C^\lambda X^I_c X^J_d f^{cd}{}_a \\
\delta H_{\mu\nu\lambda\ a} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\psi_a + i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa X^I_c\psi_d f^{cd}{}_a \\
\delta A_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_{\mu\lambda}C^\lambda\psi_d f^{db}{}_a \\
\delta C^\mu &= 0\,,
\end{aligned}$$

N.B. 3-algebra version can be rephrased in terms of

- Loop Groups [Papageorgakis, Saemann]
- Lie-Crossed-Modules of Gerbes [Palmer, Saemann]

The algebra closes with the on-shell conditions [NL, Papageorgakis]

$$
\begin{aligned}
0 &= \Gamma^\mu D_\mu \psi_a + X^I_c C^\nu \Gamma_\nu \Gamma^I \psi_d f^{cd}{}_a \\
0 &= D^2 X^I_a - \frac{i}{2}\bar{\psi}_c C^\nu \Gamma_\nu \Gamma^I \psi_d f^{cd}{}_a + C^\nu C_\nu X^J_c X^J_e X^I_f f^{ef}{}_d f^{cd}{}_a \\
0 &= D_{[\mu} H_{\nu\lambda\rho]\ a} + \frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma X^I_c D^\tau X^I_d f^{cd}{}_a + \frac{i}{8}\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\sigma \bar{\psi}_c \Gamma^\tau \psi_d f^{cd}{}_a \\
0 &= F_{\mu\nu}{}^b{}_a - C^\lambda H_{\mu\nu\lambda\ d} f^{db}{}_a \\
0 &= \partial_\mu C^\nu \\
0 &= C^\rho D_\rho X^I_d = C^\rho D_\rho \psi_d = C^\rho D_\rho H_{\mu\nu\lambda\ a}\ .
\end{aligned}
$$

Thus C^μ picks out a fixed direction in space and in the 3-algebra and $C^\mu D_\mu = 0$.

So apparently we are simply pushed back to 5D. But not so *e.g.* [NL, Richmond]

$$
\begin{aligned}
T_{\mu\nu} =& D_\mu X^I_a D_\nu X^{Ia} - \frac{1}{2}\eta_{\mu\nu} D_\lambda X^I_a D^\lambda X^{Ia} \\
&+ \frac{1}{4}\eta_{\mu\nu} C^\lambda X^I_a X^J_c C_\lambda X^I_f X^J_e f^{cda} f^{ef}{}_d + \frac{1}{4} H_{\mu\lambda\rho\, a} H_\nu{}^{\lambda\rho\, a} \\
&- \frac{i}{2}\bar{\psi}_a \Gamma_\mu D_\nu \psi^a + \frac{i}{2}\eta_{\mu\nu}\bar{\psi}_a \Gamma^\lambda D_\lambda \psi^a + \frac{i}{2}\eta_{\mu\nu}\bar{\psi}_a C^\lambda X^I_c \Gamma_\lambda \Gamma^I \psi_d f^{acd}
\end{aligned}
$$

And we also obtain 6D expressions for the central charges:

$$
\{Q_\alpha, Q_\beta\} = -2P_\mu(\Gamma^\mu C^{-1})_{\alpha\beta} + \ldots
$$

Thus the system is 6D, with a compact direction:

$$
C^\mu P_\mu = \int C^\mu T_{0\mu} \sim \mathrm{Tr}\int F\wedge F \in 8\pi^2\mathbb{Z}
$$

There are three cases for C^μ of interest:

1) Spacelike: $C^\mu = g^2\delta^\mu_5$ and the previous system reduces to 4+1D SYM [NL, Papageorgakis]

$$P_5 = -\frac{1}{2g_{YM}^2}\int \mathrm{tr}(F\wedge F) = \frac{k}{R_5}$$

So 5D MSYM knows about an extra spatial dimension through a KK-like tower of solitonic states carrying instanton number [Rozali].

Analogous to how the extra transverse momentum of M2-branes arises from magnetic flux in BLG/ABJM:

$$P_{11} = \frac{k}{2\pi}\int F$$

2) Null: $C^\mu = g^2 \delta^\mu_+$. Here $F_{ij} = g^2 H_{+ij}$ is anti-self-dual (H_{-ij} is self-dual). The previous system reduces to 1D motion on instanton moduli space with x^- acting as time [NL, Richmond]:

$$P_+ = -\frac{1}{2g_{YM}^2} \int \mathrm{tr}(F \wedge F) = \frac{k}{R_+}$$

One can then construct the Hamiltonian from T_{--} and quantize to obtain the DLCQ prescription of [Aharony, Berkooz, Kachru, Seiberg, Silverstein]

All 6D quantities (P_μ, Z^I_μ,...) have expressions in term of the ADHM data of instanton moduli space.

Timelike reduction

3) Timelike: $C^\mu = g^2\delta^\mu_0$ which we now consider

The constraint $C^\mu D_\mu = 0$ means that there is no explicit time dependence. The equations of motion all follow from the euclidean action

$$S = -\mathrm{tr}\int d^5x - \frac{1}{4g^4}F_{ij}F^{ij} + \frac{1}{2}D_iX^ID^iX^I + \frac{g^4}{4}[X^I, X^J]^2 - \frac{i}{2}\psi^T\Gamma_0\Gamma^iD_i\psi + \frac{1}{2}g^2\psi^T\Gamma^I[X^I,\psi]\,.$$

where

$$F_{ij} = g^2H_{0ij}$$

This has $ISO(5)$ euclidean and $SO(5)$ R symmetries

This action is invariant under the supersymmetry

$$\delta X^I = i\bar{\epsilon}\Gamma^I\psi$$

$$\delta\psi = \Gamma^\mu\Gamma^I\epsilon D_\mu X^I + \frac{1}{2g^2}\Gamma^{ij}\Gamma^0\epsilon F_{ij} + \frac{i}{2}g^2\Gamma_0\Gamma^{IJ}[X^I, X^J]$$

$$\delta A_i = -ig^2\epsilon^T\Gamma_i\psi \ .$$

Conserved currents associated with the symmetries

$$T_{ij} = \mathrm{tr}\left(D_iX^ID_jX^I - \frac{1}{2}\delta_{ij}D_kX^ID^kX^I - \frac{1}{4}\delta_{ij}g^4[X^I, X^J]^2 - \frac{1}{g^4}F_{ik}{F_j}^k + \frac{1}{4g^4}\delta_{ij}F_{kl}F^{kl} + fermions\right) \ ,$$

$$J^i = \mathrm{tr}\left(-\frac{1}{2g^2}F_{jk}\Gamma^{jk}\Gamma_0\Gamma^i\psi - D_jX^I\Gamma^j\Gamma^I\Gamma^i\psi + \frac{i}{2}g^2[X^I, X^J]\Gamma_0\Gamma^{IJ}\Gamma^i\psi\right)$$

In addition there is a topologically conserved current

$$K_i = \frac{1}{8g^4}\varepsilon_{ijklm}\mathrm{tr}(F^{jk}F^{lm}) \ .$$

Note that the action is Hermitian but not positive definite

However we can rotate $X^I \to iX^I$ to make the real part of the action positive definite.

- Rotating back produces real correlation functions
- Physically this corresponds to rotating to 10+0 SYM

So the theory we consider is just a different analytic continuation of the usual euclidean YM theory used to define 4+1 YM

In a theory with time conserved currents lead to dynamically conserved quantities

$$q = \int d^5x j_0 \qquad \partial_0 q = 0$$

In a euclidean theory we still find Ward identities that impose the symmetry on correlation functions. But no 'conserved charges' such as energy and momentum.

Consider an alternative evolution where one picks a direction, say $\tau = x^5$, and thinks of τ as a 'time'

$$\hat{q} = \int d^4x j_\tau \qquad \partial_\tau \hat{q} = 0$$

subject to certain boundary conditions.

These hatted charges can be physical in certain circumstances:

1) Consider an extended string-like state along $\tau = x^5$

$$\frac{P_\tau}{l_5} = \int d^4x \; \Theta_{\tau 0} = \hat{P}_0$$

2) Consider a state has been boosted along x^4 and only depends on $x^4 + vt$:

$$P_\tau = \int d^3x dx^4 dx^5 \; \Theta_{\tau 0} = |v| \int d^3x dt d\tau \; \Theta_{\tau 0} = |v| \int dt \hat{P}_\tau$$

We could examine the superalgebra of the associated $\hat{Q}$'s

$$\begin{aligned}\{\hat{Q}_\alpha, \hat{Q}_\beta\} =& -\int d^5x\ (\delta_\epsilon S_\tau C^{-1})_{\alpha\beta} \\ =& 2(\Gamma^\mu C^{-1})_{\alpha\beta}\hat{P}_\mu + (\Gamma^\mu\Gamma^I C^{-1})_{\alpha\beta}\hat{Z}^I_\mu + (\Gamma^{\mu\nu\lambda}\Gamma^{IJ}C^{-1})_{\alpha\beta}\hat{Z}^{IJ}_{\mu\nu\lambda}\end{aligned}$$

If so one finds

$$\begin{aligned}\hat{P}_\tau &= \int d^4x T_{\tau\tau} \\ \hat{P}_0 &= \int d^4x K_\tau \\ \hat{P}_i &= \int d^4x T_{i\tau}\ .\end{aligned}$$

this suggests that we identify $T_{0\tau} = K_\tau$ and more generally

$$T_{0i} = K_i$$

Alternatively, from the results of the superalgebra for generic C^μ one finds

$$\{Q_\alpha, Q_\beta\} = -\int d^5x\ (\delta_\epsilon J^0 C^{-1})_{\alpha\beta}$$
$$= 2(\Gamma^\mu C^{-1})_{\alpha\beta} P_\mu + (\Gamma^\mu \Gamma^I C^{-1})_{\alpha\beta} Z^I_\mu + (\Gamma^{\mu\nu\lambda}\Gamma^{IJ} C^{-1})_{\alpha\beta} Z^{IJ}_{\mu\nu\lambda}$$

where

$$E = \mathrm{tr}\int d^5x \frac{1}{4g^4} F_{ij}F^{ij} + \frac{1}{2} D_i X^I D^i X^I - \frac{g^4}{4}[X^I, X^J]^2$$
$$P_i = \mathrm{tr}\int d^5x \frac{1}{8g^4} \varepsilon_{0ijklm} F_{jk} F^{lm} \ .$$

and, for example,

$$Z^I_i = \frac{2}{g^2}\mathrm{tr}\int d^5x\ D^j(F_{ij}X^I)$$

This defines dynamical ’charges’ even in the euclidean case. But only formally - there is no dynamical Poisson Bracket.

In particular we will want to consider a compact time (whatever that means) with period $t \cong t + 2\pi R$ and identify the above 5D energy-momentum tensor with the Fourier zero-mode of the 5+1D one:

$$T_{\mu\nu} = \frac{1}{2\pi R} \int dt \Theta_{\mu\nu}$$

Note that by conservation the higher Fourier modes are total spatial derivatives and hence do not contribute to conserved quantities.

We can understand the role of 5+5D MSYM as follows [Hull,Khuri]:

Consider the abelian M5-brane and use

$$S = -\int d^6x \frac{1}{4\cdot 3!} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \frac{1}{2}\partial_\mu X^I \partial^\mu X^I$$

thus

$$\Theta_{\mu\nu} = \frac{1}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + \partial_\mu X^I \partial_\nu X^I - \frac{1}{2!}\eta_{\mu\nu}\partial_\lambda X^I \partial^\lambda X^I$$

Where we do the usual trick of only imposing self-duality on the equations of motion.

Reduction on time, using $F_{ij} = g^2 H_{0ij}$, gives

$$S = -\int d^5x - \frac{1}{4g^4} F_{ij}F^{ij} + \frac{1}{2}\partial_i X^I \partial^i X^I$$

which is the same as the reduction of the 5+5D theory.

$$\begin{aligned}
\Theta_{00} &= \frac{1}{4g^4} F_{ij}F^{ij} + \frac{1}{2}\partial_i X^I \partial^i X^I \\
\Theta_{0i} &= \frac{1}{8g^4} \varepsilon_{0ijklm} F^{jk} F^{lm} \\
\Theta_{ij} &= \frac{1}{g^4} F_{ik} F_j{}^k - \frac{1}{4g^4}\delta_{ij} F_{kl}F^{kl} + \frac{1}{2}\partial_i X^I \partial_j X^I - \frac{1}{2}\delta_{ij}\partial_k X^I \partial^k X^I
\end{aligned}$$

These are the abelian limits of what we have obtained above.

Matching BPS states and a Hidden $SO(5,1)$

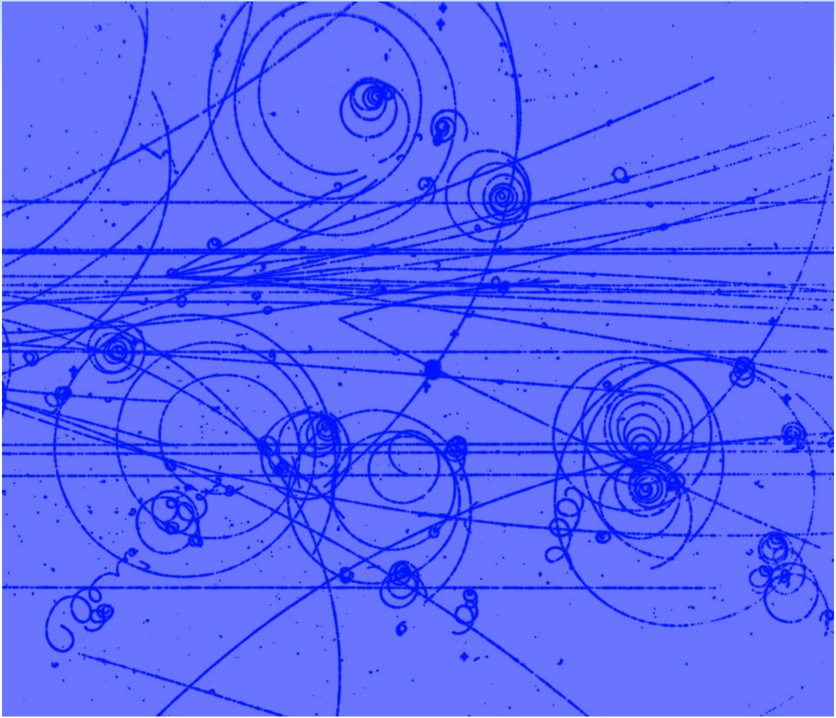
Let us look for simple solutions with energy and momentum. In the symmetric phase the simplest thing is a pure 1/2 BPS pp-wave along x^5

$$F_{ij} = \pm\frac{1}{2}\varepsilon_{ijkl}F^{kl} \qquad i,j \neq 5$$

This has

$$\frac{P_5}{l_5} = \frac{4\pi^2 n}{g^4} \qquad \frac{E}{l_5} = \frac{4\pi^2|n|}{g^4}\ .$$

Since this is a time-averaged image of the process we see the whole world line extended along x^5. As if one took a long exposure photograph of the particle as it traverses along x^5.



Next we look at the Coulomb phase with say $\langle X^6 \rangle \neq 0$. Here we expect string states, extended along say x^5 with $Z_5^6 \neq 0$.

Without momentum we just have non-solitonic states:

$$F_{i5} = \partial_i X^6 \qquad X^6 = \langle X^6 \rangle - \frac{Q_E}{4\pi^2 r^2}$$

and extended along x^5

If we are looking at a genuine 5+1D theory we should see boosted versions of these, say along x^4. What should these look like?

- still 1/2 BPS
- extended along x^5 and x^4

These are BPS Monopoles:

- $F_{ij} = \varepsilon_{ijk} D^k \Phi \qquad A_5 = \frac{1}{v}\Phi \qquad X^6 = \frac{1}{v\gamma g^2}\Phi$
- 1/2 BPS if $\gamma^2 = 1/(1-v^2)$
- Φ is the usual BPS scalar field:

$$\Phi = vg^2\gamma\langle X^6\rangle - \frac{Q_M}{2r} + \mathcal{O}(1/r^2)\ , \qquad e^{i\oint F} = e^{2\pi i Q_M} = 1\ .$$

The 'charges' are

$$\frac{Z_5^6}{l_4 l_5} = -\frac{4\pi}{vg^2}\mathrm{tr}(\langle X^6\rangle Q_M) \quad \frac{P_4}{l_4 l_5} = \frac{2\pi\gamma}{g^2}\mathrm{tr}(\langle X^6\rangle Q_M)$$
$$\frac{E}{l_4 l_5} = \frac{2\pi\alpha^2\gamma}{g^2 v}\mathrm{tr}(\langle X^6\rangle Q_M)$$

and satisfy

$$E^2 = P_4^2 + \frac{1}{4}(Z_5^6)^2\ .$$

This looks like a boosted string whose rest tension is

$$T = \frac{1}{2}\left|\frac{Z_5^6}{l_5}\right| = \frac{2\pi l_4}{g^2|v|}\left|\mathrm{tr}(\langle X^6\rangle Q_M)\right| \ ,$$

But what is the factor of v doing in the denominator?

To see this note that the 'charges' have been evaluated as densities over x^4 and x^5

- x^5 is the length along the string
- but x^4 is the distance traveled in unit time
- so $l_4 = |v|t$, where t is time
- If time is periodic with period $2\pi R$ then $l_4 = 2\pi R|v|$ and the tension is

$$T = \frac{4\pi^2 R}{g^2}\left|\mathrm{tr}(\langle X^6\rangle Q_M)\right|$$

Next we should look at 1/4 BPS excited states of strings extended along x^5 with momentum along x^5

In the rest frame these are 'dyonic instantons' [NL, Tong]

$$F_{ij} = \pm\frac{1}{2}\varepsilon_{ijkl}F^{kl} \qquad F_{i5} = D_i X^6 \qquad i,j \neq 5$$

with $D^2 X^6 = 0$

These have

$$\frac{P_5}{l_5} = \frac{4\pi^2 n}{g^4} \qquad \frac{Z_5^6}{l_5} = -\frac{4\pi^2}{g^2}\mathrm{tr}(\langle X^6\rangle Q_E) \qquad E = |\mathcal{P}_5| + \frac{1}{2}|Z_5^6|$$

So now we look for boosted versions of these:

- still 1/4 BPS
- extended along x^5 and x^4
- 1/4 BPS monopoles (for example see [Bergman],[Lee,Yi])

One finds the generic 1/4 BPS state has

$$F_{ij} = \varepsilon_{ijk} D^k \Phi \qquad X^6 = \frac{v}{g^2} A_4 + \frac{1}{\gamma g^2} A_5 \ , \qquad \Phi = \frac{1}{\gamma} A_4 - v A_5$$

with

$$D^2 \Phi = 0 \qquad D^2 X^6 = [\Phi, [\Phi, X^6]]$$

Solutions have the form

$$\Phi = \frac{\gamma}{v}Y_0 - g^2 v\gamma\langle X^6\rangle - \frac{Q_M}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right)$$
$$X^6 = \langle X^6\rangle - \frac{Q_E}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right)\ ,$$

Requiring that $Z_4^6 = 0$ gives

$$0 = \mathrm{tr}(Y_0 Q_E)$$

and we find

$$\frac{Z_5^6}{l_4 l_5} = -\frac{4\pi}{g^2}\mathrm{tr}(\langle X^6\rangle Q_M)$$

as before.

The momentum and energy are

$$P_4 = -\frac{v}{\sqrt{1-v^2}}\left(|P_5| + \frac{1}{2}|Z_5^6|\right)$$
$$\frac{P_5}{l_5} = \frac{2\pi}{g^2}\mathrm{tr}(Y_0 Q_M)$$
$$E = \frac{1}{\sqrt{1-v^2}}\left(|P_5| + \frac{1}{2}|Z_5^6|\right)$$

which are just the boosted versions of the dyonic instanton expressions $E = |P_5| + \frac{1}{2}|Z_5^6|$

Thus the spectrum of 1/2 and 1/4 BPS string states is consistent with a lorentzian 5+1 D theory

- Strings have tension

$$T = \frac{4\pi^2 R}{g^2} \left|\mathrm{tr}(\langle X^6 \rangle Q_M)\right| \qquad T = \left|\mathrm{tr}(\langle X^6 \rangle Q_E\right|$$

- Excited states carry quantized momentum along the string (at least in the rest frame)

$$\frac{P_5}{l_5} = \frac{4\pi^2 n}{g^4} \qquad n \in \mathbb{Z}$$

Consistent with an emergent time and emergent $SO(5,1)$ symmetry as $g^2 \to \infty$ with $R = g^2/4\pi^2$

Comments

We have shown how 5D euclidean super-Yang-Mills arises as the worldvolume theory of the M5.

Used supersymmetry algebra to define dynamical conserved quantities such as energy and momentum

- leads to a dynamical interpretation

Shown that it possess a hidden $SO(5,1)$ symmetry that acts on string states:

- spectrum is consistent with a 5+1D dynamical theory

We have presented a picture whereby a euclidean field theory knows about time from its topological sectors. As if one were looking at the time-independent Schrödinger equation, i.e. in a basis where E and P_i have been diagonalized.

- Euclidean Correlation functions are at fixed values of E, P_i.

Tried to give a new perspective on the emergence of role of lower-dimensional field theories to describe higher-dimensional ones.

Can be viewed as an emergent time or $SO(5,1)$ Lorentz symmetry at the strong coupling (UV) fixed point c.f. [Horava].

But there are also still various conceptual problems associated

Altogether the $(2,0)$ system of [NL, Papageorgakis] paints a consistent, interconnected picture of the M5-brane in terms of lower dimensional theories

Some important technical points remain

- In DLCQ picture instanton moduli space has singularities (but these are mild orbifold singularities)
- Is 5D SYM well-defined non-perturbatively? We have ignored this here. One approach is the conjecture of [Douglas],[NL, Papageorgakis, Schmidt-Sommerfeld]. But even without this we hope these results will stand