Non-perturbative effects in large N theory and string theory

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Non-perturbative effects in large *N* theories

The large *N* or 't Hooft expansion provides an asymptotic expansion of observables in *(S)U(N)* gauge theories:

free energy
$$F(\lambda, N) = \sum_{g\geq 0} N^{2-2g} F_g(\lambda)$$

$$\lambda = g^2 N$$
't Hooft coupling

There might be exponentially small corrections which are invisible in the 't Hooft expansion

$$\sim \exp\left(-N S(\lambda)\right)$$

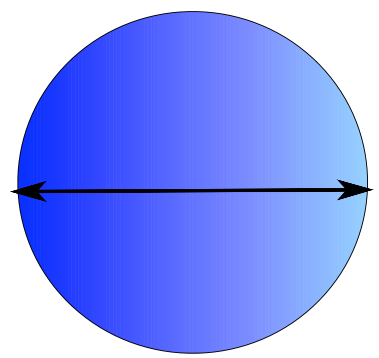
One possible source for these corrections are instantons

$$S(\lambda) = \frac{A}{\lambda} + \cdots$$

Non-perturbative effects in string theory

String theory has *two* coupling constants:

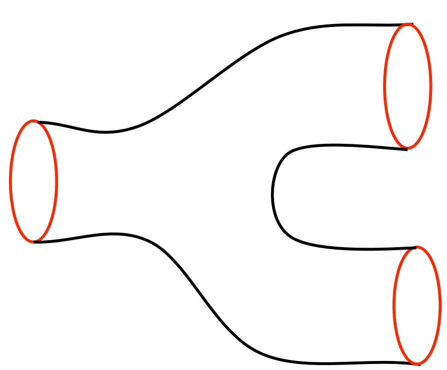
L/ℓ_s



characteristic scale of the target

The *string length* controls fluctuations with a *fixed* worldsheet topology.

g_s



The *string coupling constant* controls emission and absorption of strings, i.e. changes in worldsheet topology

There are therefore two types of non-perturbative effects in string theory:

1) *worldsheet* instantons: $\mathrm{e}^{-A_{\mathrm{ws}}/\ell_s^2}$

action of the worldsheet instanton

These are just standard field theory instantons in the non-linear sigma model describing the propagation of strings with fixed topology

2) *spacetime* instantons: $\mathrm{e}^{-A_{\mathrm{st}}/g_s}$

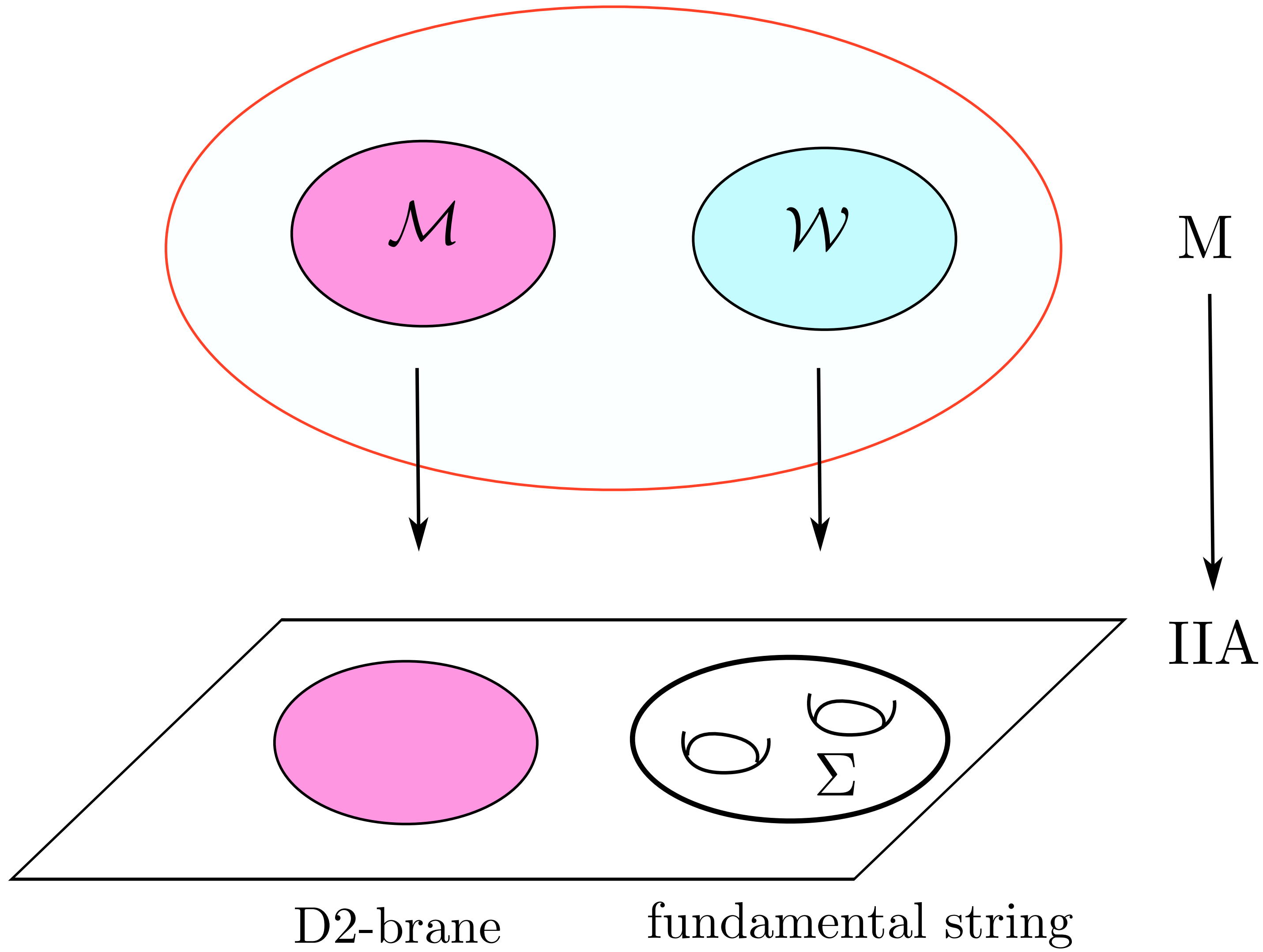
action of the spacetime instanton

Historically these instantons have been more mysterious

In 1994, Polchinski pointed out that, in type II string theory, D-branes would lead to precisely this type of non-perturbative corrections. Shortly afterwards, [Becker-Becker-Strominger] noticed that, in type IIA theory, both worldsheet and spacetime instantons arise from *membrane instantons* in M-theory/11d sugra

Membrane instantons are due to M2 branes wrapping 3-cycles $\mathcal{S}$, and lead to corrections of the form

$$\mathrm{e}^{-\mathrm{vol}(\mathcal{S})/\ell_p^3}$$



However, there are *two* types of three-cycles, depending on whether they wrap or not the M-theory circle. They lead to fundamental strings and D2 branes after reduction to IIA theory, respectively

Using the standard M-theory/type IIA dictionary, one finds

$$\ell_p = g_s^{1/3}\ell_s$$

$$R_{11} = g_s \ell_s$$

$$\mathrm{e}^{-\mathrm{vol}(\mathcal{S})/\ell_p^3}$$

$\mathcal{W}$ $\mathcal{M}$

$$\mathrm{e}^{-\mathrm{vol}(\Sigma)/\ell_s^2} \qquad \mathrm{e}^{-\frac{\mathrm{vol}(\mathcal{M})/\ell_s^3}{g_s}}$$

worldsheet instanton *spacetime* instanton

In a string theory with a large *N* dual, worldsheet instantons should be in principle visible in the ‘t Hooft expansion

$$F_g(\lambda) \to \exp(-f(\lambda)), \quad \lambda \gg 1$$

However, D2-brane instantons correspond to exponentially suppressed effects at large *N* and they can *not* be seen in the ‘t Hooft expansion

Goal of this talk: use large N duality in ABJM theory to learn about non-perturbative effects in string theory

I will derive the full series of non-perturbative effects in the free energy. They display a beautiful mathematical structure related to topological string theory and integrable models

This answer provides quantitative evidence that perturbative strings are radically insufficient, and they need to be supplemented by membranes to obtain a consistent theory

A B J M theory

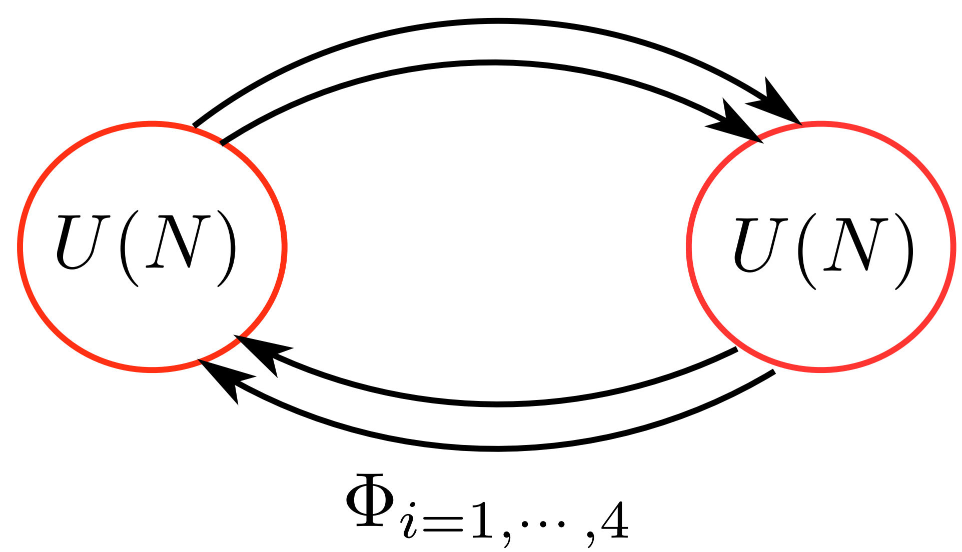
Basic building block: *Chern-Simons theory* and its supersymmetric extensions

CS level (must be an integer)

$$S_{\rm CS} = -\frac{k}{4\pi}\int_M {\rm tr}\Big(A\wedge {\rm d}A + \frac{2{\rm i}}{3}A\wedge A\wedge A\Big)$$

ABJM =2 (super)CS theories +
4 *N*=2 hypers in the bifundamental
N=6 SUSY

$$U(N)_k \times U(N)_{-k}$$



M-theory dual

A Freund-Rubin background with a metric of the form

$$ds^2 = L^2 \left(\frac{1}{4} ds^2_{\mathrm{AdS}_4} + ds^2_{X_7} \right) \qquad X_7 = \mathbb{S}^7 / \mathbb{Z}_k$$

The common rank *N* is related to the radius *L* by

$$6\,\mathrm{vol}(X_7) \left(\frac{L}{2\pi \ell_p} \right)^6 = N$$

$L/\ell_p \simeq 1$ → L/ℓ_p → $L/\ell_p \gg 1$

Planckian sizes, strong quantum gravity effects

N small

weak curvature, classical SUGRA is a good approximation

N large: "thermodynamic limit"

The natural expansion in M-theory is in powers of ℓ_p/L

This leads to the *M-theory expansion* of ABJM theory: a *1/N* expansion at *fixed k*. This is *not* the 't Hooft expansion, which is an expansion in powers of *1/N* at *fixed* 't Hooft parameter

$$\lambda = \frac{N}{k}$$

Type IIA string theory duals

This M-theory background has a type IIA reduction to a background

$$\mathrm{AdS}_4 \times \mathbb{CP}^3$$

We now have a *two-parameter theory,* since the geometric parameter *k* becomes the string coupling constant. The dictionary with the gauge theory is:

$$\left(\frac{L}{\ell_s}\right)^4 = 32\pi^2\lambda$$

$$g_{\rm st} = \frac{1}{k}\left(\frac{L}{\ell_s}\right) \propto \frac{\lambda^{5/4}}{N}$$

$$\lambda = \frac{N}{k}$$

The natural spacetime expansion in type IIA theory is the *genus expansion* (in powers of the string coupling constant) at a given curvature radius. In ABJM theory this corresponds precisely to the 't Hooft $1/N$ expansion.

We conclude that there are *two possible expansions* in ABJM theory, making contact with M-theory and type IIA theory, respectively

Localization and matrix models

In SUSY theories one can sometimes reduce the path integral to an integral over SUSY vacua [Witten]. This phenomenon is called *localization*, but it only applies to quantities which are "SUSY enough". For example, the thermal partition function breaks SUSY and it won't do.

Using this principle, it can be shown that the (Euclidean) *partition function* on the three-sphere of ABJM theory can be computed in terms of a matrix model [Kapustin-Willett-Yaakov]

$$Z(N,k) = \frac{1}{N!^2} \int \frac{\mathrm{d}^N \mu}{(2\pi)^N} \frac{\mathrm{d}^N \nu}{(2\pi)^N} \prod_{i<j} \left[2\sinh\left(\frac{\mu_i - \mu_j}{2}\right) \right]^2$$

$$\prod_{i<j} \left[2\sinh\left(\frac{\nu_i - \nu_j}{2}\right) \right]^2 \prod_{i,j} \left[2\cosh\left(\frac{\mu_i - \nu_j}{2}\right) \right]^{-2} \exp\left[\frac{\mathrm{i}k}{4\pi} \sum_{i=1}^{N} (\mu_i^2 - \nu_i^2) \right]$$

We have reduced the problem drastically, from a field theory path integral to a matrix integral. By AdS/CFT, this computes the *exact* Euclidean partition function of the dual M-theory/type IIA string theory

However, to extract "gravitational" physics, i.e. to see the *emergence of gravity and strings*, we have to study this matrix model in an (asymptotic) expansion at large *N*

Calculating at large *N*: ‘t Hooft expansion

The ABJM matrix model is closely related to the *Chern-Simons matrix models* introduced in [M.M.‘02] and it can be studied in the ‘t Hooft expansion by using well-known large *N* technology (resolvents, topological recursion, holomorphic anomaly, etc.)

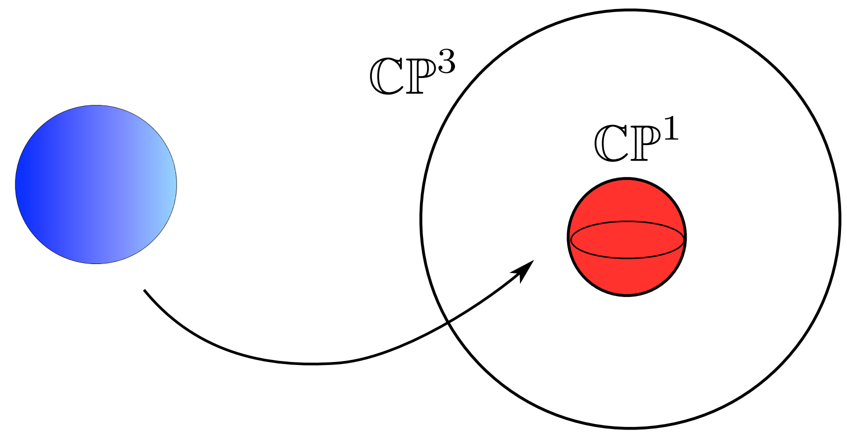
In this way one can calculate the free energies at *all genera*, i.e. the full 1/*N* expansion [Drukker-M.M.-Putrov]

$$\log Z(N,k) = \sum_{g=0}^{\infty} N^{2-2g} F_g(\lambda)$$

worldsheet instantons!

$$F_0(\lambda) = -\frac{\pi\sqrt{2}}{3\sqrt{\lambda}} + \mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\lambda}}\right) \longleftarrow \sim \mathcal{O}\left(\mathrm{e}^{-L^2/\ell_s^2}\right)$$

$N^{3/2}$ behaviour



At higher genus there is a similar structure: a finite perturbative piece in λ , plus an infinite series of worldsheet instanton corrections.

This *1/N* expansion can be however *resummed* in order to make contact with M-theory. This is because the ABJM matrix model calculates as well the free energy of topological strings on a non-compact CY manifold known as (diagonal) local $\mathbb{P}^1 \times \mathbb{P}^1$ [M.M.-Putrov]

In M-theory the most natural object turns out to be the *grand potential*, i.e. we should go to the grand canonical ensemble

$$J(\mu, k) = \log\left(1 + \sum_{N=1}^{\infty} Z(N,k) \mathrm{e}^{N\mu}\right) \longleftrightarrow \quad Z(N,k) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\mu \, \mathrm{e}^{J(\mu,k) - \mu N}$$

$$J^{\text{'t Hooft}}(\mu, k) = J^{\mathrm{p}}(\mu, k) + J^{\mathrm{WS}}(\mu, k) \longleftarrow$$ *worldsheet instantons!*

The perturbative piece encodes the $N^{3/2}$ behaviour

$$J^{\mathrm{p}}(\mu, k) = \frac{2\mu^3}{3\pi^2 k} + \mathcal{O}(\mu) \qquad \mu \sim \sqrt{kN}$$

$$J^{\mathrm{WS}}(\mu,k)=\sum_{g\geq 0}\sum_{w,d\geq 1} n_g^d \left(2\sin\frac{2\pi w}{k}\right)^{2g-2}\frac{(-1)^{dw}}{w}\mathrm{e}^{-\frac{4dw\mu}{k}}$$

$$\sim \mathrm{e}^{-2\pi\sqrt{2\lambda}}$$

GV invariants of local $\mathbb{P}^1\times\mathbb{P}^1$

This resums, order by order in $\mathrm{e}^{-2\pi\sqrt{2\lambda}}$, the genus expansion. It is the *free energy of topological string theory* on local $\mathbb{P}^1\times\mathbb{P}^1$ at large radius, and in Gopakumar-Vafa form

$\mu/k\sim$ Kahler parameter $\qquad k\sim 1/g_{\mathrm{top}}$

However, it has poles at *all integer values of k* and cannot be the final answer: the matrix model is finite for all real *k*. The poles rather indicate a *breakdown of the genus expansion*

We have to go beyond the 't Hooft expansion

The Fermi gas approach and the M-theory expansion

In this approach, $Z(N,k)$ is interpreted as the canonical partition function of an *ideal Fermi gas* of *N* particles with a non-trivial one-particle Hamiltonian defined by

$$e^{-H} = e^{-\frac{1}{2}U(\hat{q})} e^{-T(\hat{p})} e^{-\frac{1}{2}U(\hat{q})}$$

with $$U(q) = \log\left(2\cosh\frac{q}{2}\right), \quad T(p) = \log\left(2\cosh\frac{p}{2}\right)$$

$\hat{q}, \hat{p}$ are position and momentum operators

$$[\hat{q},\hat{p}] = \mathrm{i}\hbar, \quad \hbar = 2\pi k$$

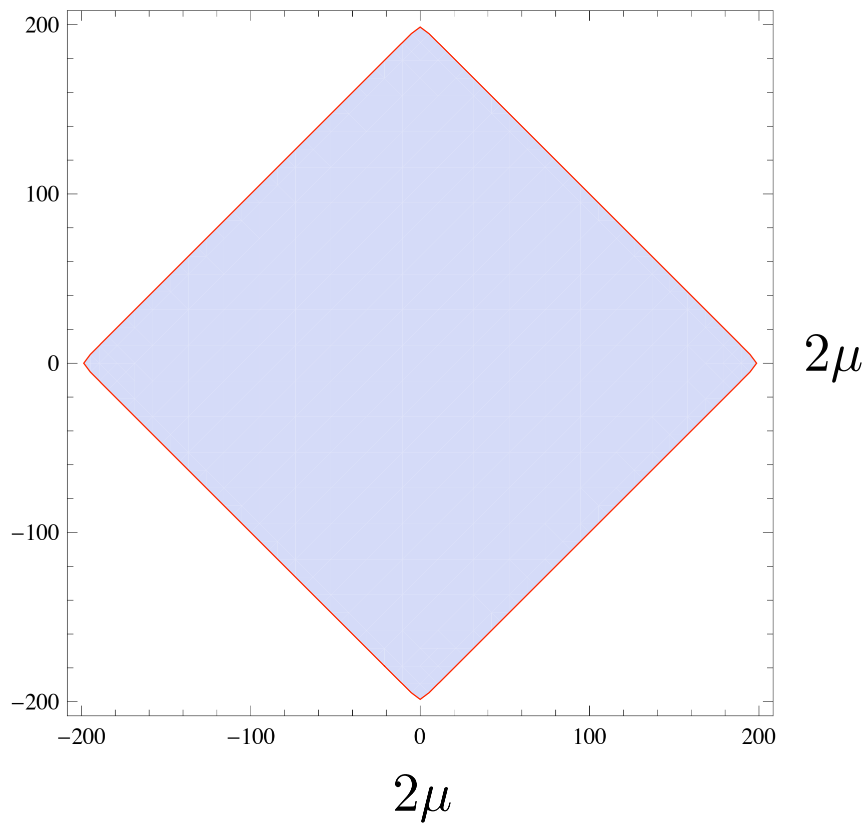
SEMICLASSICS = *STRONG* STRING COUPLING!!

The semiclassical Fermi surface of this gas is the *spectral curve* which gives the mirror geometry of local $\mathbb{P}^1 \times \mathbb{P}^1$

$$4\cosh(q/2)\cosh(p/2) - \mathrm{e}^{\mu} = 0$$

At large μ: ultra-relativistic Fermi gas in a linearly confining potential

Fermi surface: $$\frac{|p|}{2} + \frac{|q|}{2} \approx \mu$$



$$J(\mu) \approx \frac{2\mu^3}{3\pi^2 k}$$

" $N^{3/2}$ on the back of an envelope"

One can compute the grand potential of this Fermi gas by using standard methods in Statistical Mechanics (Wigner-Kirkwood expansion) or a TBA system [Zamolodchikov, Tracy-Widom]

$$J^{\rm Fermi}(\mu,k) = J^{\rm p}(\mu,k) + J^{\rm M2}(\mu,k)$$

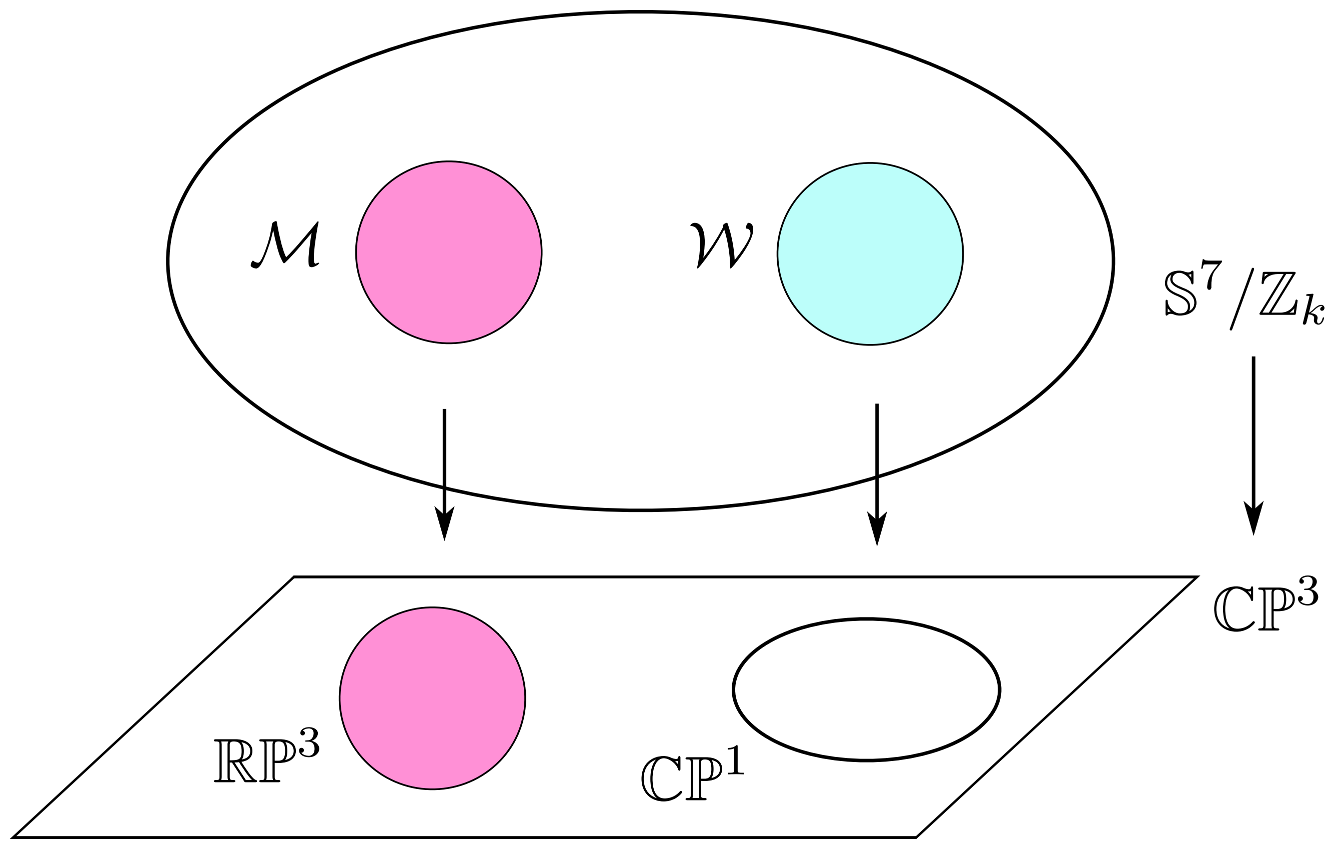
$$J^{\rm M2}(\mu,k) = \sum_{\ell\geq 1} \left(a_\ell(k)\mu^2 + b_\ell(k)\mu + c_\ell(k)\right) {\rm e}^{-2\ell\mu}$$

$\sim {\rm e}^{-N/\sqrt{\lambda}}$ non-perturbative effects in the *1/N* expansion

$\sim {\rm e}^{-(L/\ell_p)^3}$ D2-brane/membrane instantons

The above coefficients can be computed explicitly, so we have *quantitative* control of non-perturbative effects in the string coupling!

The ‘t Hooft expansion gives quantitative control of worldsheet instantons, while the Fermi gas gives control of D2-brane instantons, which have geometric realizations [Drukker-M.M.-Putrov]

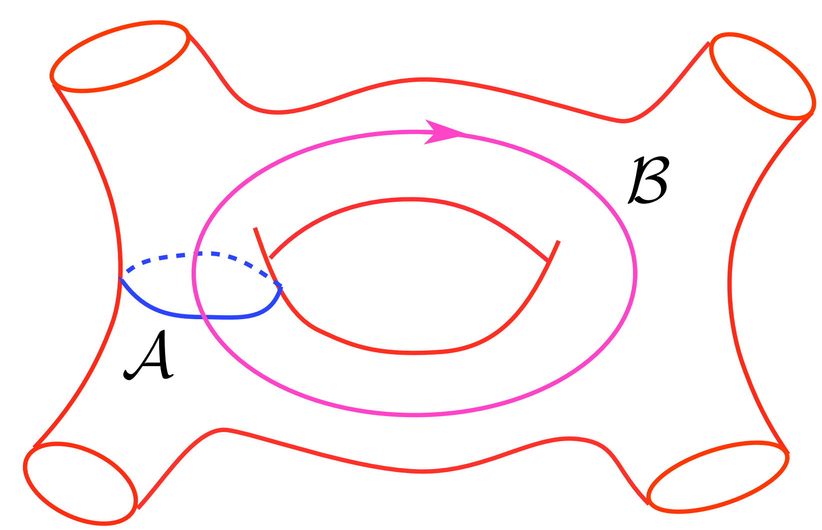


Quantum periods and membrane instantons

Since we have an ideal Fermi gas, we should focus on the Schrodinger problem for a single fermion

$$\mathrm{e}^{H(q,-\mathrm{i}\hbar\partial_q)}\psi(q) = \mathrm{e}^{E}\psi(q)$$

The WKB solution is given by periods of the *complexified* Fermi surface



Bohr-Sommerfeld

$$\frac{1}{\hbar}\oint_{\mathcal{A}+\mathcal{B}} p\mathrm{d}q = n$$

perturbative $\hbar$ corrections

$$\frac{1}{\hbar}\oint_{\mathcal{A}+\mathcal{B}} p(\hbar)\mathrm{d}q = n$$

quantum periods of the mirror spectral curve=Nekrasov-Shatashvili (NS) limit of the refined topological string!

[Mironov-Morozov, Aganagic et al.]

worldsheet instantons in ABJM: unrefined topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$

membrane instantons in ABJM: NS limit of *refined* topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$

$\mu/k \sim$ Kahler parameter $\qquad k \sim \hbar \sim 1/g_{\rm top}$

$$J^{\rm M2}(\mu, k) = \sum_{\ell \geq 1} \left(a_\ell(k)\mu^2 + b_\ell(k)\mu + c_\ell(k) \right) {\rm e}^{-2\ell\mu}$$

$a_\ell(k)$: quantum A-period

$b_\ell(k)$: quantum B-period

$$a_1(k) = -\frac{4}{\pi^2 k} \cos\left(\frac{\pi k}{2}\right),$$

$$b_1(k) = \frac{2}{\pi} \cos^2\left(\frac{\pi k}{2}\right) \csc\left(\frac{\pi k}{2}\right),$$

$$c_1(k) = \left[-\frac{2}{3k} + \frac{5k}{12} + \frac{k}{2} \csc^2\left(\frac{\pi k}{2}\right) + \frac{1}{\pi} \cot\left(\frac{\pi k}{2}\right) \right] \cos\left(\frac{\pi k}{2}\right)$$

poles (in $b_1(k)$ and $c_1(k)$)

The HMO cancellation mechanism

$$J(\mu,k) = J^{\rm p}(\mu,k) + J^{\rm WS}(\mu,k) + J^{\rm M2}(\mu,k) + J^{\rm bound}(\mu,k)$$

Poles at integer *k* *cancel*: divergences in worldsheet instantons are cured by divergences in membrane instantons and bound states! [Hatsuda-Moriyama-Okuyama]

Conceptual corollary: perturbative strings are *radically insufficient* at strong coupling. One *needs* membrane instantons to make sense of the theory.

Note: the contribution of bound states was guessed by HMO based on the cancellation mechanism, since it is non-perturbative in both approaches. It was however tested in detail against numerical studies of the large *N* asymptotics of the matrix model

‘t Hooft expansion	Fermi gas
spectral curve	Fermi surface, Hamiltonian
perturbative: worldsheet instantons	perturbative: membrane instantons
non-perturbative: membrane instantons and bound states	non-perturbative: worldsheet instantons and bound states
coupling 1/k	coupling k

Consequences for topological strings

The calculation of the grand potential of ABJM theory can be also regarded as a *first-principles* calculation of non-perturbative effects for the free energy of topological strings on local P1xP1

The non-perturbative *definition* of the free energy is given by the ABJM matrix model, and the above calculation gives its complete large radius asymptotics. It includes non-perturbative effects in the topological string coupling constant given by

$$J^{\mathrm{M2}}(\mu, k) + J^{\mathrm{bound}}(\mu, k)$$

Reminiscent of (but different from) a recent proposal by [Lockhart-Vafa]

The Fermi gas approach also suggests a new point of view on topological strings on local Calabi-Yau (CY) manifolds

Conjecture: Consider the quantized spectral curve/mirror of the local CY, encoding the NS limit of the refined topological string. This gives a well-defined Hamiltonian problem. The non-perturbative corrections (in $\hbar$) to this problem give the conventional worldsheet instantons of the topological string, with coupling $g_{\rm top} = 1/\hbar$

It follows from the analysis of the ABJM matrix model that this is *true* for local P1xP1. If true in general, this would give a beautiful non-perturbative definition of topological strings, in a sort of S-dual frame.

Conclusions and prospects

- We have determined the *full set of non-perturbative corrections to the ABJM partition function* in terms of topological string theory: the conventional topological string gives worldsheet instantons, and the NS refined topological string gives membrane instantons

- The $1/N$ or perturbative genus expansion is radically insufficient: at strong coupling, large N instantons/membrane instantons are *required* for consistency

- As a bonus, this gives a first-principles calculation of non-perturbative effects in a topological string model and suggests using the NS limit as a non-perturbative definition of topological string theory

- Can we generalize these results to other Chern-Simons-matter/topological string models?