Quantum black hole entropy and the holomorphic prepotential

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Black hole entropy is a precious clue to understand quantum gravity

$$k_B \log d_{\text{micro}} = S_{\text{BH}}^{\text{class}} + \cdots$$

(Boltzmann)

Universal law in GR

$$S_{\text{BH}}^{\text{class}} = \frac{1}{4}\frac{A_{\text{H}}}{\ell_{\text{P}}^2} = \frac{A_{\text{H}}\, c^3}{4\,\hbar\, G_N}$$

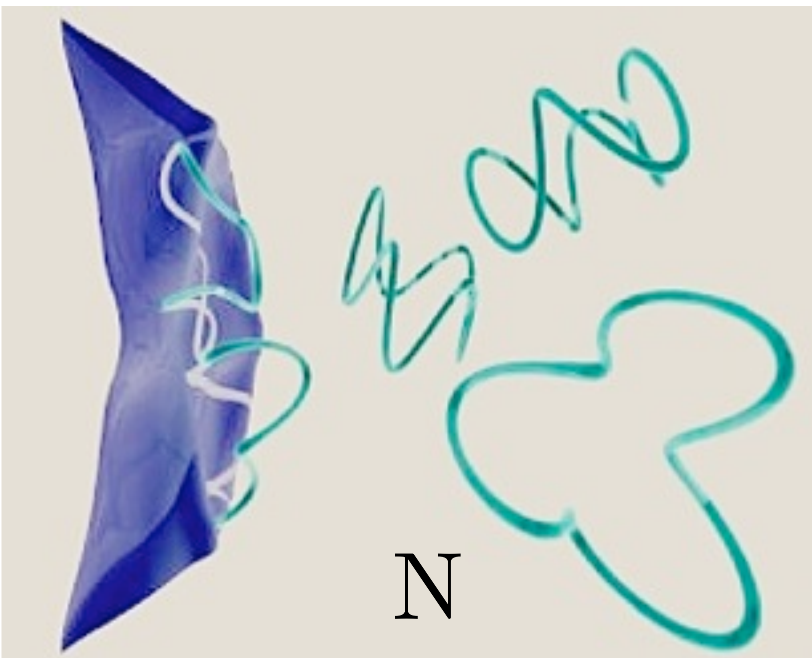
(Bekenstein-Hawking ’74)

Deviations from GR!

Recent progress on this front

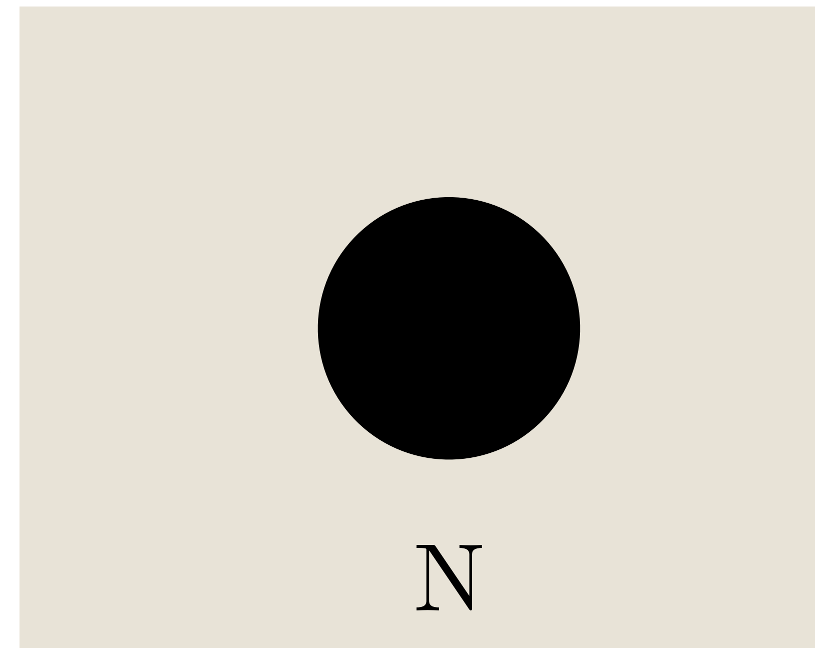
Black holes in string theory are an ensemble of microscopic excitations

Microscopic



$g_s \mathrm{N} \ll 1$ ← g_s → $g_s \mathrm{N} \gg 1$

Macroscopic



Strominger-Vafa '96

$$d_{\mathrm{micro}}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

Bekenstein-Hawking '74

$$\frac{A_{\mathrm{H}}(\mathrm{N})}{4} = \pi\sqrt{\mathrm{N}}$$

$$\log d_{\mathrm{micro}} = S_{\mathrm{BH}}^{\mathrm{class}} + \cdots \rightarrow S_{\mathrm{BH}}^{\mathrm{quant}} \text{ (finite N)}$$

What is new? Finite size quantum effects!

$$S_{\rm BH}^{\rm quant} = \frac{1}{4}A + a_0 \log(A) + a_1 \frac{1}{A} + a_2 \frac{1}{A^2} + \cdots$$
$$+ b_1(A) e^{-A} + \cdots$$

Questions

1. What is the physics of these corrections? → Exact AdS/CFT
2. How to compute them in a concrete model? → Supersymmetric Localization
3. Can we compare them to a similar expansion in the microscopic theory? → Mock modular forms

Finite size corrections arise from quantum fluctuations in the black hole

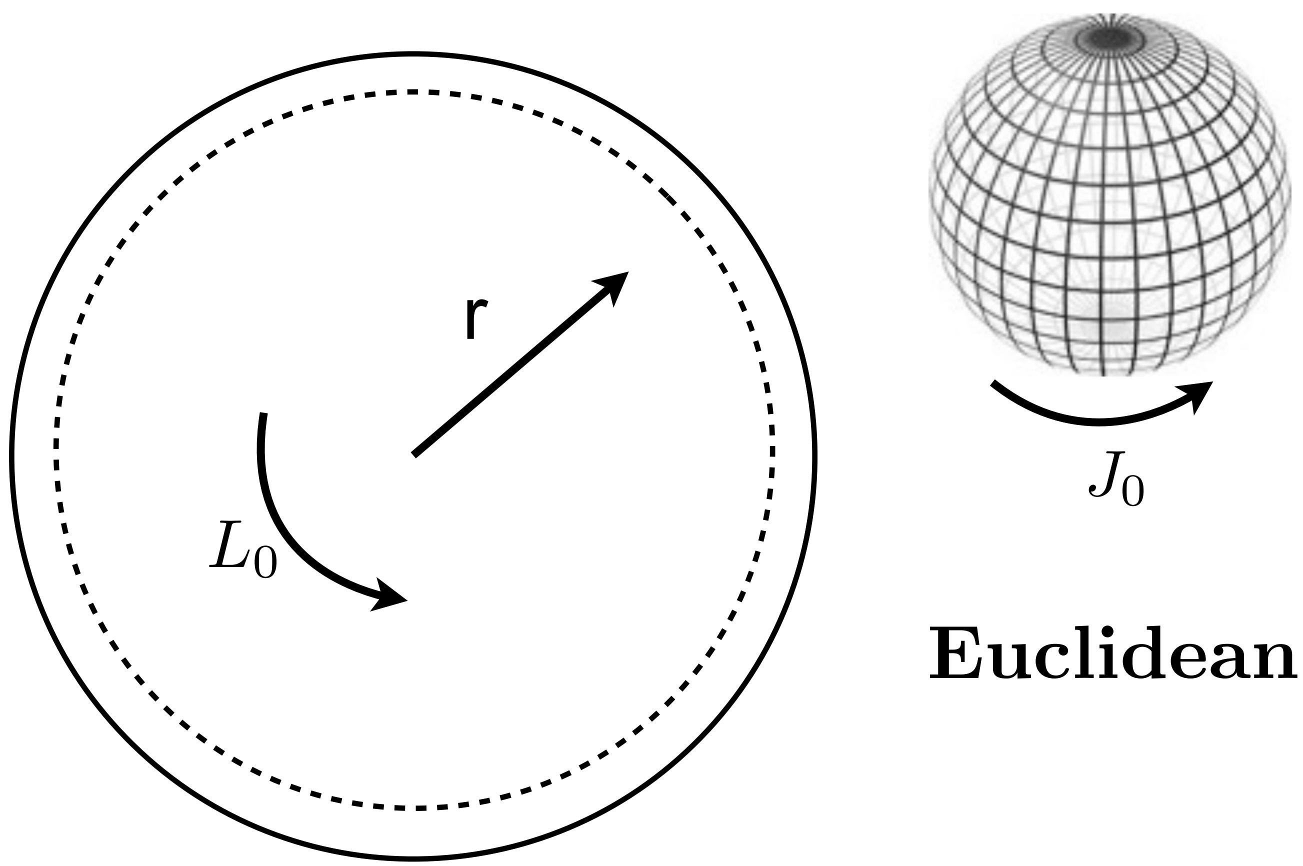
Wald Entropy formalism (Wald; Iyer,Wald; '94)

- Obeys the first law of thermodynamics
- Extends Bekenstein-Hawking area law in GR
- Applicable to any *local* effective action of gravity

We still need a good formalism to study Quantum BH entropy including non-analytic and non-local effects.

Supersymmetric black holes and $\mathbf{AdS_2}$

4d charged extremal BH (Reissner-Nordstrom) $\Rightarrow$ near-horizon attractor $AdS_2 \times S^2$.



$\mathbf{Euclidean\ AdS_2 \times S^2}$

Wald entropy of BPS BHs is found by extremizing the Lagrangian on the attractor $AdS_2 \times S^2$ solution. (Entropy function, Sen '05).

Quantum BH entropy is a functional integral over $\mathrm{AdS_2}$ configurations (Sen '08)

$$\exp(S^{\rm qu}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \Big\langle \exp\Big[- i\, q_I \oint A^I \Big] \Big\rangle^{\rm reg}_{\rm AdS_2}$$

- $AdS_2 \Rightarrow$ microcanonical ensemble with fixed charges (c.f. classical attractor mechanism).
- Saddle point evaluation gives $Z_{AdS_2}(q_I) = \exp\big(S^{\rm Wald}(q_I)\big)$.
- First quantum corrections have computed by a one-loop computation around classical background. (Sen et al '12-13)

$\mathrm{AdS/CFT}$ beyond the large N limit?

The planar $(\mathrm{N} \to \infty)$ spectral problem in $\mathrm{SU(N)}$ $\mathcal{N} = 4$ SYM theory in d=4 is completely solved.

But *Quantum* gravity = 1/N effects

Impressive progress in the last few years on computing exact quantities in supersymmetric field theory.
(Pestun; c.f. talks of Mariño, Martelli, Russo)

Dual theory for BPS BH is a collection of supersymmetric ground states

Dual $\mathbf{CFT_1}$ obtained as IR limit of brane configuration that makes up the black hole.

In d=0+1, no space for long-wavelength fluctuations.

$$Z_{\mathrm{CFT}_1}(q) \,=\, \mathrm{Tr}_{\mathcal{H}(q)}\, 1 \,=\, d_{\mathrm{micro}}(q)\,.$$

AdS/CFT correspondence

$$\Rightarrow Z_{\mathrm{AdS}_2}(q) \,=\, d_{\mathrm{micro}}(q)$$

Prototype: N=8 string theory in 4d (macro)

d=4 graviton coupled to 28 U(1) gauge fields + superpartner scalars + fermions.

1/8 BPS dyonic BH solutions. (Cvetic, Youm '96)

Charges $(q_I, p^I),\ I = 1, \ldots, 28\,,$

U-duality $E_{7,7}(\mathbb{Z})$

Quartic invariant $N = q^2 p^2 - (q.p)^2\,.$

BH Entropy $S_{BH} = \pi\sqrt{N} + \cdots$

Prototype: N=8 string theory in 4d (micro)

Type II string theory compactified on T^6.

Microscopic degeneracies $d_{\rm micro}(\rm N)$
computed using representation as D1-D5-P-K system.

(Maldacena, Moore, Strominger '99)

With $q = e^{2\pi i\tau}$,

$$\begin{aligned}\sum_{\rm N} d_{\rm micro}(\rm N)\, e^{2\pi i {\rm N}\tau} &= \theta(\tau)/\eta(\tau)^6 \\ &= q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + 56q^8 + \cdots\end{aligned}$$

Supersymmetric Localization

Witten '88, Duistermaat-Heckmann '82, Atiyah-Bott '84, Pestun '07

Consider a supermanifold $\mathcal{M}$ with an odd vector field Q and an off-shell algebra $Q^2 = H$ with H a compact $U(1)$.

We would like to evaluate an integral of a $Q-$invariant operator $\mathcal{O}$

$$I := \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

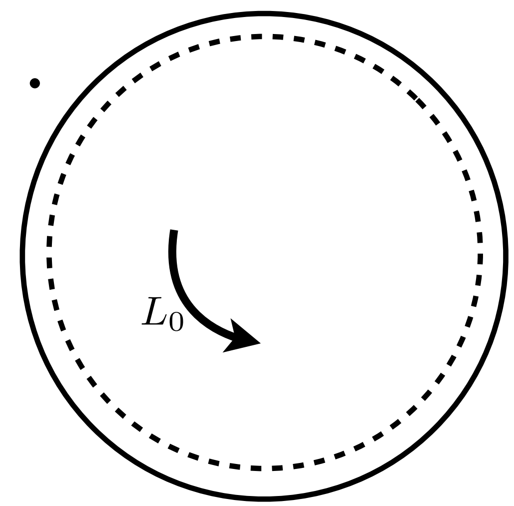
The functional integral localizes onto the submanifold $\mathcal{M}_Q$ of solutions of the off-shell BPS equations $Q \, \Psi = 0$

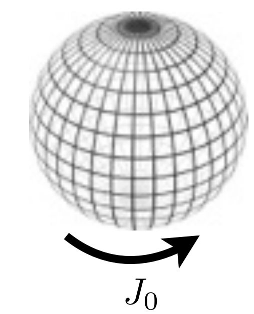
$$I = \int_{\mathcal{M}_Q} d\mu_Q \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

BPS quantum black hole entropy

$$\exp(S^{\text{qu}}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\Big[-i\, q_I \oint A^I \Big] \right\rangle^{\text{reg}}_{\text{AdS}_2} .$$

Supercharge Q with $Q^2 = L_0 - J_0$.





Euclidean $\mathbf{AdS_2 \times S^2}$

$\mathcal{M}$: Field space of supergravity.

$d\mu$: Measure on this field space.

$\mathcal{O}$: Wilson line.

$\mathcal{S}$: Supergravity action.

How to compute the BH functional integral

(A.Dabholkar, J.Gomes, S.M. '10, '11)

1. Formalism: conformal N=2 supergravity (de Wit, van Holten, Van Proeyen '80)
   Off-shell algebra $Q^2 = L_0 - J_0$.

2. Find all solutions of localization equations $Q\Psi = 0$,
   subject to $AdS_2 \times S^2$ boundary conditions.

3. Evaluate action on these solutions.
   Compute the measure.

Off-shell conformal N=2 supergravity

(de Wit, van Holten, Van Proeyen '80)

The susy transformations are specified once and for all.
They do not need to be modified as one modifies the action e.g. with higher derivative terms .

Gravity multiplet + $n_{\rm v}$ vector multiplets.

$$\delta\psi_\mu^i = 2\mathcal{D}_\mu\epsilon^i + \mathcal{V}_{\mu}^{i}{}_{j}\,\epsilon^j - \frac{1}{4}\sigma^{\rho\nu}\,T_{\rho\nu}^{ij}\,\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,$$

$$\delta\Omega_i^I = 2\gamma^\mu D_\mu X^I\epsilon_i + Y_{ij}^I\,\epsilon^j + \sigma^{\mu\nu}\mathcal{F}_{\mu\nu}^{I-}\,\varepsilon_{ij}\,\epsilon^j + 2X^I\eta_i\,.$$

Non-trivial problem since we do not know the metric and other background fields.

All BPS configurations of N=2 supergravity

Technique well-established in on-shell problems.

(Tod '88; de Wit-Nicolai, Candelas-Horowitz-Strominger-Witten,...,Gauntlett-Gutowski-Hull-Pakis-Reall '02, Meessen-Ortin '06, c.f. Talk of Martelli.)

Construct spinor bilinears $f_{\mu\nu\cdots} \equiv \bar{\psi}\gamma_{\mu\nu\cdots}\psi$
BPS equations for $\psi \Rightarrow$ first order equations for $f_{\mu\nu}$.
Identify with spacetime quantities (Killing vector etc).

BPS equations + Euclidean AdS_2 boundary conditions extremely constraining.

All BPS configurations of N=2 supergravity

In the gravity multiplet sector, the only solution is $AdS_2 \times S^2$.

Can solve for the Killing spinor ϵ_i and plug into vector multiplet equations.

Most general solution 1/2 BPS, scalar fields go off-shell:

$$X^I = X^I_* + \frac{C^I}{r}, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{r}, \qquad Y^{I1}_1 = -Y^{I2}_2 = \frac{2C^I}{r^2}.$$

Localization manifold labelled by one parameter $\phi^I \equiv X^I(0)$ for each vector multiplet X^I.

(R.Gupta, S.M., “All solutions to the localization equations for quantum black hole entropy," arXiv:1208.6221)

Evaluation of Wilson line

The Wilson line expectation value in supergravity localizes to the finite integral:

$$Z_{AdS_2}(q,p) = \int_{\mathcal{M}_Q} \exp\left(\mathcal{S}_{\text{ren}}(\phi, p, q)\right) \, [d\phi^I] \, .$$

- $\mathcal{S}_{\text{ren}}$ is the renormalized action of N=2 supergravity evaluated on the localizing manifold.
- The measure $[d\phi^I]$ is that of the supergravity scalar field space.

N=2 supergravity action (effective theory)

Chiral superspace integrals

$$\int d^4x \int d^4\theta \, \mathbf{X}$$

Holomorphic prepotential function $F(\mathbf{X^I}, \mathbf{W^2})$

Computed by topo-logical string on CY

$$F(X^I, W^2) = \sum_{n=0}^{\infty} F_g(X^I)\,(W^2)^g$$

Full superspace integrals

$$\int d^4x \int d^4\theta \, d^4\bar{\theta} \, \mathbf{e} \, \mathbf{X} \, \bar{\mathbf{X}}$$

Action governed by F

(de Wit, van Holten, Van Proeyen '80)

$$
\begin{aligned}
S \quad = \quad & (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2}R) + \big[i\nabla_\mu F_I \nabla^\mu \bar{X}^I \\
+ \quad & \frac{1}{4} i F_{IJ} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij})(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{ij}_{ab}\,\varepsilon_{ij}) \\
- \quad & \frac{1}{8} i F_I (F^{+I}_{ab} - \frac{1}{4} X^I T_{abij}\,\varepsilon^{ij}) T^{ij}_{ab}\,\varepsilon_{ij} - \frac{1}{8} i F_{IJ} Y^I_{ij} Y^{Jij} - \frac{i}{32} F\,(T_{abij}\,\varepsilon^{ij})^2 \\
+ \quad & \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik}\varepsilon^{jl} \widehat{B}_{ij} \widehat{B}_{kl} - 2 \widehat{F}^-_{ab} \widehat{F}^-_{ab}) \\
+ \quad & \frac{1}{2} i \widehat{F}^{-ab} F_{\widehat{A}I} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij}) - \frac{1}{4} i \widehat{B}_{ij} F_{\widehat{A}I} Y^{Iij} + \text{h.c.}\big] \\
- \quad & i(X^I \bar{F}_I - F_I \bar{X}^I) \cdot (\nabla^a V_a - \frac{1}{2} V^a V_a - \frac{1}{4}|M_{ij}|^2 + D^a \Phi^i{}_\alpha D_a \Phi^\alpha{}_i) \; .
\end{aligned}
$$

Simple formula for exact BH entropy

Assuming that only F-terms contribute to the action, we get a simple formula for any N=2 theory:

(A.Dabholkar, J.Gomes, S.M. '10)

$$Z_{AdS_2}(q,p) = \int_{\mathcal{M}_Q} \exp\left(\mathcal{S}_{\rm ren}(\phi,p,q)\right)\, [d\phi^I]\,.$$

$$\mathcal{S}_{\rm ren} = -\pi\, q_I\, \phi^I + {\rm Im}F(\phi^I + ip^I)\,.$$

Kinetic term for one of the modes (conformal mode of metric) has wrong sign and leads to a divergent integral. Necessary to analytically continue the contour of integration in field space.

(Gibbons-Hawking '76)

Exact BH entropy in N=8 theory

(A.Dabholkar, J.Gomes, S.M. '11)

We drop the gravitini multiplets and the hypermultiplets to get a reduced N=2 theory $\Rightarrow$ 8 vector multiplets coupled to supergravity.

The exact prepotential is (with $C_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{3\times 3}$)

$$F(X) = -\frac{1}{2}\frac{X^1 C_{ab} X^a X^b}{X^0}\,, \qquad a,b = 2,\ldots,7\,.$$

$$e^{S_{BH}^{\mathrm{qu}}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

Comparison: micro/macro

N	$d_{\text{micro}}(\text{N})$		$\exp(S^{\text{cl}}(\text{N}))$
3	8		230.76
4	12		535.49
7	39		4071.93
8	56		7228.35
11	152		33506.14
12	208		53252.29
15	513		192400.81
...	...		...
10^5	exp(295.7)		exp(314.2)

$$\log(d_{\text{micro}}) \xrightarrow{N\to\infty} S^{\text{cl}}_{BH} \,.$$

Comparison: micro/macro (A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\text{micro}}(\text{N})$	$\exp\left(S^{\text{qu}}(\text{N})\right)$	$\exp\left(S^{\text{cl}}(\text{N})\right)$
3	8	7.97	230.76
4	12	12.20	535.49
7	39	38.99	4071.93
8	56	55.72	7228.35
11	152	152.04	33506.14
12	208	208.45	53252.29
15	513	512.96	192400.81
...	...	...	...
10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\text{micro}}(N) = e^{S^{\text{qu}}_{BH}(N)}\left(1 + O(e^{-\pi\sqrt{N}/2})\right)$$

What about non-F-terms in the action?

Does the holomorphic prepotential F capture the full perturbative quantum BH entropy in any N=2 theory?

Non-renormalization of classical entropy

(Maldacena-Strominger-Witten '97)

- Recall that classical Wald entropy is found by extremizing the Lagrangian on the full-BPS attractor solution.

- Full BPS solution is a constant superfield configuration. Any full superspace integral and its first derivative vanish on this solution. (de Wit-Katmadas-van Zalk '10)

Localization and holomorphic prepotential

Off-shell formalism: BPS equations do not change when the action is changed.

The set of BPS solutions $\mathcal{M}_Q$ is thus also universal!

$$Z_{AdS_2}(q,p) = \int_{\mathcal{M}_Q} \exp\left(\mathcal{S}_{\rm ren}(\phi,p,q)\right)\,[d\phi^I]\,.$$

A full-superspace integral can:

(a) Change the value of the renormalized action $\mathcal{S}_{\rm ren}$.

(b) Change the definition of electric charges.

(c) Change the induced measure on the localizing manifold.

Generically, the full-superspace integrals do not vanish on non-constant configurations (even 1/2-BPS).

In our case, the AdS_2 boundary conditions force the solutions to have a one-dimensional nature.

A large class of full-superspace integrals (based on kinetic multiplets) vanish on the localizing manifold.

(V. Reys, S.M. arXiv:1306.3796)

$\mathcal{S}_{\rm ren}$ is invariant $\Rightarrow$ measure and charges also invariant.

Comparison: micro/macro (A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\text{micro}}(\text{N})$	$\exp\left(S^{\text{qu}}(\text{N})\right)$	$\exp\left(S^{\text{cl}}(\text{N})\right)$
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10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\text{micro}}(N) = e^{S_{BH}^{\text{qu}}(N)}\left(1 + O(e^{-\pi\sqrt{N}/2})\right)$$

Why does this work so well?

The microscopic partition function is a modular form

$$Z_{\text{micro}}(\tau) \equiv \sum_N d_{\text{micro}}(N)\, e^{2\pi i N\tau} = \frac{\theta(\tau)}{\eta(\tau)^6}$$

Strong-weak coupling symmetry:

$$Z_{\text{micro}}(-1/\tau) = \tau^{5/2} Z_{\text{micro}}(\tau)$$

$$\tau \to \tau + 1$$
$$\tau \to -1/\tau\,.$$

$\Rightarrow$ $SL_2(\mathbb{Z})$ Modular symmetry group

Highly constraining

Exact formula for degeneracies

Hardy-Ramanujan-Rademacher expansion

$$d_{\text{micro}}(N) = \sum_{c=1}^{\infty} c^{-9/2}\, K_c(N)\, \widetilde{I}_{7/2}\left(\frac{\pi\sqrt{N}}{c}\right)$$

$$= \widetilde{I}_{7/2}(\pi\sqrt{N}) + O(e^{-\pi\sqrt{N}/2})$$

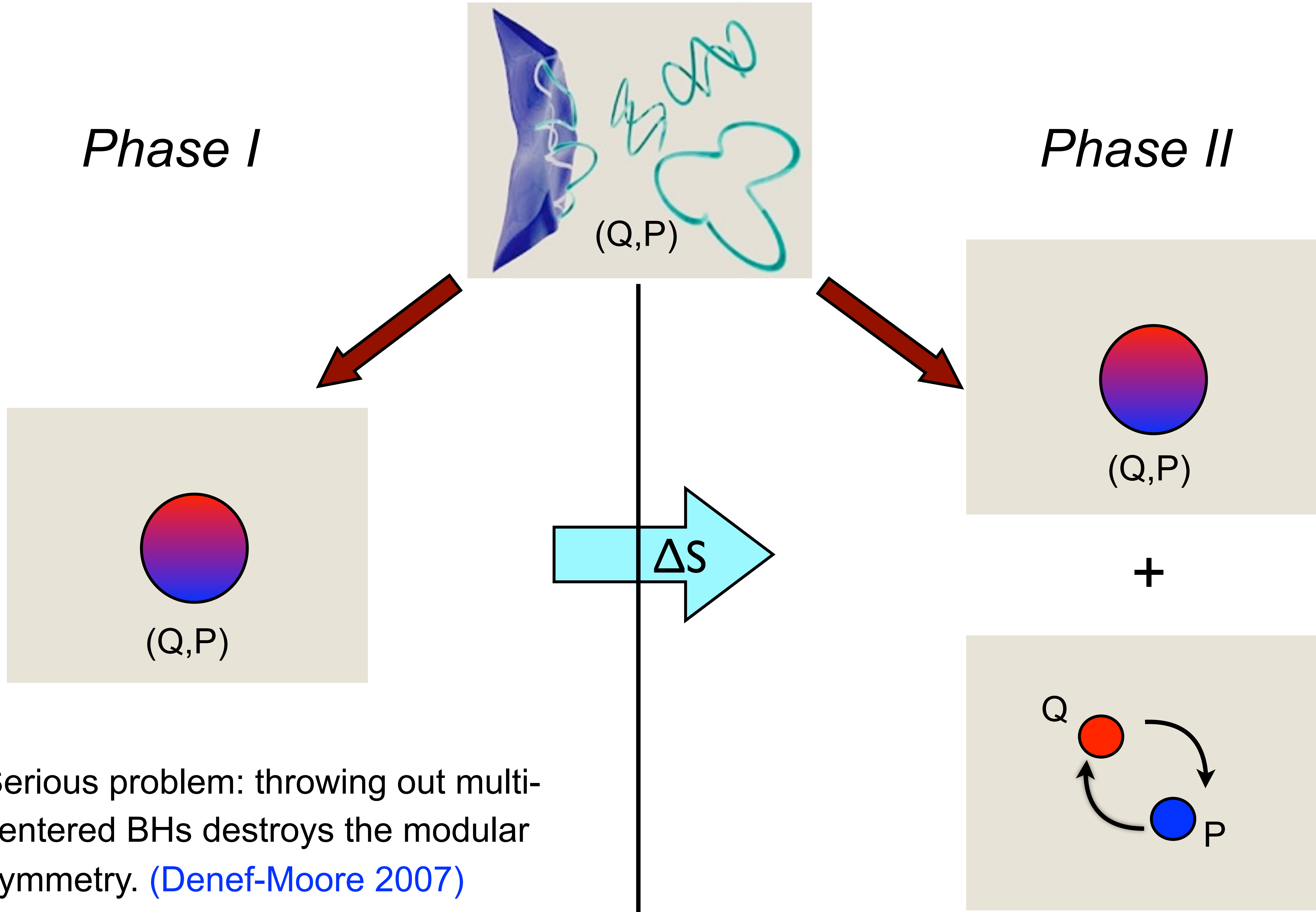
$$= e^{\pi\sqrt{N}}\left(1 - \frac{15}{4}\log N + O\left(\frac{1}{N}\right)\right).$$

Orbifolds of AdS_2

Bekenstein-Hawking

One-loop corrections

Wall-crossing and BH phase transitions



Serious problem: throwing out multi-centered BHs destroys the modular symmetry. (Denef-Moore 2007)

Mock modular forms provide the solution

(A.Dabholkar, S.M., D.Zagier ’12)

List of examples by Ramanujan (1920), but no definition!

Definition and structural properties now finally understood.

S. Zwegers (2000)

Exactly what we need to solve the black hole wall-crossing problem.

For N=4 string theory, we could solve it fully, and explicitly compute the partition function of a single BH.

This resolves the tension between modular symmetry and wall-crossing.

Canonical decomposition of the partition function:

$$Z_{\mathrm{micro}}(\tau) \;=\; Z_{\mathrm{BH}}(\tau) + Z_{\mathrm{multi}}(\tau)$$

Partition function of the isolated BH is a *mock modular form*.

All the wall-crossing information.

Many new quantitative explorations now within reach. e.g. Large discrete symmetry groups (moonshine) of BHs in string theory (J. Harvey, S.M., *in prep*.)

Quantum black hole thermodynamics opens a new window into quantum gravity

- Finite size effects in BH thermodynamics.

- Convergent perturbation expansions in quantum gravity via localization.

- A concrete example of exact finite N holography.

- Mathematical structures: New mock modular symmetries play a key role. Interaction both ways!

Comments and/or questions?

Mock modular forms

Mock modular form $f(\tau)$

Shadow $g(\tau)$

Completion $\widehat{f}(\tau,\bar{\tau}) := f(\tau) + g^*(\tau,\bar{\tau})$

Holomorphic anomaly equation

$$(4\pi\tau_2)^k \; \frac{\partial \widehat{f}(\tau,\bar{\tau})}{\partial\bar{\tau}} = -2\pi i \, \overline{g(\tau)} \; .$$