Gauge Dynamics and Topological Insulators

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Back in Swansea 2013

Based on arXiv:1305.2414
with Benjamin Béri and Kenny Wong

Story 1: Massless Fermions

SU(2) Yang-Mills in *d=2+1*

$$S = -\int d^3x \;\; \frac{1}{2e^2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}_1 \not{D} \psi_1 + i\bar{\psi}_2 \not{D} \psi_2$$

Global Symmetries	ψ_1	ψ_2
$U(1)_F$	+	+
$U(1)_A \subset SU(2)_F$	+	−

Background Magnetic Field

$$F_{12} = \frac{B}{2}\sigma^3$$

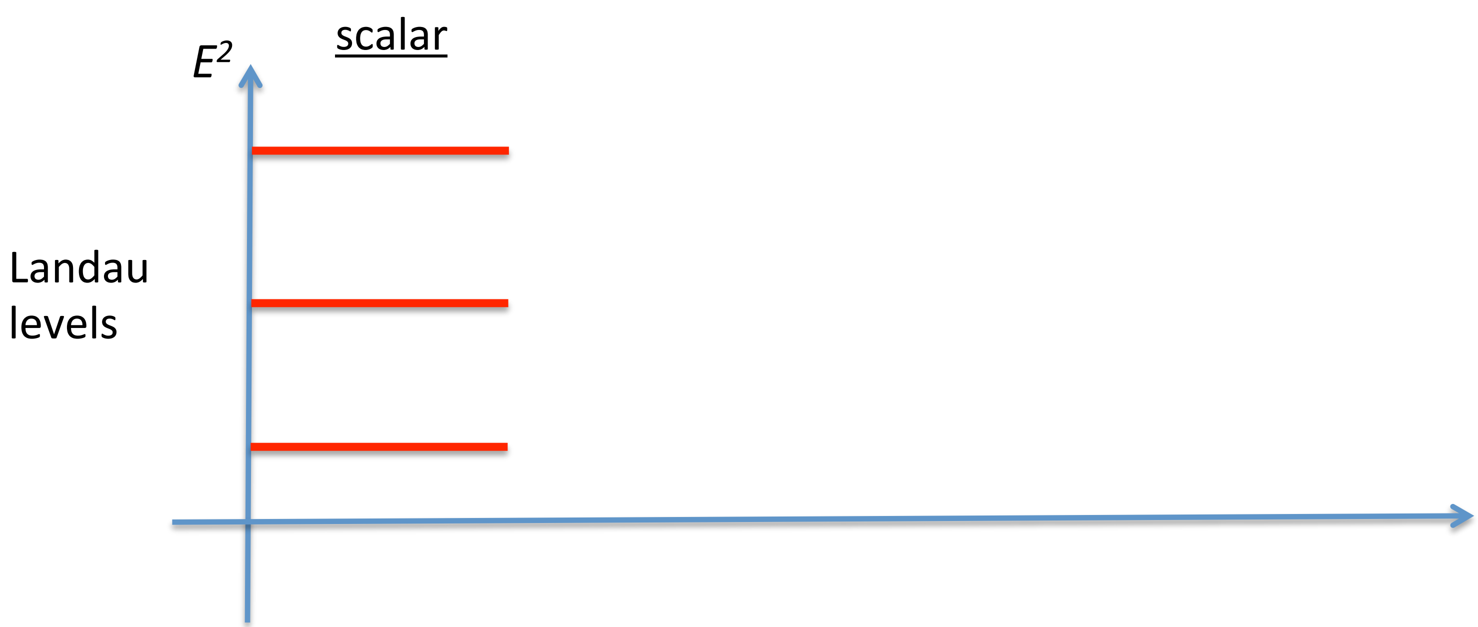
Note

- Preserves time reversal *T*
- Semi-classical limit $B \gg e^2$

Dynamics in a Magnetic Field: Spin 0

Massless relativistic scalar: $$E^2 = B(2n+1)$$

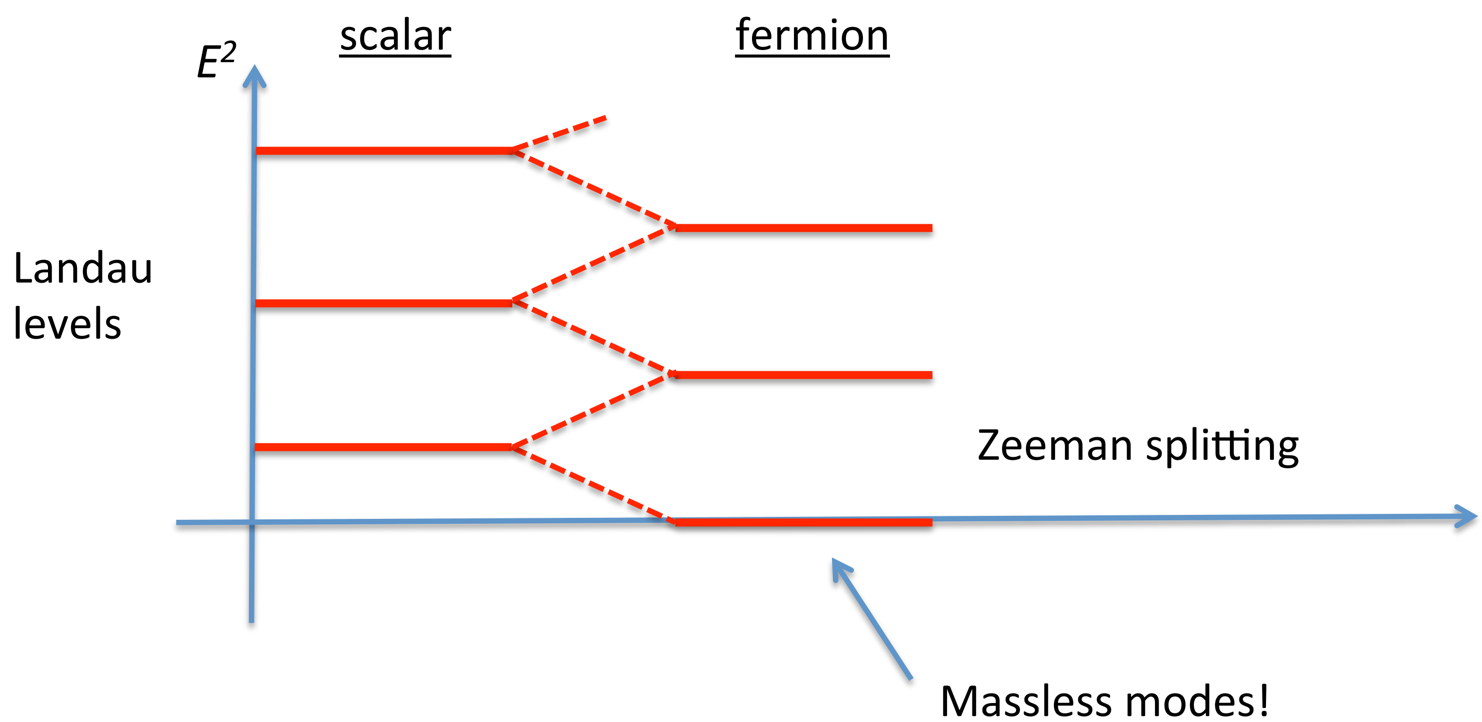
$$n = 0, 1, 2, \ldots$$



Dynamics in a Magnetic Field: Spin 1/2

Massless relativistic fermion: $E^2 = B(2n+1) + 2Bs$

$$s = \pm\frac{1}{2}$$



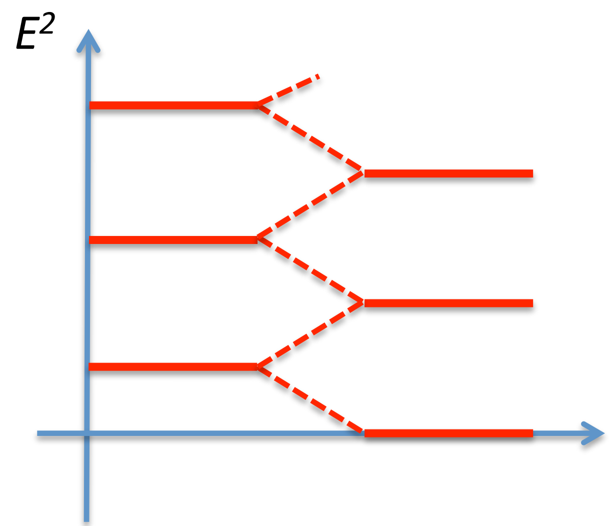
Degeneracy of Lowest Landau Level

- 8 species of fermion
  - Dirac spinor is 2 component; SU(2) gauge; SU(2) flavour
- Lowest Landau has 4 excitations
  - All others have 8 excitations

- Pick gauge $A_y = \frac{B}{2} x \sigma^3$

$$\psi_k(x,y) \sim e^{-iky} e^{-B/4(x+2k/B)^2} \begin{pmatrix} \xi(k) \\ 0 \end{pmatrix}$$

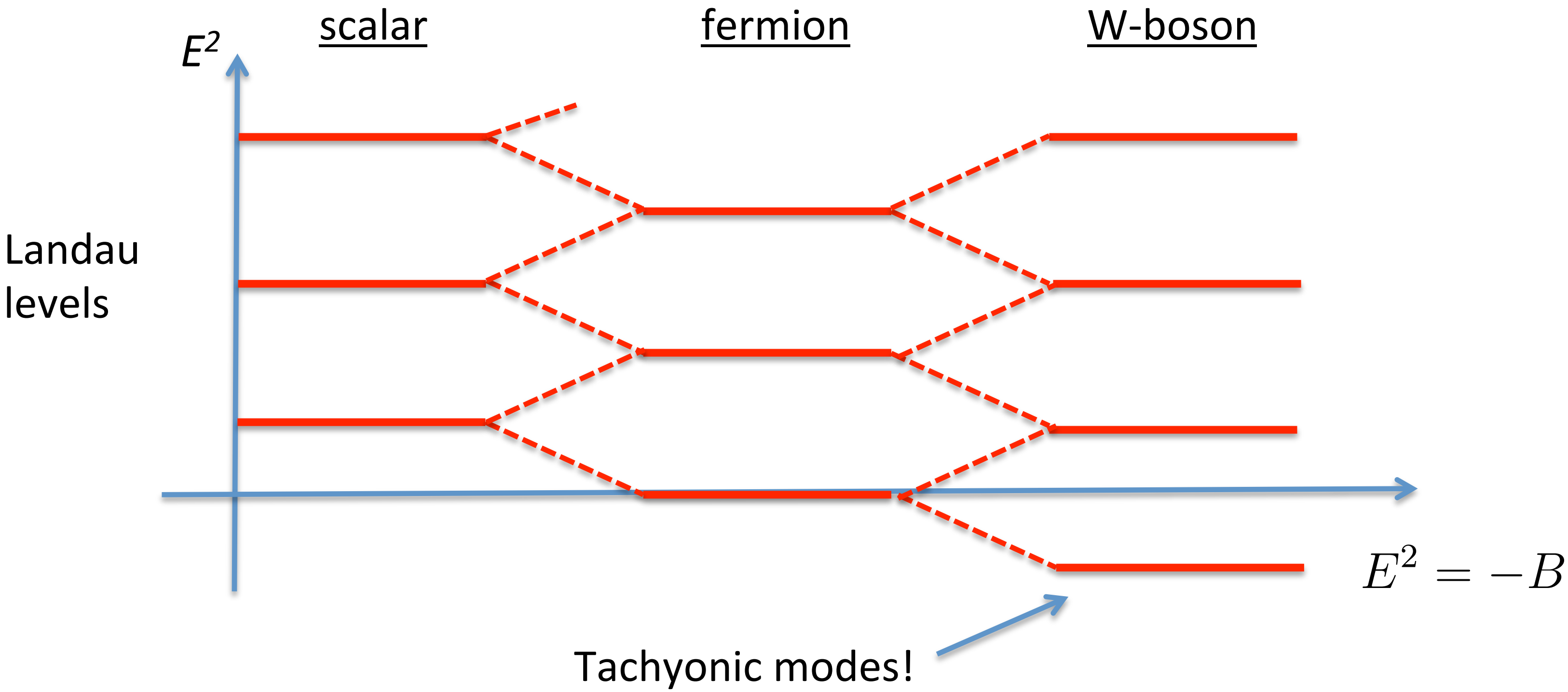
k labels degeneracy of Landau level



Dynamics in a Magnetic Field: Spin 1

Massless relativistic W-boson: $E^2 = B(2n+1) + 2Bs$

$s = -1, 0, +1$



What Becomes of the W-Bosons?

- W-bosons are tachyonic
- They condense and form a lattice

Nielsen and Olesen; Ambjorn and Olesen

W-Boson Lattice

Plan: Work to linear order

- This is not valid
- It's easy to make it valid
- But it doesn't matter
  - We will compute topological quantities

W-Boson Lattice

$$W_x = -iW_y = W \sum_{n \in \mathbf{Z}} e^{-\pi i n^2/2} e^{-2\pi i n y/d} e^{-B/2(x+2\pi n/Bd)^2}$$

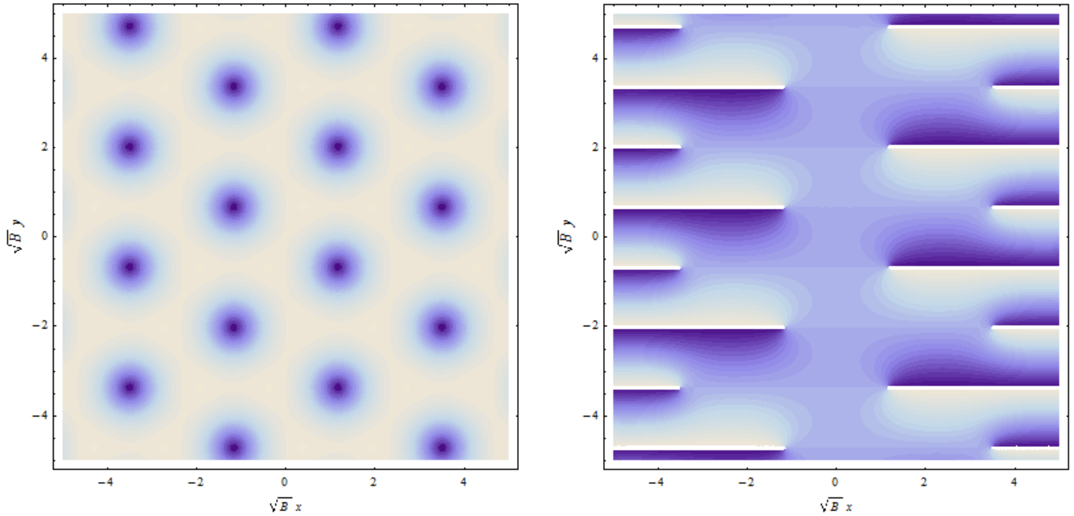
$$Bd^2 = 4\pi \quad \Longrightarrow \quad \text{Square lattice}$$

$$Bd^2 = \frac{4\pi}{\sqrt{3}} \quad \Longrightarrow \quad \text{Triangular lattice}$$

W-Boson Lattice

$$W_x = -iW_y = W \sum_{n \in \mathbf{Z}} e^{-\pi i n^2/2} e^{-2\pi i n y/d} e^{-B/2(x + 2\pi n/Bd)^2}$$

W-boson winds around vortex cores



What Becomes of the Fermions?

Need to do (very) degenerate perturbation theory

$$\Delta H = \int d^2x \sum_{k,k'} \psi_{1,k}(x)\gamma^\mu W_\mu(x)\psi_{1,k'}(x) \;+\; (1 \leftrightarrow 2)$$

where $\psi_k(x,y) \sim e^{-iky} e^{-B/4(x+2k/B)^2} \begin{pmatrix} \xi(k) \\ 0 \end{pmatrix}$

$$W_x = -iW_y = W \sum_{n\in\mathbf{Z}} e^{-\pi i n^2/2} e^{-2\pi i n y/d} e^{-B/2(x+2\pi n/Bd)^2}$$

What Becomes of the Fermions?

Diagonalise with

$$\zeta(p_1, p_2) = \sum_n e^{-2\pi i n p_2/Bd}\psi_{k=p_1/2+2\pi n/d}$$

$$p_1 \in [0, 4\pi/d)$$

$$p_2 \in [0, Bd)$$

$$\Longrightarrow \quad \vec{p} \in \mathbf{T}^2$$

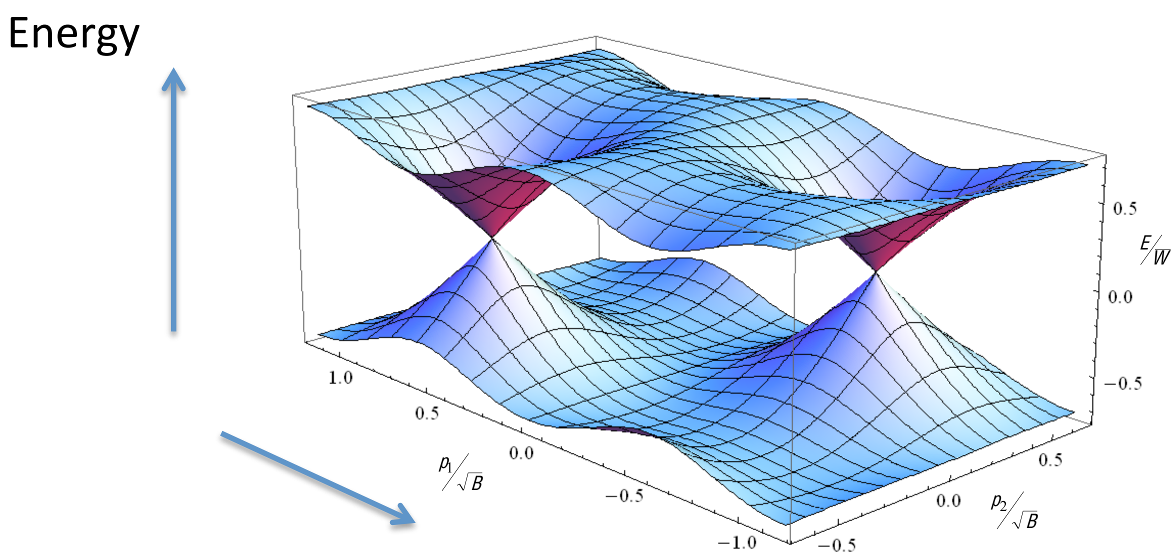
This is the (magnetic) Brilliouin zone (BZ)

And something nice happens....

$$\Delta H = \int d^2p \; \zeta_1^\dagger(\vec{p})\, W(p_1, p_2)\, \zeta_1(\vec{p}) \;+\; (1 \leftrightarrow 2)$$

Physical W-boson lattice becomes
energy function over BZ!

Dirac Cones in the Brillouin zone



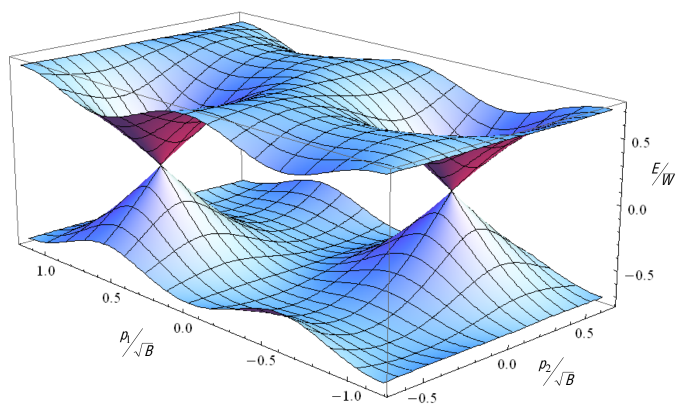
The *E<0* states become filled.

Robust Dirac Cones in the Brillouin zone

Comments:
- The Dirac points are vortices in momentum space!
  - They are protected by topology and symmetry
- The position Dirac points lie at non-zero Bloch momenta

$$\vec{p} \sim B$$

  - They too are protected by symmetries



Volovik; Wen and Zee; Beri

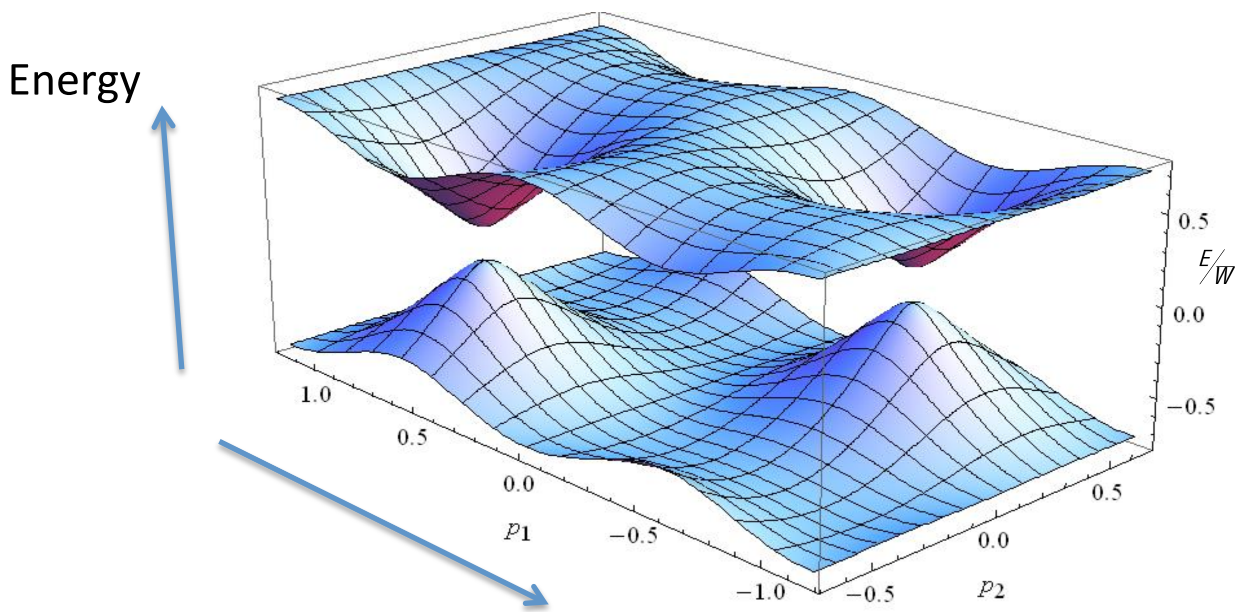
Story 2: Massive Fermions

Adding a Mass for the Fermion

$$S_{\text{mass}} = \int d^3x \; im(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)$$

- Preserves both charge conjugation *C* and time reversal *T*
  - both must be accompanied by $(1 \leftrightarrow 2)$
- Flavour symmetry $SU(2)_F \rightarrow U(1)_A$

Now we have an insulator



Again, the *E<0* states become filled.

Topology of Gapped Fermions

U(1) gauge connection over the BZ

$$A_i^{\text{Berry}}(p) = i\langle\zeta(p)|\frac{\partial}{\partial p_i}|\zeta(p)\rangle$$

Quantum state with momentum *p*

Note:

$$|\zeta(p)\rangle \to e^{i\alpha(p)}|\zeta(p)\rangle \quad \Rightarrow \quad A_i^{\text{Berry}} \to A_i^{\text{Berry}} - \partial_i\alpha$$

TKNN Invariant

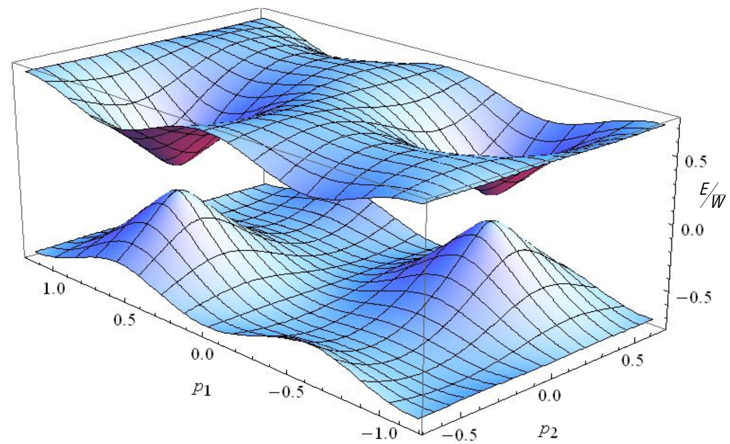
$$C = \frac{1}{2\pi} \int_{\mathrm{BZ}} F^{\mathrm{Berry}}$$

A lovely result

Hall conductivity: $$\sigma_{xy} = \frac{e^2}{2\pi} \sum_{\text{filled bands}} C$$

TKNN Invariants for Yang-Mills

Two flavours of fermions:



$$\psi_1 : \quad C = -1$$

$$\psi_2 : \quad C = +1$$

Two flavour symmetries: $U(1)_F \times U(1)_A$

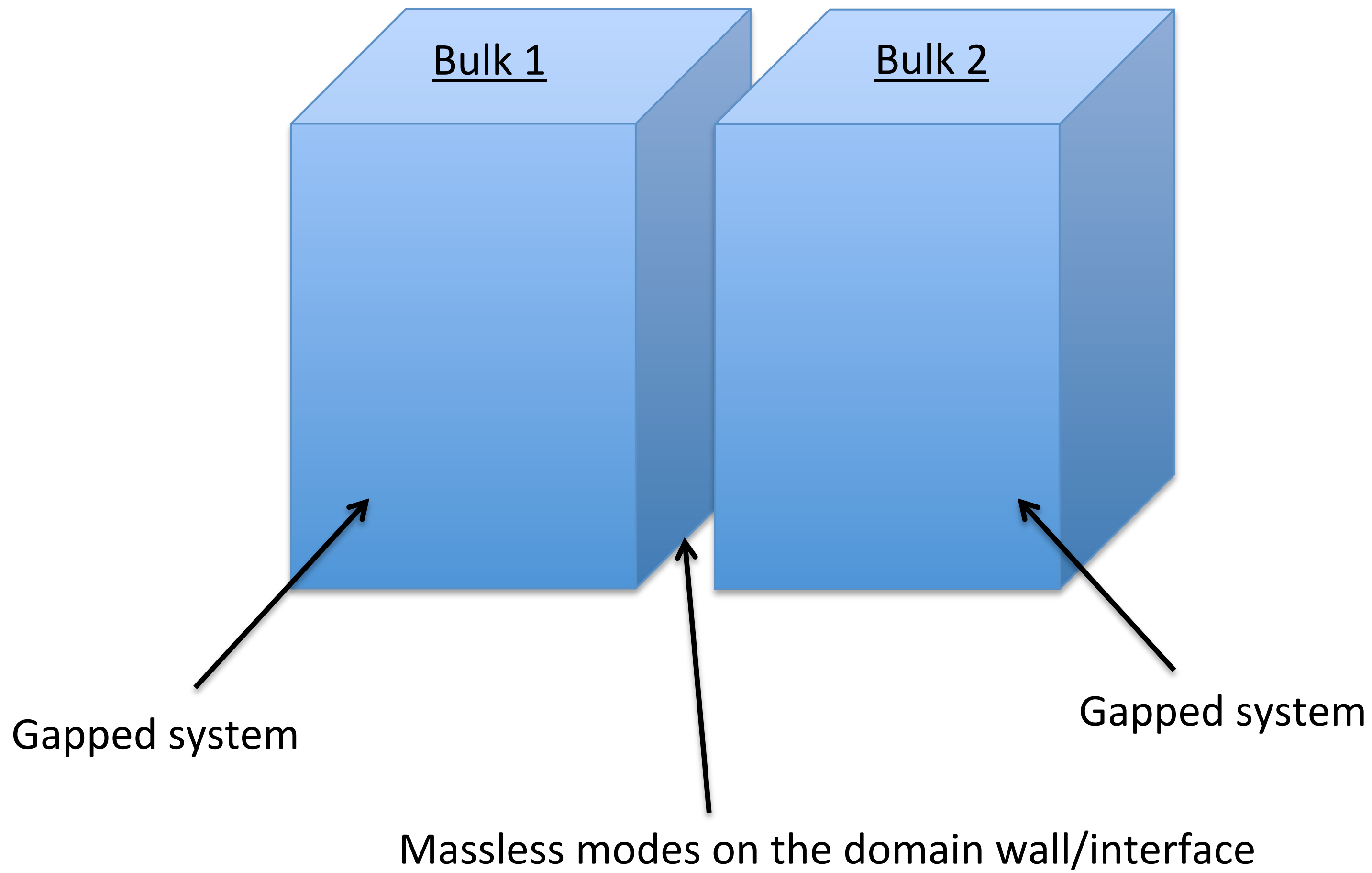
$$\sigma_{xy}^F = \sigma_{xy}^A = 0 \quad \text{and} \quad \sigma_{xy}^{F-A} = \frac{e^2}{\pi}$$

(Quantum Spin-Hall Effect)

(This result also follows from usual Chern-Simons calculation)

Story 3: Edge Modes and Topological Insulators

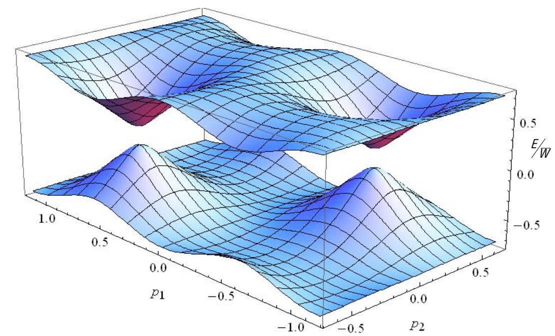
Edge Modes

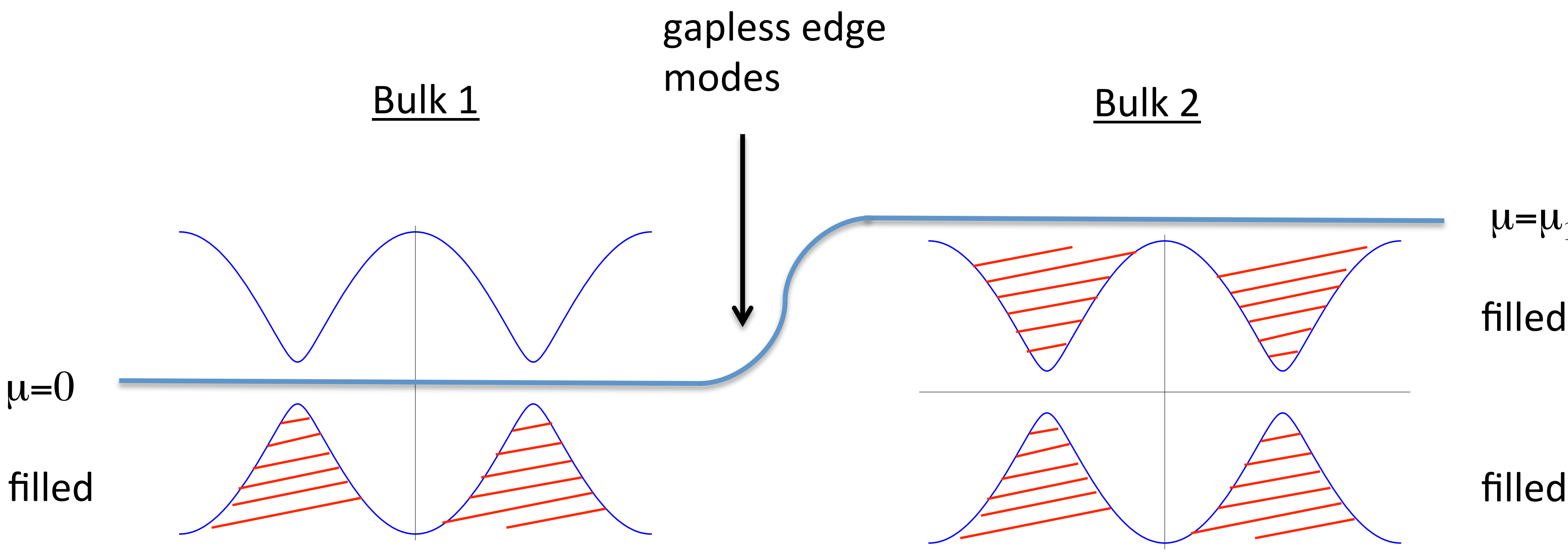


(It's just Jackiw-Rebbi to you and me)

Edge Modes in Yang-Mills

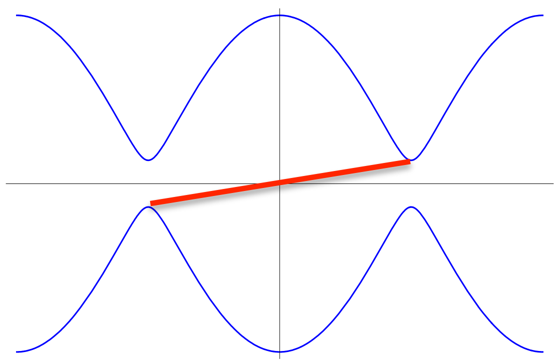
- Vary the chemical potential
    - This breaks charge conjugation *C*



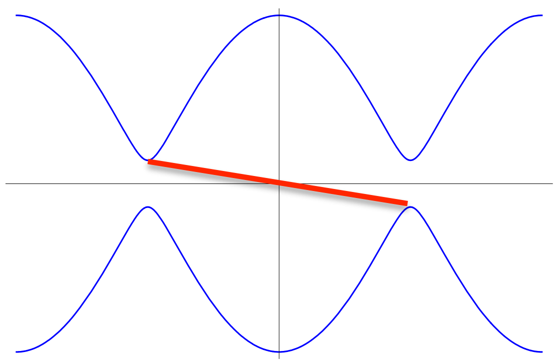


Edge Modes in Yang-Mills

ψ_1



ψ_2



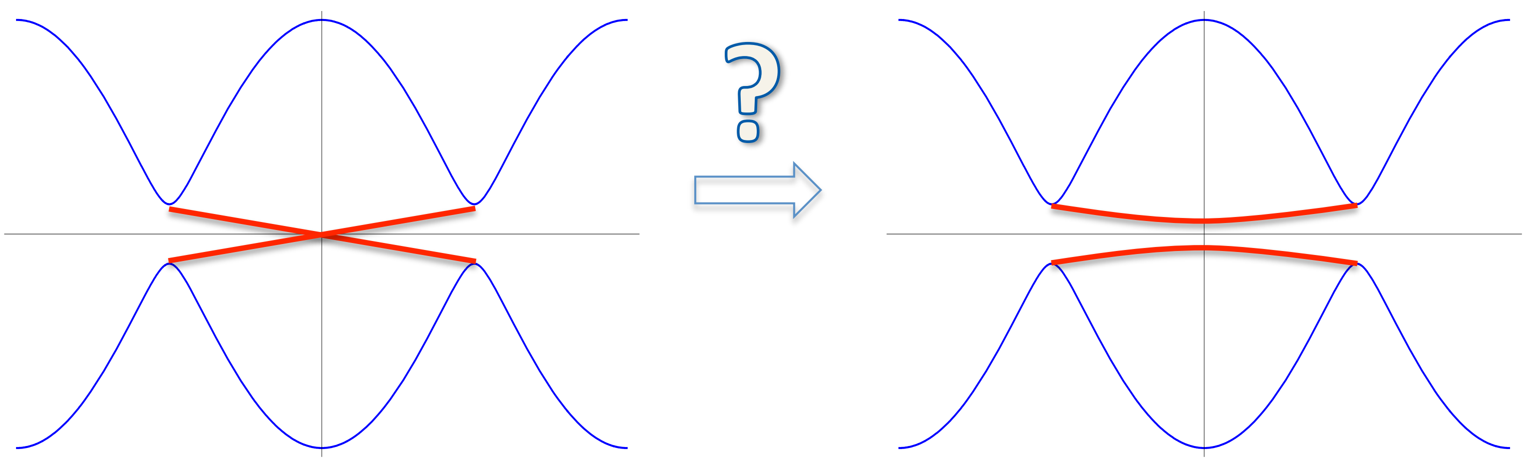
We get a chiral edge mode from the first fermion

And a chiral edge mode going the other way from the second

Can we Kill the Edge Modes?

- Suppose we now add some interaction between the fermions
  - Or impurities
  - Just something that kills the *U(1)* flavour symmetries.

- What protects the massless modes?



The $\mathbf{Z}_2$ Invariant

- The TKNN Chern all now vanish. They offer no protection.
- But there is a more subtle $\mathbf{Z}_2$ invariant
  - Relies on existence of time reversal with

$$\mathcal{T}^2 = -1$$

Kane and Mele (2005)

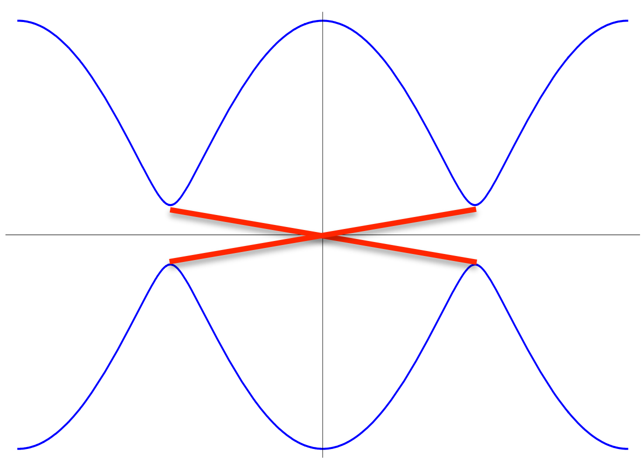
The Importance of Time Reversal

- Kramers Theorem: $\mathcal{T}^2 = -1$ implies that all states come in pairs with

$$\mathcal{T} : \vec{p} \rightarrow -\vec{p}$$

At $\vec{p} = 0$ states must be degenerate ⇨ massless edge states remain.

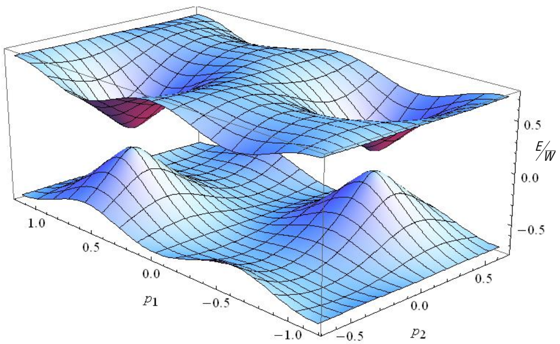
- This is the essence of the $\mathbf{Z}_2$ topological insulator



Summary

Fermions in a the background of a W-boson lattice:

- Massless fermions have protected Dirac cones
- Massive fermions have TKNN invariants
- Adding a chemical potential gives rise to the simplest $\mathbf{Z}_2$ topological insulator.



The End

A Better W-Boson Lattice

$$S = S_{YM} - \int d^3x \ \ \mathrm{Tr}\, D_\mu \phi D^\mu \phi - \frac{\lambda}{4}\left(\mathrm{Tr}\,\phi^2 - \frac{v^2}{2}\right)^2$$

- $SU(2) \to U(1)$
- W-bosons tachyonic for $B > e^2 v^2$
- Numerical solutions found (triangular lattice preferred)
  - (But under assumption of existence of a lattice structure)
- Solution is certainly only meta-stable in quantum theory
  - Monopoles destroy magnetic field
- Is solution stable classically?