STOCHASTIC QUANTIZATION
AT FINITE CHEMICAL POTENTIAL:
SIGN AND SILVER BLAZE PROBLEMS

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Outline

- Sign problem at finite chemical potential
- A revived approach: stochastic quantization
- Three QCD inspired models
- Relativistic Bose gas
- The Silver Blaze problem is not a problem
fermion determinant is complex

\[ [\det M(\mu)]^* = \det M(-\mu) \]

avoid fluctuating sign?

simulations at \( \mu = 0 \) or with \( |\det M(\mu)| \)

but ...
important configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$ cancelation between configurations with ‘positive’ and ‘negative’ weight

how to pick the dominant configurations in the path integral?
QCD at finite $\mu$

Sign Problem

- Important configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- Cancellation between configurations with ‘positive’ and ‘negative’ weight
- How to pick the dominant configurations in the path integral?

Radically different approach:

- Complexifying all degrees of freedom: $SU(3) \rightarrow SL(3, \mathbb{C})$
- Stochastic quantization and complex Langevin dynamics
based on

- with I.O.S.: stochastic quantization at finite chemical potential, 0807.1597 [hep-lat], JHEP

- can stochastic quantization evade the sign problem? – the relativistic Bose gas at finite chemical potential 0810.2089 [hep-lat], PRL

- complex Langevin dynamics at finite chemical potential: mean field analysis in the relativistic Bose gas, 0902.4686 [hep-lat]

more reading

- with I.O.S.: Lattice proceedings, 0809.5527 [hep-lat]

- SEWM proceedings: 0811.1850 [hep-ph]
STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

Parisi & Wu '81

field theory

- path integral \( Z = \int D\phi e^{-S} \)
- Langevin dynamics in “fifth” time direction

\[
\frac{\partial \phi(x, \theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x, \theta)} + \eta(x, \theta)
\]

Gaussian noise

\[
\langle \eta(x, \theta) \rangle = 0 \quad \langle \eta(x, \theta) \eta(x', \theta') \rangle = 2\delta(x - x')\delta(\theta - \theta')
\]

equilibrium distribution \( P[\phi] \sim e^{-S} \)
**Stochastic Quantization**

**Langevin Dynamics**

force $\partial S/\partial \phi$ complex: complexify Langevin dynamics

- example: real scalar field $\phi \rightarrow \phi^R + i\phi^I$
- Langevin eqs

\[
\frac{\partial \phi^R}{\partial \theta} = -\text{Re} \left. \frac{\delta S}{\delta \phi} \right|_{\phi \rightarrow \phi^R + i\phi^I} + \eta
\]

\[
\frac{\partial \phi^I}{\partial \theta} = -\text{Im} \left. \frac{\delta S}{\delta \phi} \right|_{\phi \rightarrow \phi^R + i\phi^I}
\]

observables: analytic extension

\[
\langle O(\phi) \rangle \rightarrow \langle O(\phi^R + i\phi^I) \rangle
\]
STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- associated Fokker-Planck equation

\[
\frac{\partial P[\phi, \theta]}{\partial \theta} = \int d^d x \frac{\delta}{\delta \phi(x, \theta)} \left( \frac{\delta}{\delta \phi(x, \theta)} + \frac{\delta S[\phi]}{\delta \phi(x, \theta)} \right) P[\phi, \theta]
\]

- stationary solution: \( P[\phi] \sim e^{-S} \)

- real action: formal proofs of convergence

\[
P[\phi, \theta] = \frac{e^{-S[\phi]}}{Z} + \sum_{\lambda > 0} e^{-\lambda \theta} P_\lambda[\phi]
\]

- complex action: theoretical status less clear cut

but all other methods fail!
consider three models with a partition function

\[ Z = \int DU e^{-S_B} \det M \quad \det M(\mu) = [\det M(-\mu)]^* \]

- QCD with static quarks
- SU(3) one link model
- U(1) one link model

observables:
- (conjugate) Polyakov loops
- density
- phase of determinant
THREE MODELS

I: QCD with static quarks

\[ Z = \int DU e^{-S_B} \det M \]

- bosonic action: standard SU(3) Wilson action

\[ S_B = -\beta \sum_P \left( \frac{1}{6} \left[ \text{Tr} U_P + \text{Tr} U_P^{-1} \right] - 1 \right) \]

- determinant \( \det M \) for Wilson fermions

fermion matrix:

\[ M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left( e^\mu \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_4 \right) \]
THREE MODELS

I: QCD WITH STATIC QUARKS

hopping expansion:

$$\det M \approx \det \left[ 1 - \kappa \left( e^{\mu \Gamma_{+4} U_{x,4} T_4} + e^{-\mu \Gamma_{-4} U_{x,4}^{-1} T_4} \right) \right]$$

$$= \prod_x \det \left( 1 + h e^{\mu/T} \mathcal{P}_x \right)^2 \det \left( 1 + h e^{-\mu/T} \mathcal{P}_x^{-1} \right)^2$$

with $$h = (2\kappa)^{N_\tau}$$ and (conjugate) Polyakov loops $$\mathcal{P}_x^{(-1)}$$

- static quarks propagate in temporal direction only: Polyakov loops
- full gauge dynamics included
THREE MODELS

II: SU(3) ONE LINK MODEL

\[ Z = \int dU e^{-S_B} \det M \quad \text{link } U \in \text{SU}(3) \]

\[ S_B = -\frac{\beta}{6} (\text{Tr } U + \text{Tr } U^{-1}) \]

determinant:

\[ \det M = \det \left[ 1 + \kappa \left( e^\mu \sigma_+ U + e^{-\mu} \sigma_- U^{-1} \right) \right] \]

\[ = \det (1 + \kappa e^\mu U) \det (1 + \kappa e^{-\mu} U^{-1}) \]

with \( \sigma_\pm = (\mathbb{1} \pm \sigma_3)/2 \)

- det in colour space remaining
- exact evaluation by integrating over the Haar measure

ECT*, March 2009 – p.8
THREE MODELS

III: U(1) ONE LINK MODEL

U(1) model: link \( U = e^{ix} \) with \(-\pi < x \leq \pi\)

\[ S_B = -\frac{\beta}{2} \left( U + U^{-1} \right) = -\beta \cos x \]

determinant:

\[ \det M = 1 + \frac{1}{2} \kappa \left[ e^{\mu U} + e^{-\mu U}^{-1} \right] = 1 + \kappa \cos(x - i\mu) \]

partition function:

\[ Z = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{\beta \cos x} \left[ 1 + \kappa \cos(x - i\mu) \right] \]

- all observables can be computed analytically
COMPLEX Langevin dynamics

Langevin update:

\[ U(\theta + \epsilon) = R(\theta) U(\theta) \]

\[ R = \exp \left[ i\lambda_a (\epsilon K_a + \sqrt{\epsilon} \eta_a) \right] \]

- **drift term**

\[ K_a = -D_a S_{\text{eff}} \]

\[ S_{\text{eff}} = S_B + S_F \]

\[ S_F = -\ln \det M \]

- **noise**

\[ \langle \eta_a \rangle = 0 \]

\[ \langle \eta_a \eta_b \rangle = 2\delta_{ab} \]

real action: \( \Rightarrow K^\dagger = K \iff U \in \text{SU}(3) \)

complex action: \( \Rightarrow K^\dagger \neq K \iff U \in \text{SL}(3, \mathbb{C}) \)
(CONJUGATE) POLYAKOV LOOPS

U(1) ONE LINK MODEL

- data points: complex Langevin stepsize $\epsilon = 5 \times 10^{-5}$, $5 \times 10^7$ time steps
- lines: exact results

excellent agreement for all $\mu$
(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL

- data points: complex Langevin stepsize $\epsilon = 5 \times 10^{-5}$, $5 \times 10^{7}$ time steps
- lines: exact results

excellent agreement for all $\mu$
(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL

scatter plot of $P$ during Langevin evolution
(CONJUGATE) POLYAKOV LOOPS

QCD WITH STATIC QUARKS

first results on $4^4$ lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase
**Density**

**U(1) one link model**

**SU(3) one link model**

- Linear increase at small $\mu$
- Saturation at large $\mu$

Excellent agreement for all $\mu$
Density QCD with static quarks

first results on $4^4$ lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase
SU(3) → SL(3, \mathbb{C})

QCD with static quarks

- complex Langevin dynamics: no longer in SU(3)
- instead \( U \in \text{SL}(3, \mathbb{C}) \)
- in terms of gauge potentials \( U = e^{i\lambda_a A_a / 2} \)
- \( A_a \) is now complex
- how far from SU(3)?

Consider

\[
\frac{1}{N} \text{Tr} U^\dagger U \begin{cases} 
= 1 & \text{if } U \in \text{SU}(N) \\
\geq 1 & \text{if } U \in \text{SL}(N, \mathbb{C})
\end{cases}
\]
\[
\frac{1}{3} \text{Tr} U^\dagger U \geq 1 = 1 \text{ if } U \in \text{SU}(3)
\]
Complexification of Phase Space

Why does it work?

- Most approaches start from $\mu = 0$ or $|\det M(\mu)|$.
- Complex Langevin dynamics radically different.
- $\Rightarrow$ complexification of degrees of freedom.
- Visualization in U(1) model.
- Understanding in terms of classical fixed points.
CLASSICAL FLOW

U(1) ONE LINK MODEL

- link \( U = e^{ix} \)
- complexification \( x \rightarrow z = x + iy \)

- Langevin dynamics:
  \[
  \dot{x} = K_x + \eta \quad \dot{y} = K_y
  \]

- classical forces:
  \[
  K_x = -\text{Re}\left(\frac{\partial S}{\partial x}\right)_{x \rightarrow z} \quad K_y = -\text{Im}\left(\frac{\partial S}{\partial x}\right)_{x \rightarrow z}
  \]

- classical fixed points: \( K_x = K_y = 0 \)
**CLASSICAL FLOW**

**U(1) ONE LINK MODEL**

flow diagrams and Langevin evolution

- black dots: classical fixed points
- $\mu = 0$: dynamics only in $x$ direction
- $\mu > 0$: spread in $y$ direction
CLASSICAL FLOW

U(1) ONE LINK MODEL

imag part of gauge potential
→
real part of gauge potential
intriguing questions:

- how severe is the sign problem?
- thermodynamic limit?
- phase transitions?
- Silver Blaze problem?

... study in a model with a phase diagram with similar features as QCD at low temperature

⇒ relativistic Bose gas at nonzero $\mu$ or scalar O(2) model
**Relativistic Bose gas at nonzero $\mu$**

**Phase transitions and the Silver Blaze**

- Continuum action

\[
S = \int d^4x \left[ |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 \\
+ \mu (\phi^* \partial^4 \phi - \partial^4 \phi^* \phi) + \lambda |\phi|^4 \right]
\]

- Complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu)$ as in QCD

ECT*, March 2009 – p.17
Relativistic Bose gas at nonzero $\mu$

Phase transitions and the Silver Blaze

- lattice action

$$S = \sum_x \left[ (2d + m^2) \phi_x^* \phi_x + \lambda (\phi_x^* \phi_x)^2 - \sum_{\nu=1}^{4} (\phi_x^* e^{-\mu \delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_x^* e^{\mu \delta_{\nu,4}} \phi_{x+\hat{\nu}}) \right]$$

- complex scalar field, $d = 4, m^2 > 0$

$$S^*(\mu) = S(-\mu)$$ as in QCD
tree level potential in the continuum

\[ V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4 \]

condensation when \( \mu^2 > m^2 \), SSB
Relativistic Bose gas at nonzero $\mu$

Complex Langevin

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \Rightarrow \phi_a \ (a = 1, 2)$

- complexification $\phi_a \rightarrow \phi^R_a + i\phi^I_a$

- complex Langevin equations

$$\frac{\partial \phi^R_a}{\partial \theta} = -\text{Re} \left. \frac{\delta S}{\delta \phi_a} \right|_{\phi_a \rightarrow \phi^R_a + i\phi^I_a} + \eta_a$$

$$\frac{\partial \phi^I_a}{\partial \theta} = -\text{Im} \left. \frac{\delta S}{\delta \phi_a} \right|_{\phi_a \rightarrow \phi^R_a + i\phi^I_a}$$

- straightforward to solve numerically, $m = \lambda = 1$

- lattices of size $N^4$, with $N = 4, 6, 8, 10$

- no instabilities etc
Relativistic Bose gas

Field modulus squared

$$|\phi|^2 \rightarrow \frac{1}{2} \left( \phi_R^2 \ - \ \phi_I^2 \right) \ + \ i\phi_R \phi_I$$

Silver Blaze!
field modulus squared  \[ |\phi|^2 \rightarrow \frac{1}{2} \left( \phi_a^R \phi_a^R - \phi_a^I \phi_a^I \right) + i\phi_a^R \phi_a^I \]

second order phase transition in thermodynamic limit
**RELATIVISTIC BOSE GAS**

**COMPLEX Langevin**

Density \( \langle n \rangle = \frac{1}{\Omega} \frac{\partial \ln Z}{\partial \mu} \)

![Graph](image)

Silver Blaze
**RELATIVISTIC BOSE GAS**

**COMPLEX LANGEVIN**

Density \( \langle n \rangle \) = \( (1/\Omega) \partial \ln Z / \partial \mu \)

second order phase transition in thermodynamic limit
Silver Blaze and sign problems are intimately related

- complex action
  \[ e^{-S} = |e^{-S}|e^{i\varphi} \]

- phase quenched theory
  \[ Z_{pq} = \int D\phi |e^{-S}| \]

Different physics

QCD: phase quenched = finite isospin chemical potential

different onset: \( m_N/3 \) versus \( m_\pi/2 \)
phase quenched theory in this case:

- real action
- chemical potential appears only in the mass parameter (in continuum notation)

\[ V = (m^2 - \mu^2) |\phi|^2 + \lambda |\phi|^4 \]

- dynamics of symmetry breaking, no Silver Blaze
**Silver Blaze and the Sign Problem**

**Complex vs Phase Quenched**

*density*

\[ \text{complex phase quenched} \]

\[ e^{i\varphi} = e^{-S}/|e^{-S}| \]

does precisely what is expected
HOW SEVERE IS THE SIGN PROBLEM?

- complex action $e^{-S} = |e^{-S}|e^{i\varphi}$
- full and phase quenched partition functions
  
  $Z_{\text{full}} = \int D\phi \, e^{-S}$ \quad $Z_{\text{pq}} = \int D\phi |e^{-S}|$

- average phase factor in phase quenched theory
  
  $\langle e^{i\varphi} \rangle_{\text{pq}} = \frac{Z_{\text{full}}}{Z_{\text{pq}}} = e^{-\Omega \Delta f} \rightarrow 0 \quad \text{as} \quad \Omega \rightarrow \infty$

- exponentially hard in thermodynamic limit
HOW SEVERE IS THE SIGN PROBLEM?

AVERAGE PHASE FACTOR

average phase factor $\langle e^{i\varphi} \rangle_{pq}$
How severe is the sign problem?

Average phase factor

- Phase factor behaves exactly as expected for larger $\mu$: phase factor $\to 0$ on all volumes
- In the condensed phase: phase factor $= 0$
- At small $\mu$, sign problem gets exponentially worse with increasing volume

Yet, no problem in practice
free Langevin dynamics

real Fokker-Planck distribution

include interactions with mean field approximation

see 0902.4686 [hep-lat] for more details
RELATIVISTIC BOSE GAS
FREE LANGEVIN DYNAMICS IN MOMENTUM SPACE

\[
\frac{\partial}{\partial \theta} \phi_{a,p}^R(\theta) = K_{a,p}^R(\theta) + \eta_{a,p}(\theta)
\]

\[
\frac{\partial}{\partial \theta} \phi_{a,p}^I(\theta) = K_{a,p}^I(\theta)
\]

\[
K_{a,p}^R = -A_p \phi_{a,p}^R + iB_p \varepsilon_{ab} \phi_{b,p}^I
\]

\[
K_{a,p}^I = -A_p \phi_{a,p}^I - iB_p \varepsilon_{ab} \phi_{b,p}^R
\]

\[
A_p = m^2 + 4 \sum_{i=1}^{3} \sin^2 \frac{p_i}{2} + 2 \left(1 - \cosh \mu \cos p_4 \right)
\]

\[
B_p = 2 \sinh \mu \sin p_4
\]
Relativistic Bose gas

Free Langevin dynamics

Solution:

\[ \phi_R^a(\theta, p) = e^{-A_p \theta} \left[ \cos(B_p \theta) \phi_R^a(0, p) + i \sin(B_p \theta) \epsilon_{ab} \phi_I^b(0, p) \right] \]

\[ + \int_0^\theta ds \ e^{-A_p (\theta - s)} \cos[B_p (\theta - s)] \eta_a(s, p) \]

\[ \phi_I^a(\theta, p) = e^{-A_p \theta} \left[ \cos(B_p \theta) \phi_I^a(0, p) - i \sin(B_p \theta) \epsilon_{ab} \phi_R^b(0, p) \right] \]

\[ -i \int_0^\theta ds \ e^{-A_p (\theta - s)} \sin[B_p (\theta - s)] \epsilon_{ab} \eta_b(s, p) \]

Convergence provided \( A_p > 0 \Rightarrow 4 \sinh^2 \frac{\mu}{2} < m^2 \)

Standard (in)stability for free Bose gas
convergence of two-point functions (provided $A_p > 0$)

\[
\lim_{\theta \to \infty} \langle \phi^R_{a, -p}(\theta) \phi^R_{b, p'}(\theta) \rangle = \delta_{ab} \delta_{pp'} \frac{1}{2A_p} \frac{2A_p^2 + B_p^2}{A_p^2 + B_p^2}
\]

\[
\lim_{\theta \to \infty} \langle \phi^I_{a, -p}(\theta) \phi^I_{b, p'}(\theta) \rangle = \delta_{ab} \delta_{pp'} \frac{1}{2A_p} \frac{B_p}{A_p^2 + B_p^2}
\]

\[
\lim_{\theta \to \infty} \langle \phi^R_{a, -p}(\theta) \phi^I_{b, p'}(\theta) \rangle = \varepsilon_{ab} \delta_{pp'} \frac{i}{2} \frac{B_p}{A_p^2 + B_p^2}
\]

structure agrees with symmetry of Langevin dynamics

observables constructed with these two-point functions agree with standard expressions
Relativistic Bose gas

Discretized Langevin equations

\[ \phi(n+1) = [1 + \epsilon M] \phi(n) + \sqrt{\epsilon} \eta(n) \]

are stable provided that \(|1 + \epsilon M| < 1\)

In this theory:

\[ A_p - \frac{\epsilon}{2} \left( A_p^2 + B_p^2 \right) > 0 \]

Constraint from high momentum modes

\[ \epsilon < \frac{2}{4d + m^2 + 2(cosh \mu - 1)} \]

\( \mu < m \): modest bound on \( \epsilon \)

\( \mu \gg m \): eventually \( \epsilon < e^{-\mu} \) (however, in region where lattice artefacts are severe)
complex distribution

\[ \langle O[\phi, \theta] \rangle_\eta = \int D\phi \, P[\phi, \theta] O[\phi] \]

satisfies the Fokker-Planck equation

\[ \frac{\partial P[\phi, \theta]}{\partial \theta} = \sum_x \frac{\delta}{\delta \phi_{a,x}(\theta)} \left( \frac{\delta}{\delta \phi_{a,x}(\theta)} + \frac{\delta S[\phi]}{\delta \phi_{a,x}(\theta)} \right) P[\phi, \theta] \]

stationary solution \( P[\phi] \sim e^{-S[\phi]} \)

not appropriate for the real Langevin process
Fokker-Planck equations

Real and complex distributions

Real distribution

\[ \langle O[\phi, \theta]\rangle_{\eta} = \int D\phi^R D\phi^I \rho[\phi^R, \phi^I, \theta] O[\phi^R + i\phi^I] \]

satisfies the extended Fokker-Planck equation

\[ \frac{\partial \rho[\phi^R, \phi^I, \theta]}{\partial \theta} = \sum_x \left[ \frac{\delta}{\delta \phi^R_{a,x}(\theta)} \left( \frac{\delta}{\delta \phi^R_{a,x}(\theta)} - K^R_{a,x}(\theta) \right) \right] \rho[\phi^R, \phi^I, \theta] \]

Stationary solutions not known in general!
Fokker-Planck equation

look for stationary solution ignoring interactions

\[
\sum_p \left[ \frac{\delta}{\delta \phi_{a,p}^R} \frac{\delta}{\delta \phi_{a,-p}^R} + \left( A_p \phi_{a,p}^R - i B_p \varepsilon_{ab} \phi_{b,p}^I \right) \frac{\delta}{\delta \phi_{a,p}^R} \right] + \left( A_p \phi_{a,p}^I + i B_p \varepsilon_{ab} \phi_{b,p}^R \right) \frac{\delta}{\delta \phi_{a,p}^I} + 2 A_p \right] \rho[\phi^R, \phi^I] = 0
\]

Gaussian problem
Fokker-Planck equation

**REAL DISTRIBUTION**

- solution

\[
\rho[\phi^R, \phi^I] = N \exp \left[ -\sum_p \left( \alpha_p \phi^R_{a,-p} \phi^R_{a,p} + \beta_p \phi^I_{a,-p} \phi^I_{a,p} + 2i \varepsilon_{ab} \gamma_p \phi^R_{a,-p} \phi^I_{b,p} \right) \right]
\]

\[
\alpha_p = A_p \quad \beta_p = \frac{A_p}{B_p^2} \left( 2A_p^2 + B_p^2 \right) \quad \gamma_p = \frac{A_p^2}{B_p}
\]

- generalized partition function

\[
Z = \prod_p \int d\phi^R_p d\phi^I_p \rho[\phi^R, \phi^I] = \mathcal{N} \prod_p \frac{1}{\alpha_p \beta_p - \gamma_p^2}
\]

- Gaussian integrals converge provided \(A_p > 0\)
correlation functions

\[ \langle \phi^R_{a,-p} \phi^R_{a,p} \rangle = - \frac{\partial \ln Z}{\partial \alpha_p} = \frac{\beta_p}{\alpha_p \beta_p - \gamma^2_p} = \frac{1}{A_p} \frac{2A_p^2 + B_p^2}{A_p^2 + B_p^2} \]

\[ \langle \phi^I_{a,-p} \phi^I_{a,p} \rangle = - \frac{\partial \ln Z}{\partial \beta_p} = \frac{\alpha_p}{\alpha_p \beta_p - \gamma^2_p} = \frac{1}{A_p} \frac{B_p^2}{A_p^2 + B_p^2} \]

\[ 2i \varepsilon_{ab} \langle \phi^R_{a,-p} \phi^I_{b,p} \rangle = - \frac{\partial \ln Z}{\partial \gamma_p} = \frac{-2\gamma_p}{\alpha_p \beta_p - \gamma^2_p} = \frac{-2B_p}{A_p^2 + B_p^2} \]

agree with solution of Langevin process when \( \theta \to \infty \)
distribution is singular as $p_4 \rightarrow 0$ or $\mu \rightarrow 0$

why?

- mode with $p_4 = 0$ is purely real, not complexified
- when $\mu = 0$, no need for complexification

$$\rho[\phi^R, \phi^I] = P[\phi^R] \delta(\phi^I)$$

include interactions?
**Mean field approximation**

**Langevin dynamics**

- include interactions on the mean field level
- Langevin equations contain terms of form $\lambda\phi^3$
- Gaussian factorization: example

$$\phi_{b,x}^R \phi_{b,x}^I \phi_{a,x}^R \rightarrow \langle \phi_{b,x}^R \phi_{b,x}^I \phi_{a,x}^R \rangle \phi_{a,x}^R + \langle \phi_{b,x}^R \phi_{a,x}^R \phi_{b,x}^I \rangle \phi_{b,x}^I + \langle \phi_{b,x}^I \phi_{a,x}^R \phi_{b,x}^R \rangle \phi_{b,x}^R$$

- solve for fixed points, etc
- when all the dust settles:

$$A_p \rightarrow A_p = A_p + 4\lambda \langle |\phi|^2 \rangle$$

as expected (mean field mass/tadpole resummation)
- solve mass from self-consistent gap equation
Relativistic Bose gas

Mean Field Comparison

\[
\text{complex phase quenched lines are mean field predictions}
\]
Relativistic Bose gas

Mean field comparison

density

Complex phase quenched lines are mean field predictions
Relativistic Bose Gas

Mean Field Analysis

- Mean field analysis in noncondensed phase (Silver Blaze region)
- Can be analyzed in detail
- Agreement with numerical results

- Extension to condensed phase
- Include \( \langle \phi_{a,x}^R \rangle \neq 0 \)
- In progress....
many stimulating results

- one link models: excellent agreement
- relativistic Bose gas: phase transition and Silver Blaze
- QCD with static quarks: encouraging

why does it work?
partly understood in simple models and relativistic Bose gas
in progress:

- more analytical insight in the relativistic Bose gas
- QCD with static and dynamical quarks
- ...