Issues in analytic continuation: the critical line, spectral properties and the average phase factor.

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In collaboration with P. Cea (Bari), L. Cosmai (Bari), C. Manneschi (Genoa), A. Papa (Cosenza) and F. Sanfilippo (Rome)

1 – OUTLINE

Waiting for a definite solution to the sign problem, analytic continuation from imaginary chemical potentials remains one of the few available methods to investigate QCD at finite density. More efforts are needed to test its reliability and to push its predictivity. In this talk we focus on the following issues:

- **Analytic continuation of the (pseudo)critical line: precision tests in sign problem free theories.** with P. Cea (Bari), L. Cosmai (Bari), C. Manneschi (Genoa), A. Papa (Cosenza)
  - **Is it reliable?**
  - **What is the best strategy to perform it?**

- $N_f = 2$ **QCD thermodynamics and analytic continuation from two independent imaginary chemical potentials** with F. Sanfilippo (Rome)
  - **validity of the HRG model for** $T < T_c$
  - **computation of generalized susceptibilities**
  - **analytic continuation of the average phase factor** $\langle e^{i2\theta} \rangle$

The sign problem disappears for imaginary values of the chemical potential

$$Z(T,\mu^2) \equiv \int \mathcal{D}U e^{-S_G} \det M[\mu] \longrightarrow \int \mathcal{D}U e^{-S_G} \det M[\mu = i\mu_I]$$

$$M[\mu]_{i,j} = am\delta_{i,j} + \frac{1}{2}\sum_{\nu=1}^{3} \eta_{i,\nu}\left(U_{i,\nu}\delta_{i,j-\hat{\nu}} - U^{\dagger}_{i-\hat{\nu},\nu}\delta_{i,j+\hat{\nu}}\right) + \eta_{i,4}\left(e^{a\mu}U_{i,4}\delta_{i,j-\hat{4}} - e^{-a\mu}U^{\dagger}_{i-\hat{4},4}\delta_{i,j+\hat{4}}\right)$$

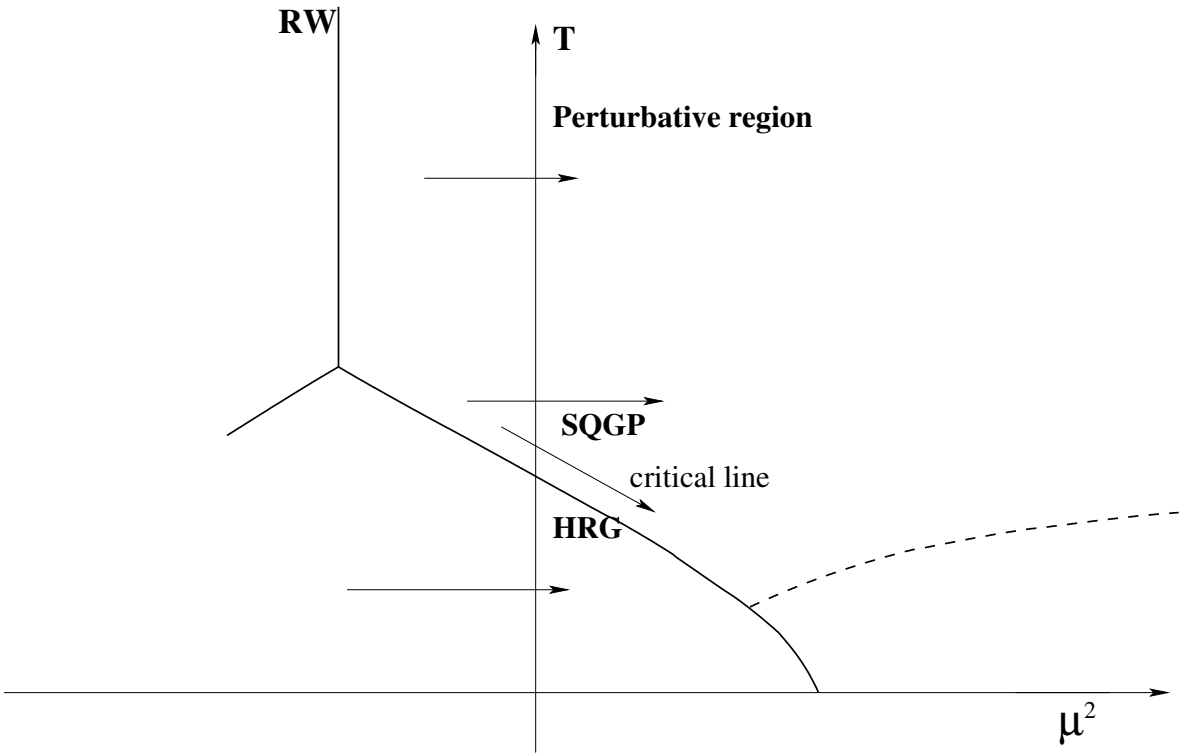
$$M[i\mu_I]_{i,j} = am\delta_{i,j} + \frac{1}{2}\sum_{\nu=1}^{3} \eta_{i,\nu}\left(U_{i,\nu}\delta_{i,j-\hat{\nu}} - U^{\dagger}_{i-\hat{\nu},\nu}\delta_{i,j+\hat{\nu}}\right) + \eta_{i,4}\left(e^{ia\mu_I}U_{i,4}\delta_{i,j-\hat{4}} - e^{-ia\mu_I}U^{\dagger}_{i-\hat{4},4}\delta_{i,j+\hat{4}}\right)$$

Z is an even function of μ and periodic in μ_I/T with period $2\pi/N_c$ (by center symmetry).

ANALYTIC CONTINUATION $\equiv$ a given ansatz for the dependence of physics on μ^2 can be continued to $\mu^2 < 0$ and checked (fitted) against numerical data at imaginary chemical potentials

Predictivity restricted by domains of analyticity

Systematics affected by the choice of the ansatz



- **Below T_c: the whole range of imaginary μ is available, predictions at real μ restricted by the physical (pseudo)-transition line.**
- **Above T_c: imaginary μ available for analytic continuation restricted by physical or unphysical (RW) transitions.**
- **Assuming the physical transition line is continuous at $\mu^2 = 0$, also the critical line can be continued.**

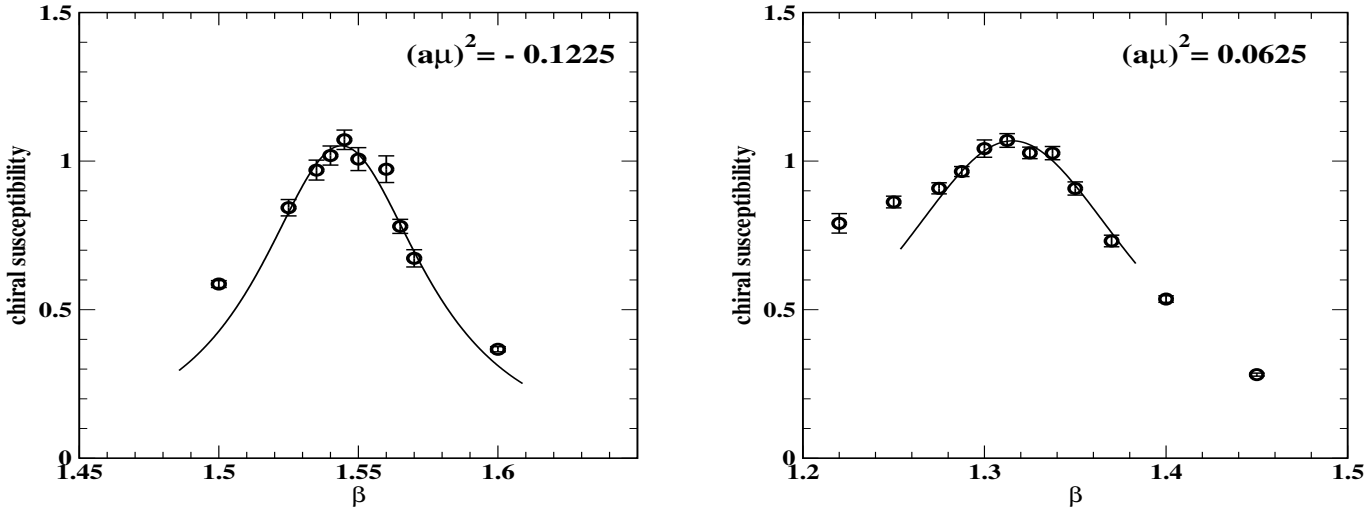
2 – Analytic continuation of the critical line

Careful checks of possible systematics in analytic continuation can be performed in theories which are free of the sign problem. An example is QCD with two colors: Each loop contribution to the hopping expansion is real because of the $SU(2)$ gauge group, no cancellations required.

We have considered the continuation of the critical line in $SU(2)$ with 8 flavors, standard staggered fermions, $am = 0.07$, $L_s = 16$ and $L_t = 4$, standard HMC algorithm.

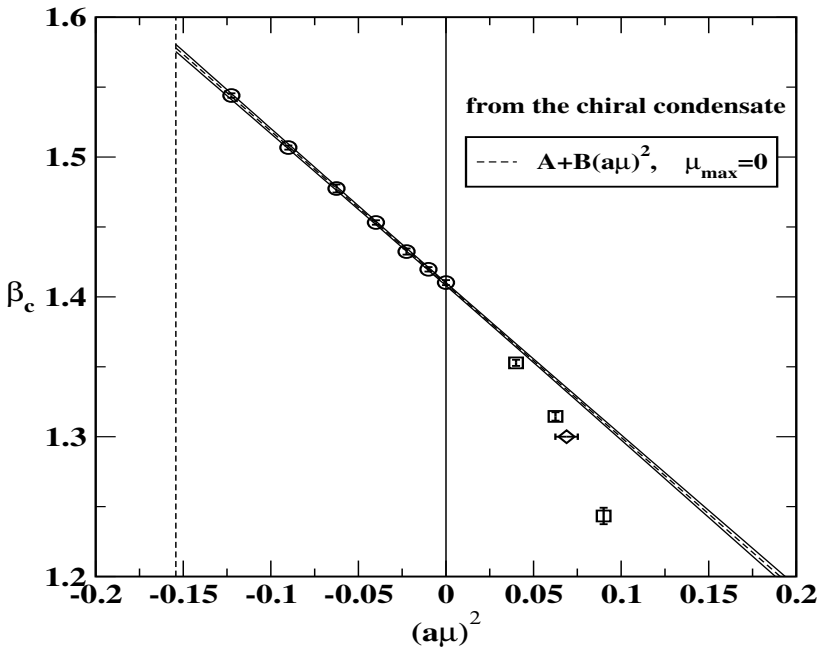
P. Cea, L. Cosmai, M. D'E., A. Papa, PRD 77, 051501(R) (2008)

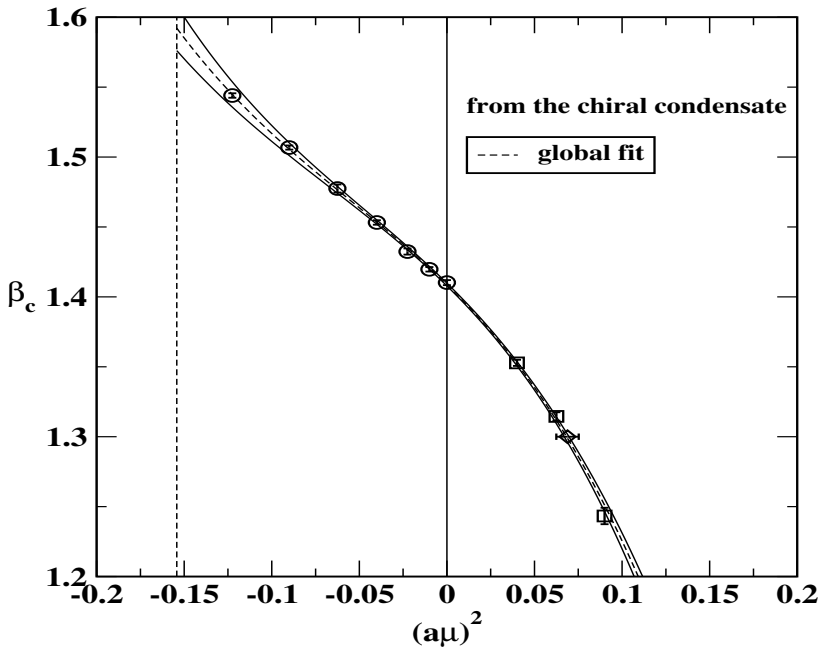
Location of pseudocritical couplings performed by looking at susceptibility peaks



$(a\mu)^2$	$\langle\overline{\psi}\psi\rangle$	χ^2/d.o.f.	$\langle L\rangle$	χ^2/d.o.f.	$\langle P\rangle$	χ^2/d.o.f.
-0.1225	1.5440(16)	1.34	1.5349(43)	0.85	1.5418(24)	0.93
-0.09	1.5068(15)	0.65	1.5019(29)	0.25	1.5046(21)	1.06
-0.0625	1.4775(29)	0.88	1.4665(32)	0.31	1.4755(37)	0.65
-0.04	1.4532(16)	0.50	1.4453(36)	0.76	1.4522(26)	1.21
-0.0225	1.4324(22)	1.20	1.4240(28)	0.80	1.4300(39)	0.80
-0.01	1.4197(16)	1.86	1.4104(33)	0.43	1.4199(26)	1.45
0.	1.4102(18)	0.07	1.3989(61)	0.49	1.4117(32)	0.07
0.04	1.3528(22)	2.91	1.3388(72)	1.01	1.3631(46)	1.16
0.0625	1.3145(30)	1.34	1.2976(62)	0.87	1.3286(50)	1.28
0.09	1.2433(59)	1.09	1.2508(62)	0.98	1.2864(109)	0.60

Table of (pseudo)critical couplings at different μ^2, obtained by fitting the peaks of different observables.





Data at $\mu^2 < 0$ cannot predict terms beyond the linear one in μ^2, $\beta_c(\mu) = A + B\mu^2$

$A = 1.4091(17), B = -1.095(15), \tilde{\chi}^2 = 0.27$ **but that fails to reproduce data at real μ.**

But a sixth order polynomial, $\beta_c(\mu) = A + B\mu^2 + C\mu^4 + D\mu^6$, nicely fits all data!

$A = 1.4088(99), B = -1.230(25), C = -3.77(25) D = -22.7(3.6), \tilde{\chi}^2 = 1.0$

Analyticity at $\mu^2 = 0$ not contradicted, but analytic continuation not predictive enough!

Suppressed, hardly visible contributions ($C\mu^4 + D\mu^6$ in our case) becoming important in different regions are a typical problem of analytic continuation.

Similar problems could apply to real QCD as well: non-linear terms in μ^2 in the critical line could be missed by analytic continuation.

It is worth checking that in sign problem free theories which are closer to QCD: we are investigating QCD in presence of a finite isospin chemical potential:

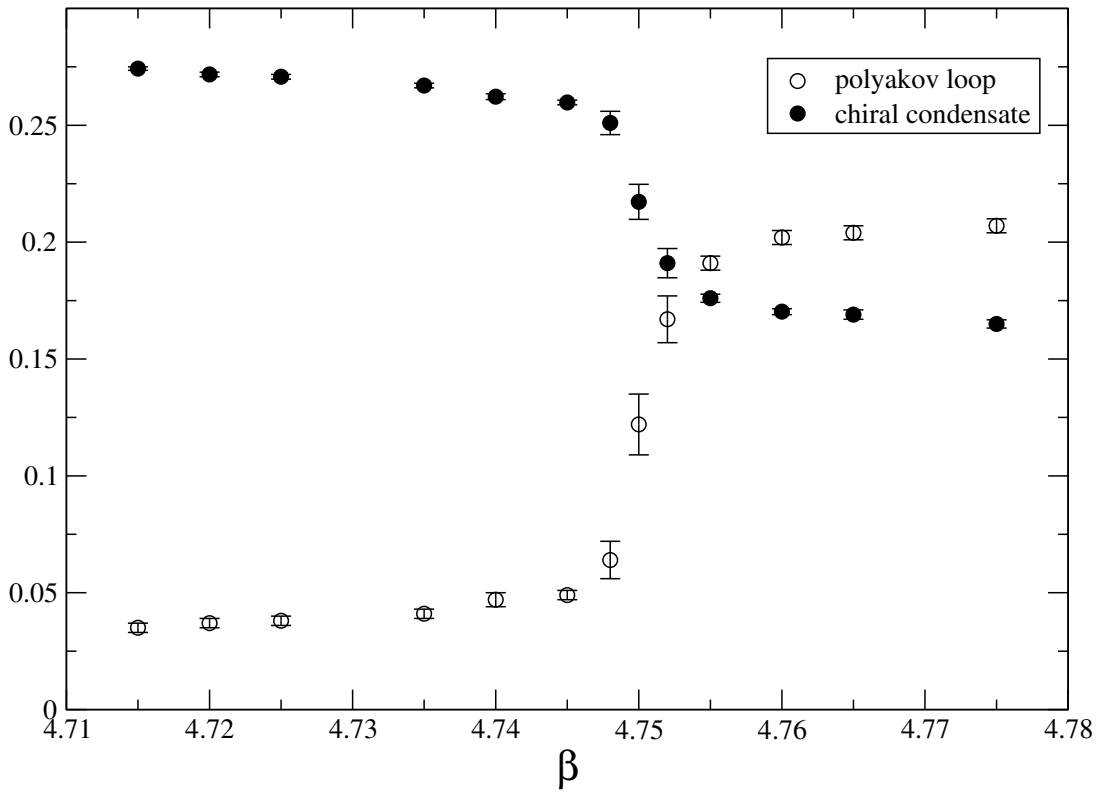
P. Cea, L. Cosmai, C. Manneschi, A. Papa, in progress

$$Z(T,\mu,-\mu) \equiv \int \mathcal{D}U e^{-S_G} \det M[\mu] \det M[-\mu] = \int \mathcal{D}U e^{-S_G} |\det M[\mu]|^2$$

because of $\det M[-\mu] = \det M[\mu]^*$

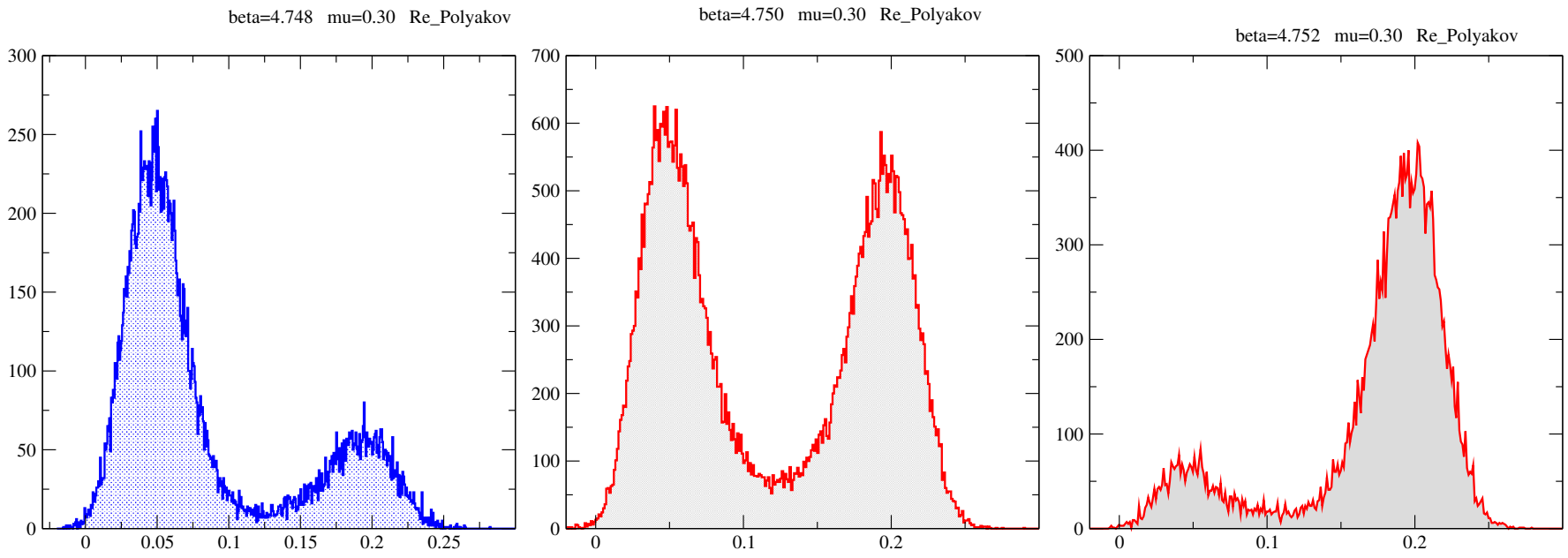
We have considered QCD with 8 flavors (4+4), standard staggered fermions, $am = 0.1$**. The transition at** $\mu^2 = 0$ **is strong first order in this case, and remains so also at** $\mu^2 \neq 0$**.**

Simulations have been done on a relatively small lattice ($8^3 \times 4$**) to avoid too long tunneling times. Critical couplings** $\beta_c(\mu^2)$ **have been determined by looking at double peak distributions around the transition.**



We show as an example how we determine the transition location for $\mu/(\pi T) = i\,0.30$**.**

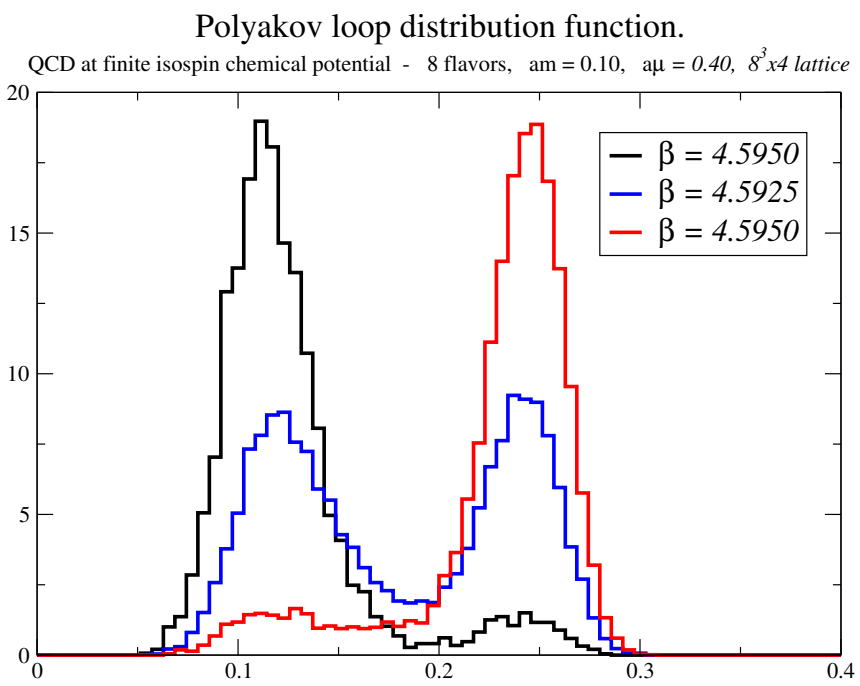
A rough idea about β_c **is obtained by looking at the behaviour of the observables**

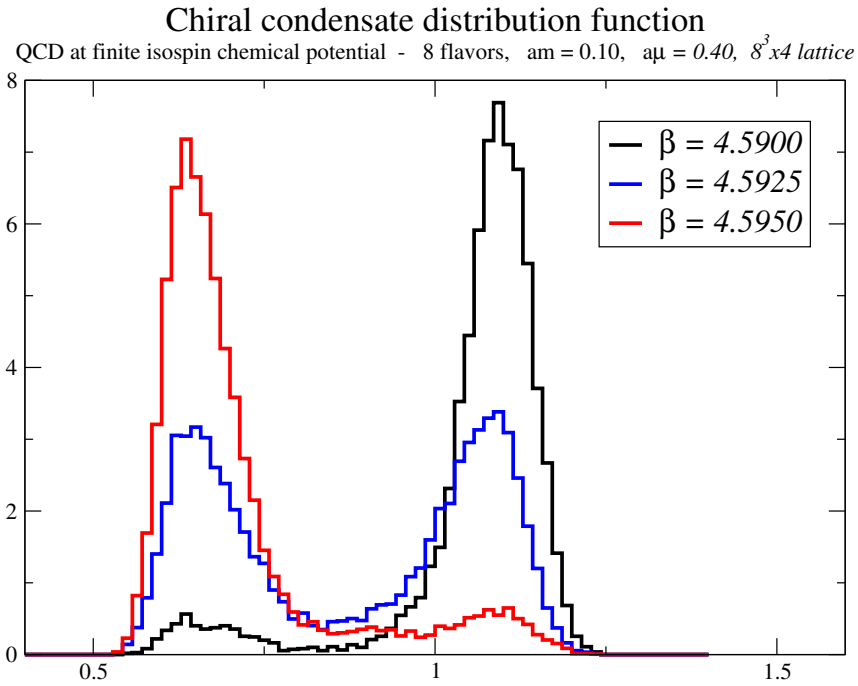


$\beta = 4.748$ $\beta = 4.750$ $\beta = 4.752$

β_c is then obtained by looking at double peak distributions (Re(Polyakov) is shown), which are typically taken over $5 - 10 \cdot 10^4$ trajectories (about 10-20 tunneling events observed at β_c)

we estimate $\beta_c = 4.750(1)$ in this case

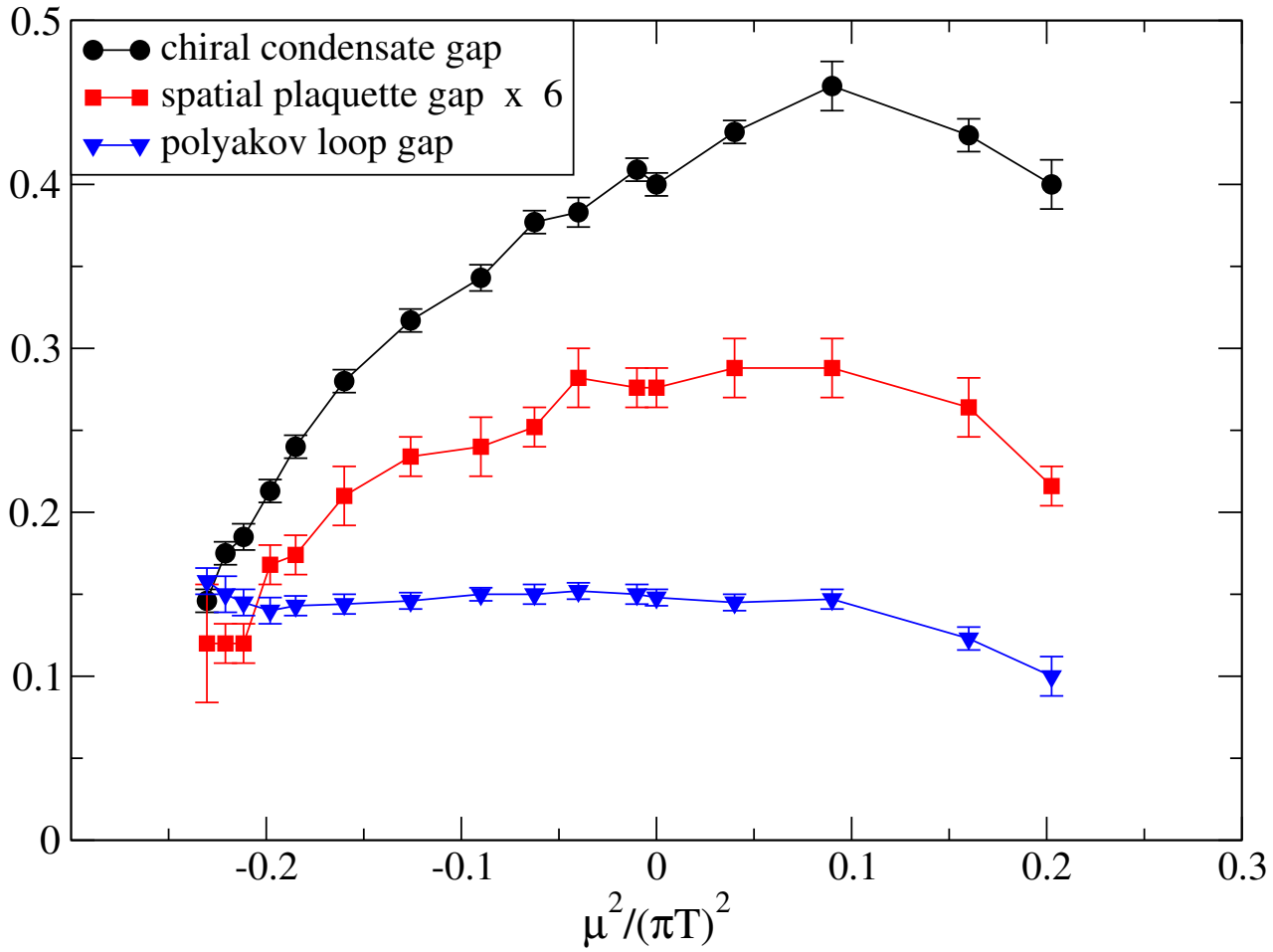




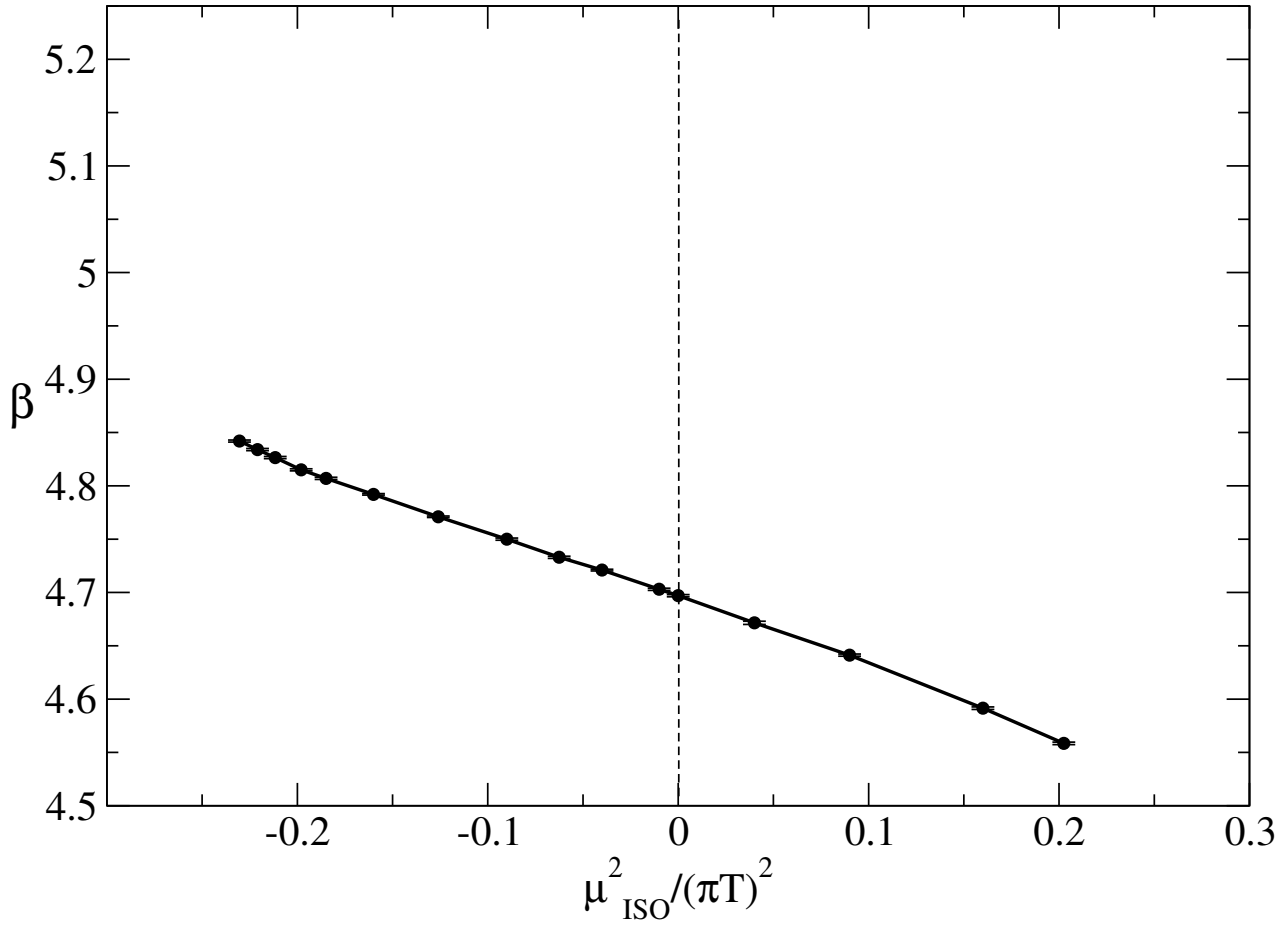
Re(Polyakov) **chiral condensate**

A similar analysis applies to other chemical potentials, $\mu/(\pi T) = 0.40$ is shown in the figures.

Of course different observables jump at the same point (first order transition).

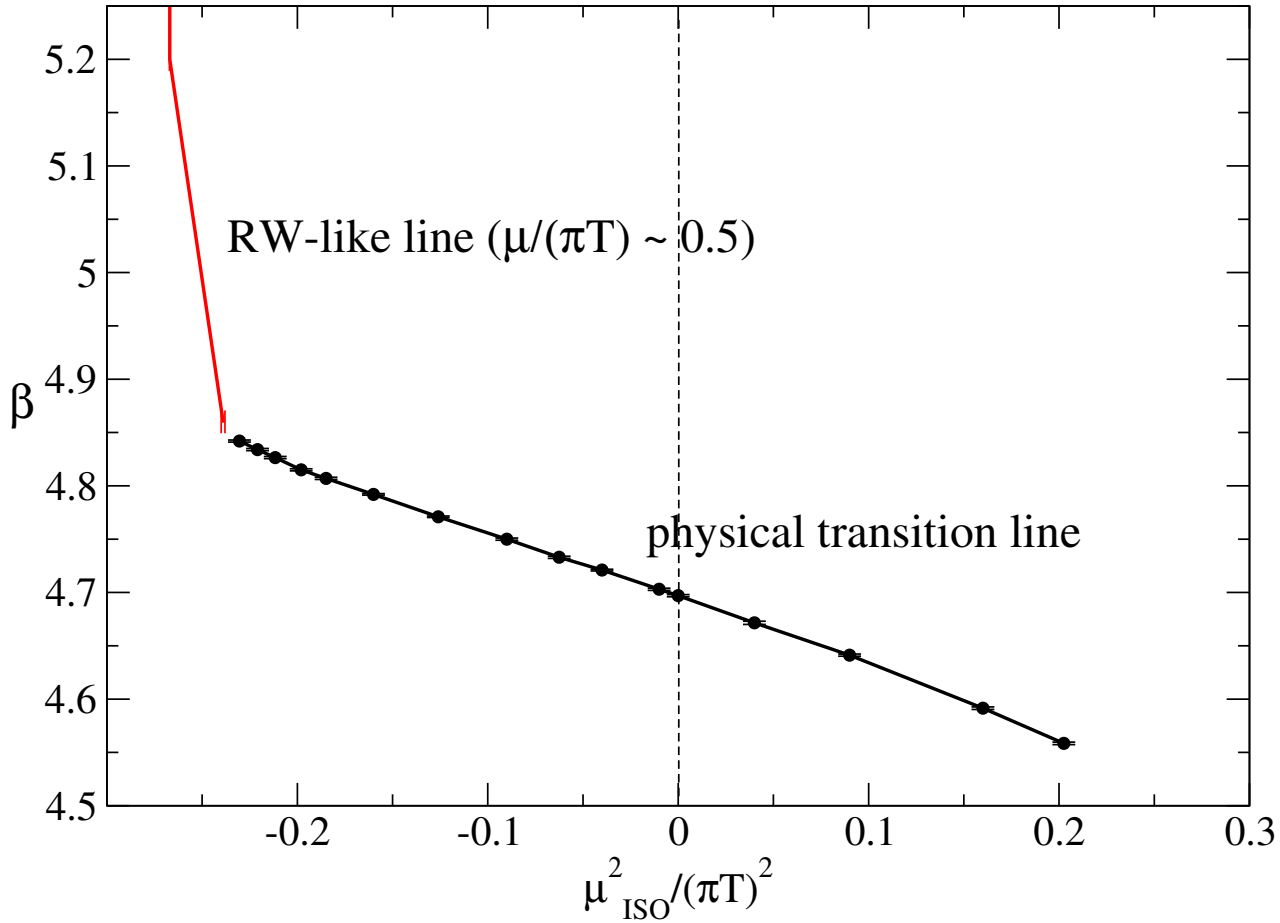


This is the behaviour of the gap (jump) at the transition as a function of μ^2 for different observables. The transition stays first order over the whole range explored, its strength smoothly changes with μ.



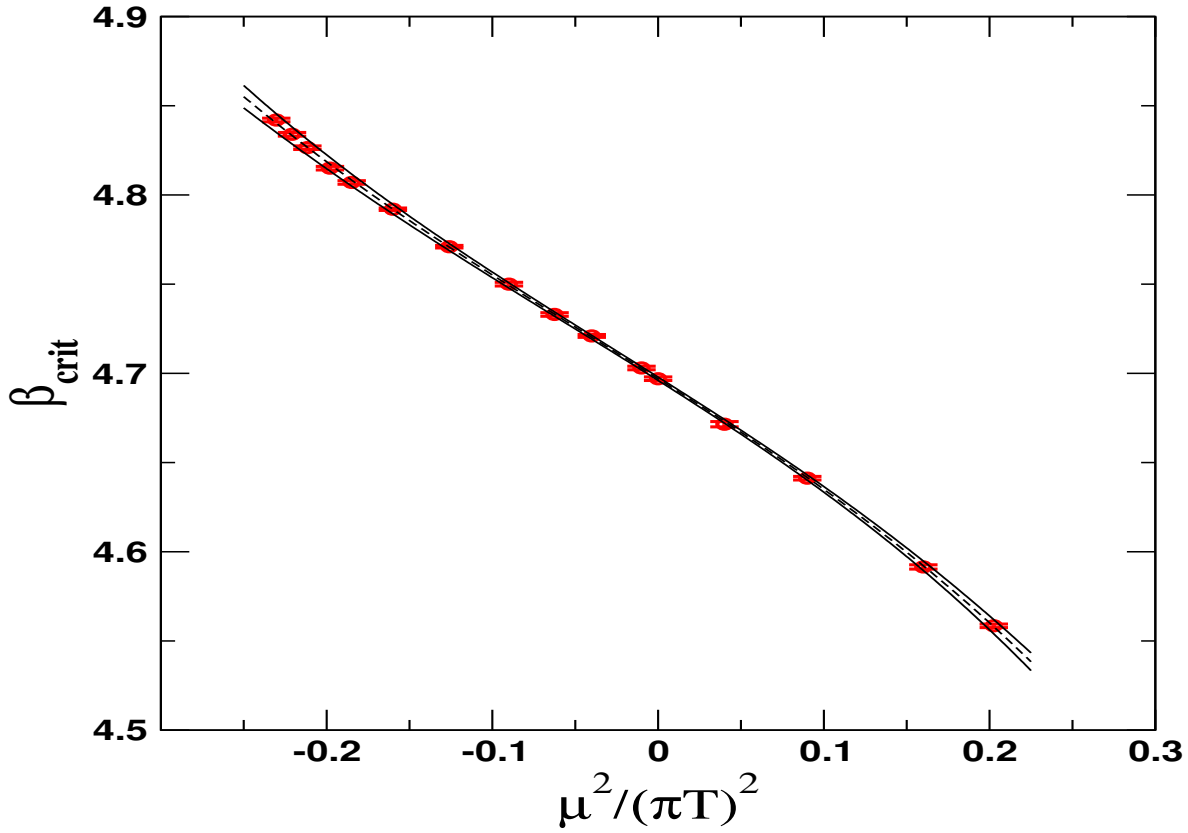
This is the whole collection of our determinations of $\beta_c(\mu^2)$, which will be used to test analytic continuation.

The range of μ^2 is limited on the right hand side by the increasing computational power needed to simulate large real chemical potential (Dirac eigenvalues get close to the origin)



On the left hand side the range is limited by an unphysical transition similar to the RW-transition present for imaginary baryon chemical potentials ($\mu^2/(\pi T)^2 \sim -0.25$).

We have taken many determinations of $\beta_c(\mu^2)$ close to this RW-like line, in order increase sensitivity to non-linear terms.



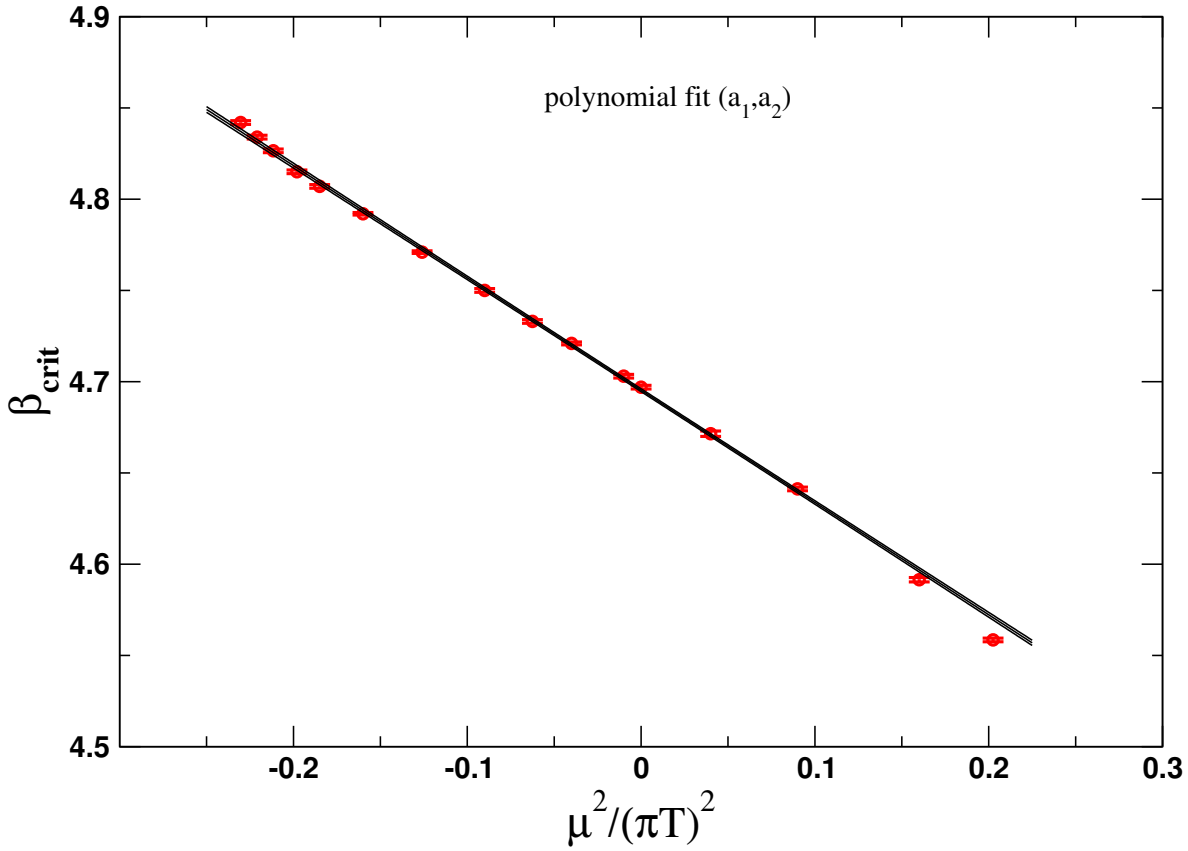
The whole set of data is again nicely fitted by a sixth order polynomial:

$\beta_c(\mu) = A + B\mu^2 + C\mu^4 + D\mu^6$

$A = 4.6970(10), B = -0.585(12), C = -0.19(4), D = -1.53(35), \tilde{\chi}^2 = 1.1$

Features common to $SU(2)$: all coefficients are negative, μ^4 and μ^6 contributions tend to cancel at $\mu^2 < 0$ also in this case.

Let us see whether we can predict the right behaviour from imaginary μ's only.

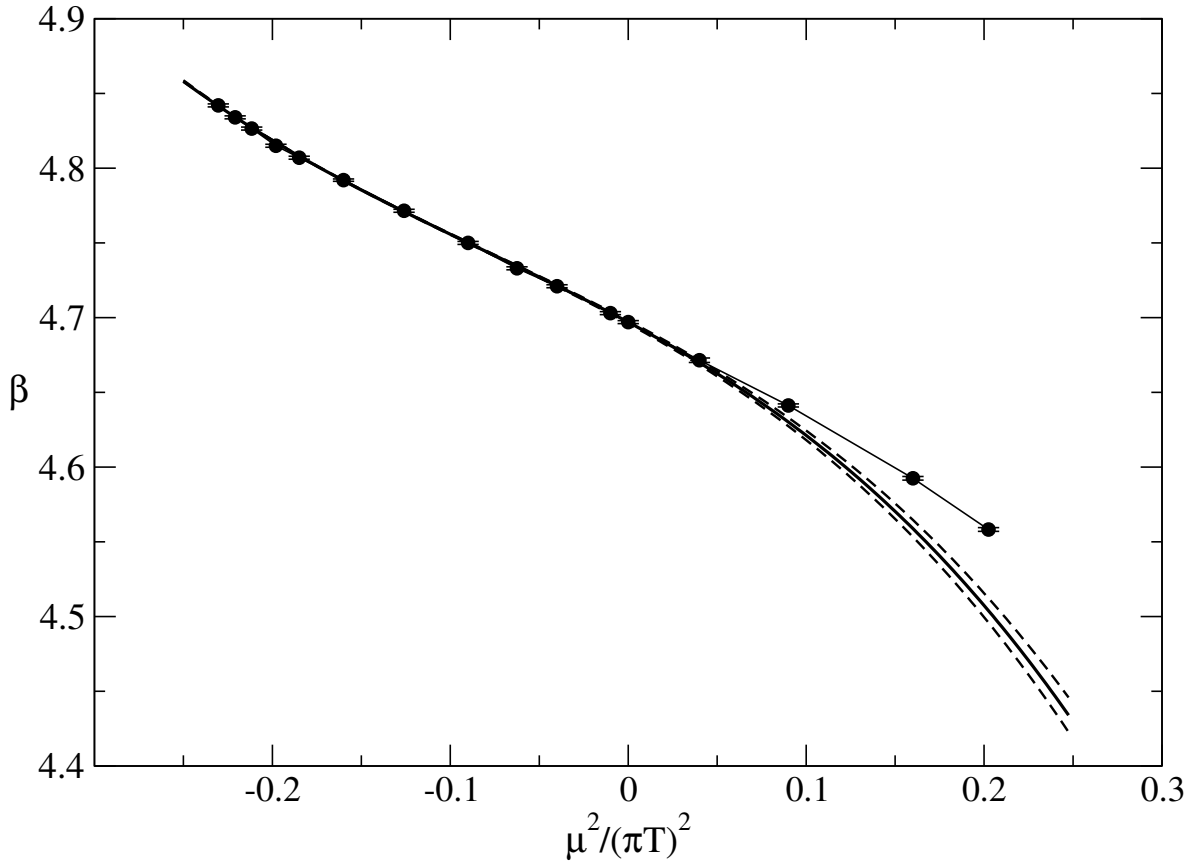


Good news: the simple linear ansatz in μ^2 does not work.

$\beta_c(\mu) = A + B\mu^2$

$A = 4.6953(8)$, $B = -0.615(5)$, $\tilde{\chi}^2 = 6.3$

Data at imaginary chemical potentials contain enough information in this case, also about non-linear terms in μ^2



Unfortunately a simple sixth order fit does not work and fails reproducing data at real μ's

$$\beta_c(\mu) = A + B\mu^2 + C\mu^4 + D\mu^6$$

$$A = 4.6967(17), B = -0.636(7), C = -0.83(74), D = -3.4(2.1), \tilde{\chi}^2 = 0.88$$

PROBLEM:

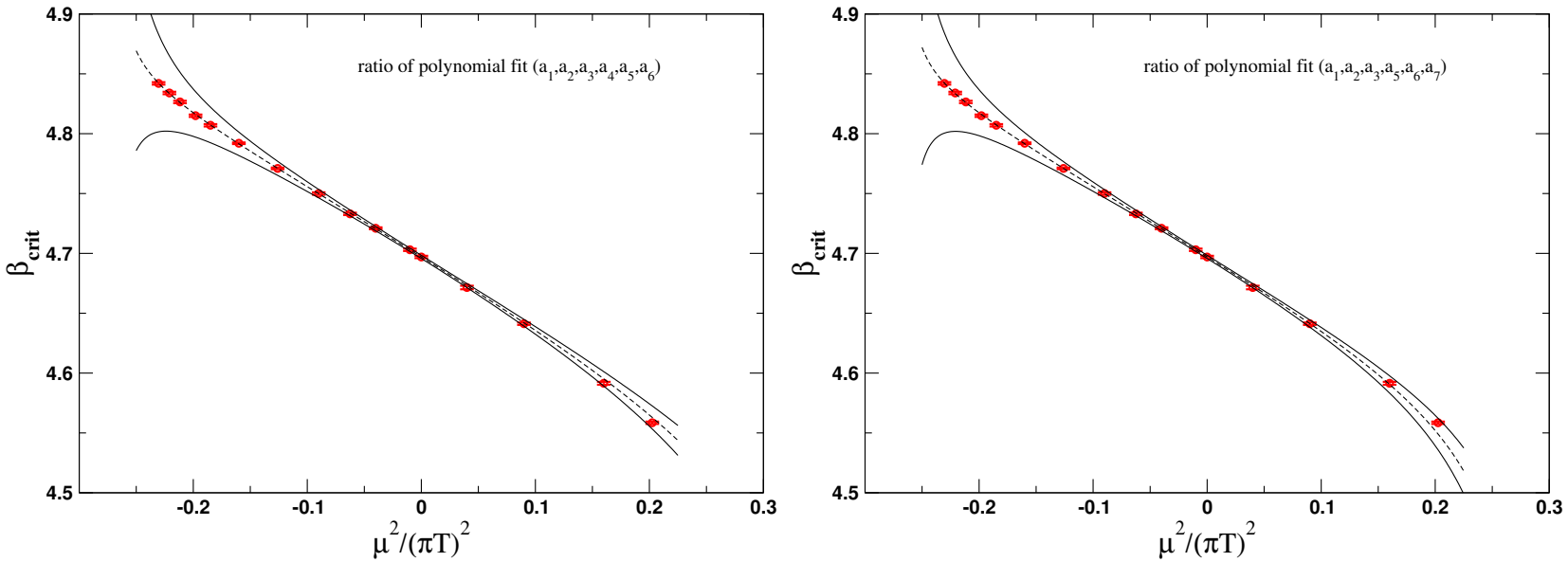
Differents sixth order polynomials exist, almost coincident in the available $\mu^2 < 0$ region, but with significant differences at $\mu^2 > 0$.

POSSIBLE SOLUTIONS?

- **Try different functional forms, for instance Padé approximants $P(n,m)$ (ratio of two polynomials of order n and m), which may reveal successful (M.P.Lombardo, Lattice 2005, Cea-Cosmai-D'Elia-Papa, JHEP02(2007)066)**

$$P(n,m) = \frac{a_0 + a_1\mu^2 + \ldots a_n\mu^{2n}}{1 + b_1\mu^2 + \ldots b_m\mu^{2m}}$$

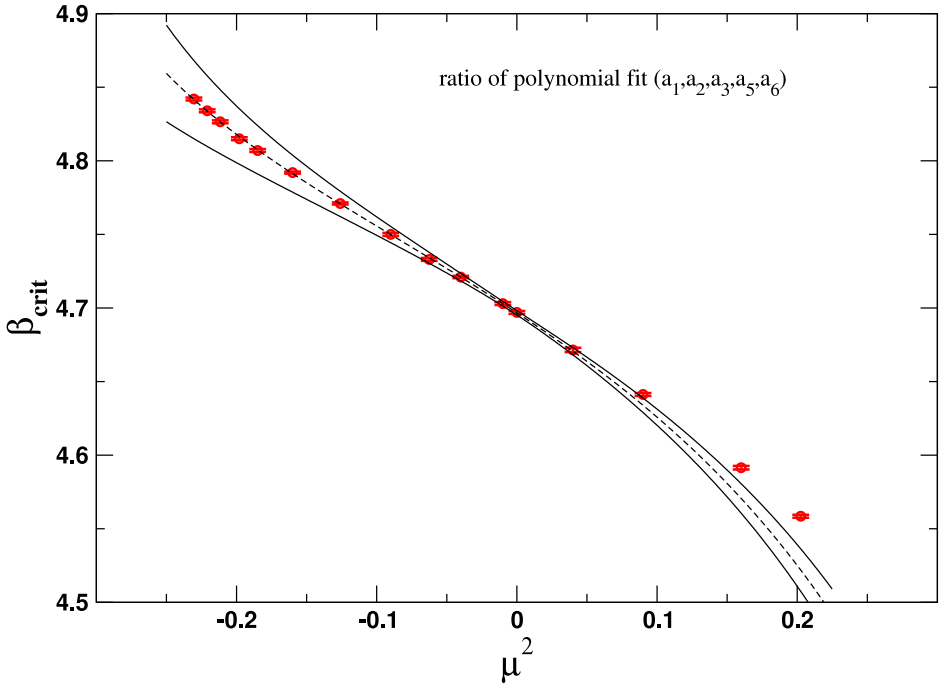
- **Try to constrain the linear term in μ^2 by looking at small chemical potentials only.**



Pade approximants seem to work fine, but at least a $P(6,4)$ (center) or a $P(4,6)$ (right) are needed.

Why two more parameters necessary?

Ratio of polynomials (Pade approximants) instead work perfectly, but at least (6,4) or (4,6) is needed. Could be a possibility, but why two more parameters?

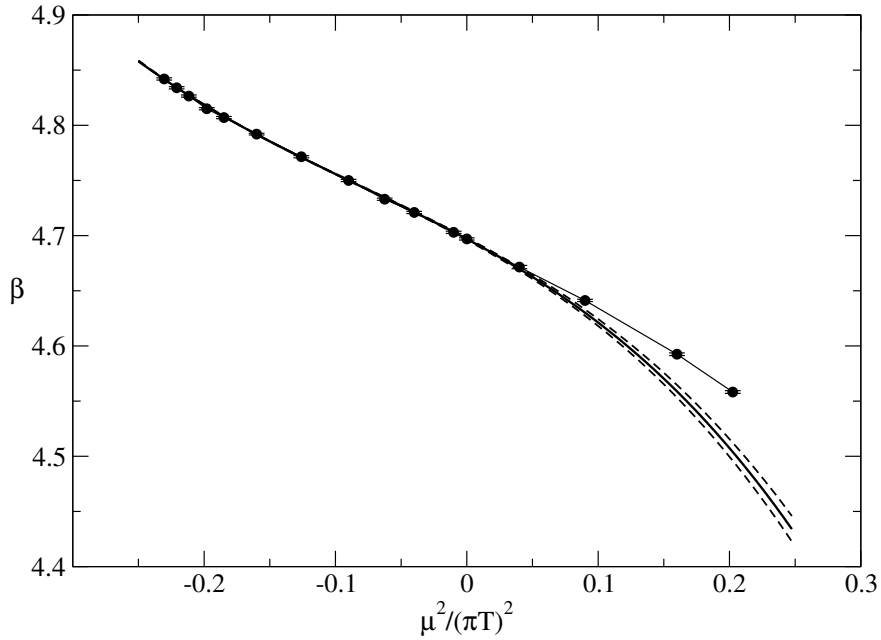


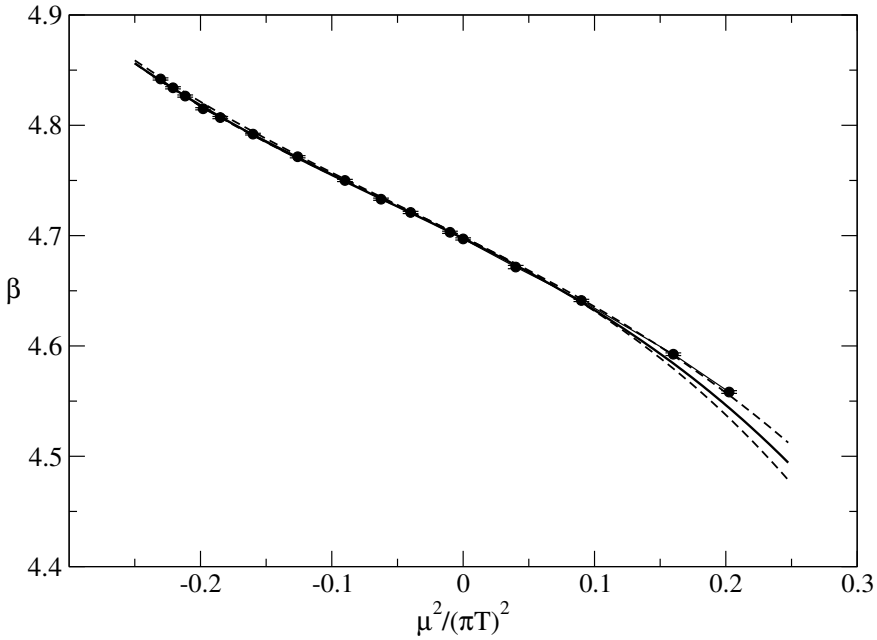
We show for comparison the case of a $P(4,4)$

Try to fix the linear term according to the information from small chemical potentials, look at $A + B\mu^2$ fits in limited ranges:

$(\mu)^2_{\rm min}$	B	$\chi^2/{\rm d.o.f.}$
-0.04	-0.60(3)	0.000007
-0.0625	-0.58(2)	0.3
-0.09	-0.585(13)	0.25
-0.126025	-0.589(9)	0.24
-0.16	-0.593(6)	0.25
-0.1849	-0.594(6)	0.24
-0.198025	-0.595(5)	0.23
-0.2116	-0.602(4)	1.5

A reasonable choice is to constrain $-0.596 < B < -0.584$ also in fits with higher order polynomials





Now things work much better: here we compare the unconstrained (left) with the constrained fit (right).

$A = 4.6974(8), B = -0.5960(12), C = -0.427(14), D = -2.4(6), \tilde{\chi}^2 = 1.07$

LESSON: fix linear term by simulations at small chemical potentials, or by other methods (reweighting or Taylor expansion), then analytic continuation is predictive enough to correctly reproduce non-linear terms.

Alternatively, Pade approximants work fine, but need more free parameters.

3 – Analytic continuation from two imaginary chemical potentials

In collaboration with F. Sanfilippo

Next we switch from tests of analytic continuation to predictions for theories with a sign problem.

We are investigating $N_f = 2$ QCD with two independent chemical potentials coupled to each quark flavor

$$Z(T, \mu_1, \mu_2) \equiv \int \mathcal{D}U e^{-S_G} \det M^{\frac{1}{4}}[\mu_1] \det M^{\frac{1}{4}}[\mu_2]$$

μ_1 and μ_2 can be rewritten in terms of a quark number and of an isospin chemical potential

$$\mu_q = \frac{\mu_1 + \mu_2}{2} = \frac{\mu_B}{3}$$

$$\mu_I = \frac{\mu_1 - \mu_2}{2}$$

While the original theory is invariant under both charge conjugation and isospin rotations, the theory in presence of finite chemical potentials obviously is not. However the original invariance is reflected in the fact that the free energy $F = -T \ln Z$ must be an even function of μ_q and μ_I separately, i.e. invariant under

$$(\mu_q, \mu_I) \to (\mu_q, -\mu_I) \qquad \text{i.e.} \qquad (\mu_1, \mu_2) \to (\mu_2, \mu_1)$$

and

$$(\mu_q, \mu_I) \to (-\mu_q, \mu_I) \qquad \text{i.e.} \qquad (\mu_1, \mu_2) \to (-\mu_2, -\mu_1)$$

Apart from the case $\mu_1 = -\mu_2$ ($\mu_B = 0$, i.e. QCD at finite isospin chemical potential) numerical simulations are not feasibile but at imaginary values of μ_1 and μ_2. Analytic continuation is in this case continuation from negative to positive values of μ_q^2 and μ_I^2.

We are interested in reconstructing the dependence of the free energy on both chemical potentials. To that aim we determine free energy first derivatives (quark number densities) at imaginary μ's

$$\begin{aligned}
\hat{n}_q &\equiv \frac{\langle N_q \rangle}{VT^3} = \frac{\partial}{\partial \mu_q}(p/T^4) = \hat{n}_1 + \hat{n}_2 \\
\hat{n}_I &\equiv \frac{\langle N_{I_3} \rangle}{VT^3} = T\frac{\partial}{\partial \mu_I}(p/T^4) = \hat{n}_1 - \hat{n}_2
\end{aligned}$$

where N_q and N_{I_3} are the quark number and isospin charge operators, while

$$\hat{n}_i \equiv \frac{\langle N_i \rangle}{VT^3} = \frac{1}{VT^2}\frac{\partial \ln Z}{\partial \mu_i} = -\frac{1}{VT^3}\frac{\partial F}{\partial \mu_i} = \frac{N_t^2}{4N_s^3}\left\langle \mathrm{Tr}\left(M^{-1}[U,\mu_i]\frac{\partial}{\partial a\mu_i}M[U,\mu_i]\right)\right\rangle$$

Once the functional dependence of first derivatives has been fixed (fitted by a proper ansatz), the free energy is known up to a constant term.

Knowing the free energy up to a constant we can:

- **Compare free energy dependence with the prediction of the Hadron Resonance Gas model below T_c**
- **Compute generalized susceptibilities at zero chemical potential and compare with the Taylor expansion method (check consistency and efficiency)**

$$\chi_{i,j} = \frac{\partial^{i+j}}{\partial\mu_1^i \partial\mu_2^j}\left(-\frac{F}{VT^4}\right) = \frac{\partial^{i+j}}{\partial\mu_1^i \partial\mu_2^j}(p/T^4)$$

- **Determine the average phase factor measured in the phase quenched theory**

$$\langle e^{i2\theta}\rangle_\mu \equiv \left\langle \frac{\det M(\mu)}{\det M(-\mu)} \right\rangle_{(\mu,-\mu)} = \frac{Z(\mu,\mu)}{Z(\mu,-\mu)} = \exp\left(-\frac{F(\mu,\mu) - F(\mu,-\mu)}{T}\right)$$

Expected periodicities in the imaginary chemical potentials:

It is useful to define

$$\theta_q = \mathrm{Im}(\mu_q)/T = N_t a \mathrm{Im}(\mu_q)$$
$$\theta_I = \mathrm{Im}(\mu_I)/T = N_t a \mathrm{Im}(\mu_I)$$

and

$$\theta_1 = \mathrm{Im}(\mu_1)/T = \theta_q + \theta_I$$
$$\theta_2 = \mathrm{Im}(\mu_2)/T = \theta_q - \theta_I$$

introducing imaginary chemical potentials can be viewed as a rotation of temporal boundary conditions of the two quarks by θ_1 and θ_2 respectively.

The free energy is trivially periodic in θ_1 and θ_2 with period 2π. That means periodicity 2π in both θ_q and θ_I plus invariance for $(\theta_q, \theta_I) \to (\theta_q + \pi, \theta_I + \pi)$.

Furthermore, following the argument given by Roberge and Weiss, a rotation $\theta_q \to \theta_q + 2\pi k/N_c$ can be cancelled by a center transformation of gauge links, hence the actual periodicity in θ_q is $2\pi/N_c = 2\pi/3$.

At low temperatures ($T < T_c$) F is smooth function of θ_q, θ_I, hence its most natural parametrization (even in θ_q and θ_I) is as follows

$$F(\theta_q, \theta_I) = V \sum_{h,l} W_{h,l} \cos(3h\theta_q) \cos(l\theta_I)$$

where h, l run over all couples of integers having the same parity.

Further constraints on the number of terms may be predicted by particular effective models of strong interactions below T_c, like for instance the HRG model (more later).

In such regime information valid for analytic continuation can be gathered in the whole θ_q, θ_I plane.

At high temperatures unphysical (RW-like) phase transitions or the continuation of the physical (pseudo)-critical surface are met.

$\Longrightarrow$ **above T_c information for analytic continuation can be gathered from a restricted region around $\theta_q = \theta_I = 0$. We shall write an expression for the free energy valid in such region.**

That may be a polynomial like

$$F(\theta_q, \theta_I) = \sum_{i,j} c_{i,j} \frac{\theta_q^{2i}}{(2i)!} \frac{\theta_I^{2j}}{(2j)!}$$

with i, j non negative integers, or as a ratio of polynomials of the same kind:

$$F(\theta_q, \theta_I) = \frac{\sum_{i,j} n_{i,j} \frac{\theta_q^{2i}}{(2i)!} \frac{\theta_I^{2j}}{(2j)!}}{\sum_{k,l} d_{k,l} \frac{\theta_q^{2k}}{(2k)!} \frac{\theta_I^{2l}}{(2l)!}\Big|_{d_{00}=1}}$$

4 – Numerical Results

Parameters are fixed according to R.V. Gavai and S. Gupta, Phys. Rev. D71, 114014 (2005) **and correspond to a fixed pion mass** $m_\pi \sim 280$ **MeV,** $m_\rho \sim 900$ **MeV and** $a^{-1} \sim 700$ **MeV.**

T/T_c	$a\,m_q$	β	n_{pairs}	n_{traj}
0.9	0.02778	5.26	95	2300
0.951	0.02631	5.275	95	2460
1	0.025	5.2875	95	2200
1.048	0.0238	5.30	24	3120
1.25	0.02	5.35	77	2270

We have made simulations on a $16^3 \times 4$ **lattice using a RHMC algorithm.**

n_{traj} **is the average number of trajectories for each** μ_q, μ_I **pair**

n_{pairs} **the total number of simulated pairs for each** T

For $T \leq T_c$ **we have made simulations at about 100 different pairs** (θ_q, θ_I) **distributed non-uniformly in the whole interesting range** $[0, \pi] \times [0, \pi]$

Less pairs in a restricted region around the origin have been taken for $T > T_c$**.**

Validity of the Hadron Resonance Gas model

A quite successful phenomenological description of the thermal medium below T_c is that of a gas of hadron resonances with free particle spectrum for all constituents.

The partition function for species i of spin s_i, mass m_i, baryon number B_i, isospin I_{3i}, is given by

$$\ln Z_i = \mp \frac{g_i V}{2\pi^2} \int_0^\infty \ln\left(1 \mp z_i e^{\frac{\sqrt{m_i^2+k^2}}{T}}\right) k^2 dk = \frac{VTm_i^2}{2\pi^2} \sum_{l=1}^{\infty} \left[\frac{(\pm 1)^{l+1}}{l^2} z_i^l K_2\left(\frac{m_i l}{T}\right)\right]$$

where $g_i = 2s_i + 1$, the minus (plus) sign applies to mesons (baryons) and

$$z_i = e^{\mu_i/T} = \exp\left(\frac{3B_i\mu_q + 2I_{3i}\mu_I}{T}\right)$$

The Bessel function K_2 is exponentially suppressed for large values of the argument, hence for $m_i \gg T$ we can take the first term $l = 1$ in the l expansion.

Summing up over all known particles and resonances and grouping together all charge conjugation and isospin partners we get

$$\ln Z(T,V,\mu_q,\mu_I) = VT^3 \sum_{B,I,m} W(m,g,T)\bar{\delta}(B)\cosh\left(3B\frac{\mu_q}{T}\right)\left(\sum_{I_3\geq 0}\bar{\delta}(I_3)\cosh\left(2I_3\frac{\mu_I}{T}\right)\right)$$

where $\bar{\delta}(n) = 1 - 1/2\delta_{n,0}$ **and** $W(m,g,T) = (2/\pi^2)(m/T)^2 g K_2(m/T)$**.**

Such prediction is easily continued to imaginary chemical potentials

$$\ln Z(T,V,\theta_q,\theta_I) = VT^3 \sum_{B,I} W_{B,I}(T)\bar{\delta}(B)\cos(3B\theta_q)\sum_{I_3\geq 0}\bar{\delta}(I_3)\cos(2I_3\theta_I)$$

$$\hat{n_q} \equiv \mathrm{Im}(\langle N_q\rangle/VT^3) = \sum_{B,I} 3BW_{B,I}(T)\sin(3B\theta_q)\sum_{I_3\geq 0}\bar{\delta}(I_3)\cos(2I_3\theta_I)$$

$$\hat{n_I} \equiv \mathrm{Im}(\langle N_I\rangle/VT^3) = \sum_{B,I} W_{B,I}(T)\bar{\delta}(B)\cos(3B\theta_q)\sum_{I_3\geq 0} 2I_3\sin(2I_3\theta_I)$$

where $W_{B,I}(T) = \sum_{m|_{B,I}} W(m,g,T)$**.**

Predictions of the HRG model to be tested in lattice QCD simulations can be classified as follows

1. **Only a few terms contribute to free energy corresponding to known resonances, i.e.** $(B, I) = (0, 0), (0, 1), (1, 1/2), (1, 3/2)$
   $\Longrightarrow$ **only** $W_{0,0}, W_{0,1}, W_{1,1/2}, W_{1,3/2} \neq 0$

2. **Also the values of the coefficients can be predicted from the known experimental resonance mass spectrum**
   of course since we measure first derivatives we cannot look at $W_{0,0}$

The second prediction is easily affected by lattice artifacts and the presence of unphysical quark masses which change the details of the hadron spectrum.

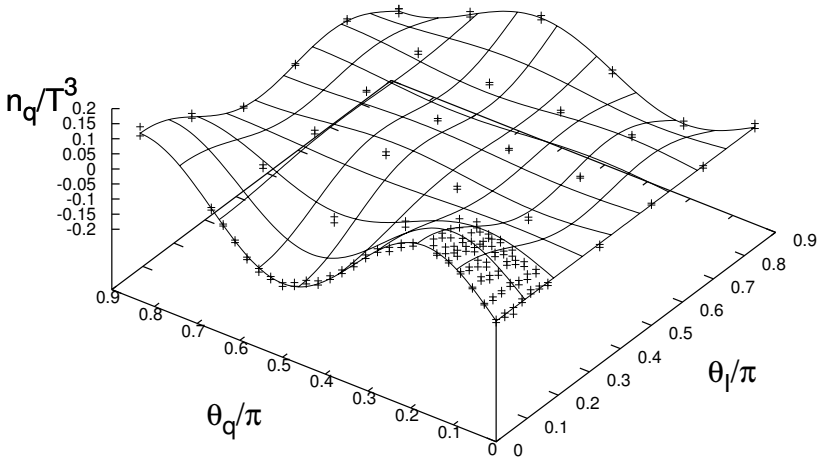
First prediction is instead more robust and spectrum independent.

Analytic continuation has been already used to test the HRG at $\theta_I = 0$

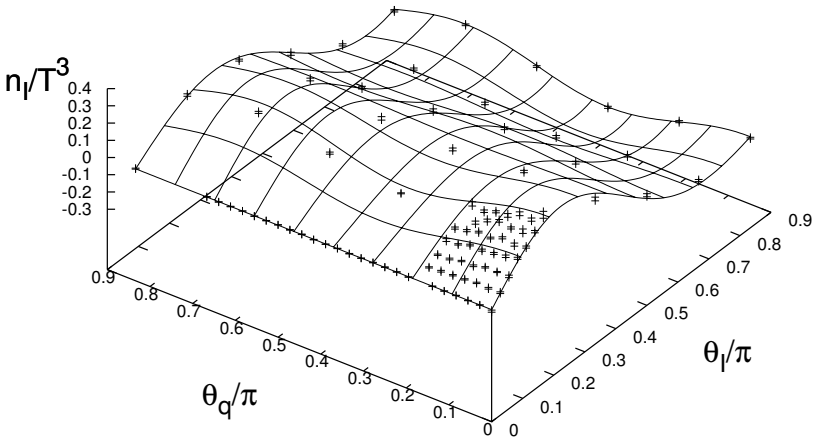
M. D'E., M.P. Lombardo, 2002, 2004; Ph. de Forcrand, S. Kratochvila, 2006

As an example we show data for

$$\underline{T = 0.9T_c}$$



normalized quark density $\hat{n}_q$



normalized isospin density $\hat{n}_I$

Parameter for HRG inspired fits at all temperatures $T \leq T_c$

$W_{0,1}$	$W_{0,2}$	$W_{1,\frac{1}{2}}$	$W_{1,\frac{3}{2}}$	$W_{1,\frac{5}{2}}$	$W_{1,\frac{7}{2}}$	$W_{2,1}$	$W_{2,2}$	χ^2/d.o.f.
$T = 0.9\,T_c$								
0.2284(11)	-	0.0110(6)	0.0202(3)	-	-	-	-	284/187
0.2157(18)	0.0050(6)	0.0115(6)	0.0198(3)	-	-	-	-	206/186
0.2156(18)	0.0051(6)	0.0111(6)	0.0197(3)	-	-	0.00043(13)	-	196/185
$T = 0.951\,T_c$								
0.2862(13)	-	0.0199(7)	0.0305(4)	-	-	-	-	640/187
0.258(2)	0.0114(6)	0.0212(7)	0.0292(4)	-	-	-	-	281/186
0.257(2)	0.0117(6)	0.0203(8)	0.0290(4)	-	-	0.00084(18)	-	259/185
0.256(2)	0.0114(6)	0.0210(8)	0.0264(6)	0.0017(3)	-	0.00088(18)	-	230/184
0.257(2)	0.0106(7)	0.0212(8)	0.0265(6)	0.0009(4)	0.0006(2)	0.00090(18)	-	222/183
$T = T_c$								
0.321(3)	0.0225(10)	0.0367(11)	0.0464(6)	-	-	-	-	724/186
0.318(3)	0.0236(10)	0.0336(11)	0.0460(6)	-	-	0.0030(3)	-	628/185
0.312(2)	0.0234(10)	0.0337(11)	0.0392(9)	0.0051(5)	-	0.0034(3)	-	506/184
0.317(3)	0.0189(11)	0.0342(11)	0.0381(9)	0.0019(6)	0.0036(3)	0.0037(3)	-	398/183
0.318(3)	0.0190(11)	0.0345(11)	0.0384(9)	0.0020(6)	0.0032(4)	0.0021(8)	0.0011(5)	392/182

Validity of HRG model at $T = 0.9\,T_c$

A $W_{0,2}$ is needed. It does not correspond to any known physical state, but it is easily shown to correspond to the first neglected term ($l = 2$) in the Boltzmann approximation (two pion permutations). That would be $W_{0,2} \sim 0.0045$ with our pion mass, to be compared with $W_{0,2} \sim 0.0050(6)$ in the table.

A term with $B = 2$ is marginally visible but not necessary.

Comparing the numerical values of the coefficients is not trivial: $W_{0,1} = 0.216(2)$ is about half the expected experimental value, but is compatible within errors with the value expected if only pion of mass $m\pi \sim 280$ MeV contribute ($W_{0,1} \sim 0.225(15)$). On the other hand $m_\rho >$ UV cutoff on our lattice ...

On the whole, the HRG model is a very good approximation at $T = 0.9T_c$.

$T = 0.95 T_c$

A term $B = 2$ is now strictly needed. This term is really not expected from the HRG model: first correction from the Boltzmann approximation would lead a term a factor 10^2 smaller with an opposite sign! (baryons are fermions)

Contribution from bound states with $B = 2$ or rather the HRG model is starting to break down.

$W_{0,2}$ term this time is not compatible with $l = 2$ corrections

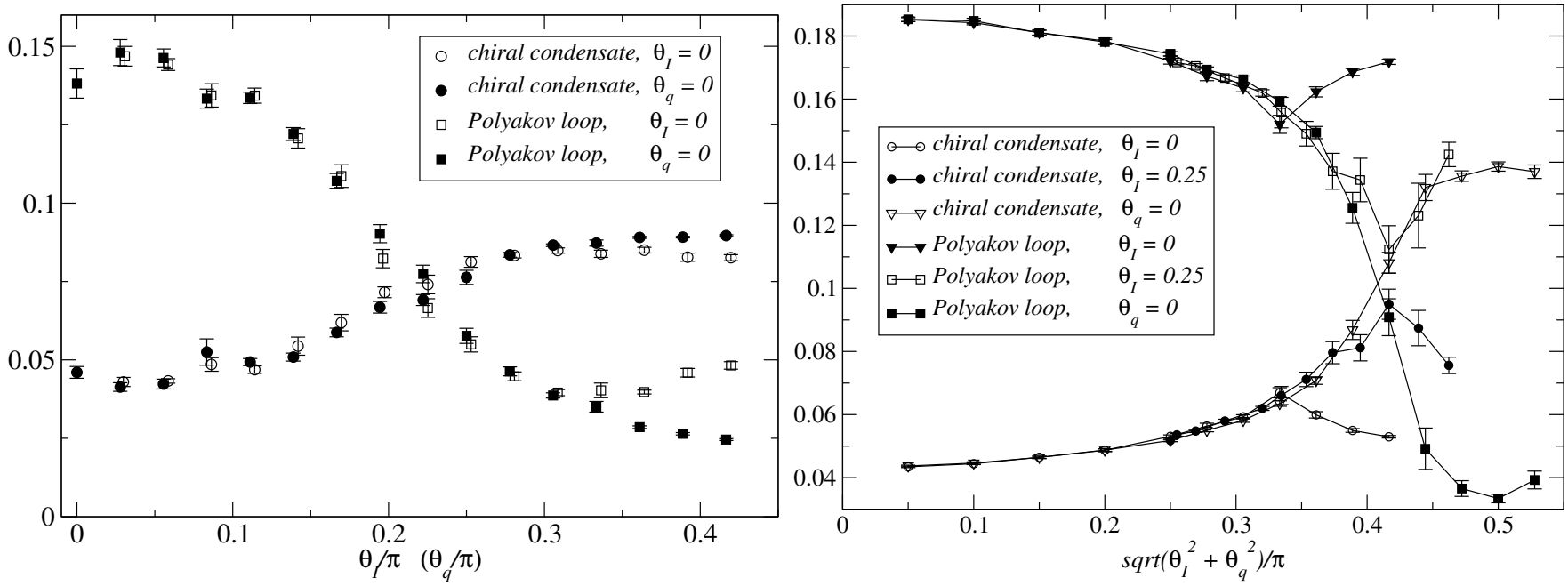
further $W_{1,5/2}$ and $W_{1,7/2}$ terms are needed

$\Longrightarrow$ **deviations from HRG are clearly visible at $T = 0.95 T_c$**

The situations is even worse at $T = T_c^-$

This is in agreeement with similar findings by the Bielefeld-BNL collaboration

$T > T_c$

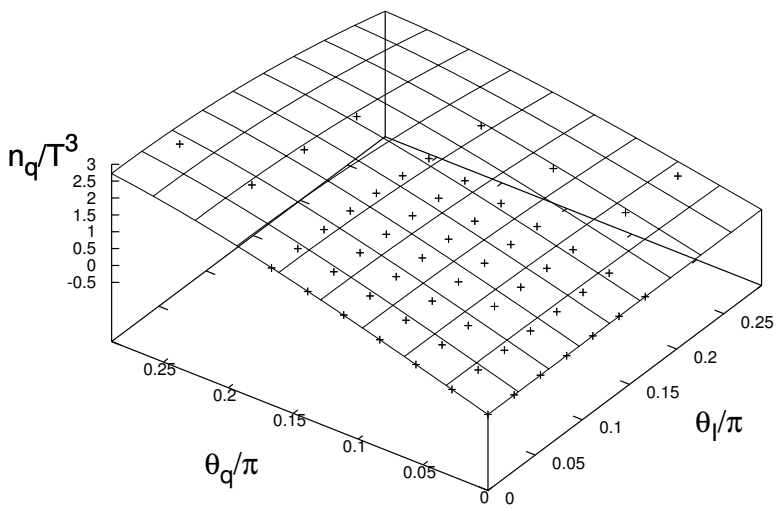


The regions of available imaginary chemical potentials is limited roughly by

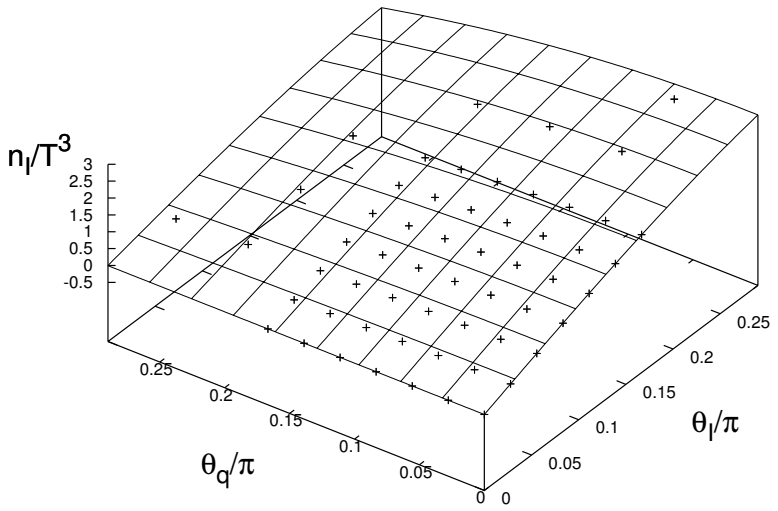
$\sqrt{\theta_I^2 + \theta_q^2} < 0.12\pi$ **at** $T = 1.048\, T_c$ **(left) and by**

$\sqrt{\theta_I^2 + \theta_q^2} < 0.3\pi$ **at** $T = 1.25\, T_c$ **(right)**

Above T_c **polynomials or ratios of polynomials are best suited to fit data.**



normalized quark density $\hat{n}_q$



normalized isospin density $\hat{n}_I$

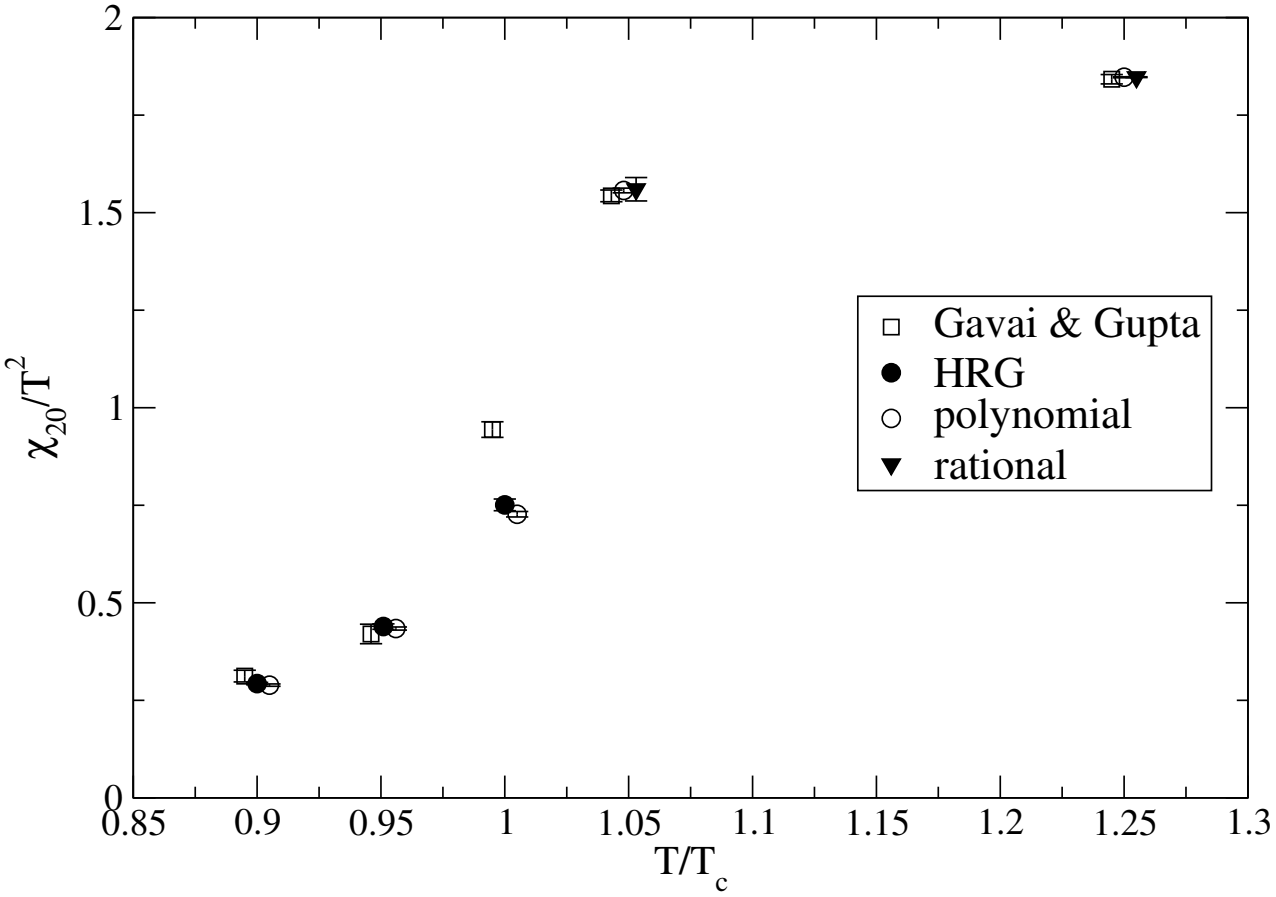
At $T = 1.25\,T_c$ (shown in the figure) a sixth order polynomial or a Pade $P(4,2)$ are the best interpolating functions.

At $T = 1.048\,T_c$ instead a fourth order polynomial provides a good fit, and a marginally good fit is obtained with a Pade $P(2,2)$.

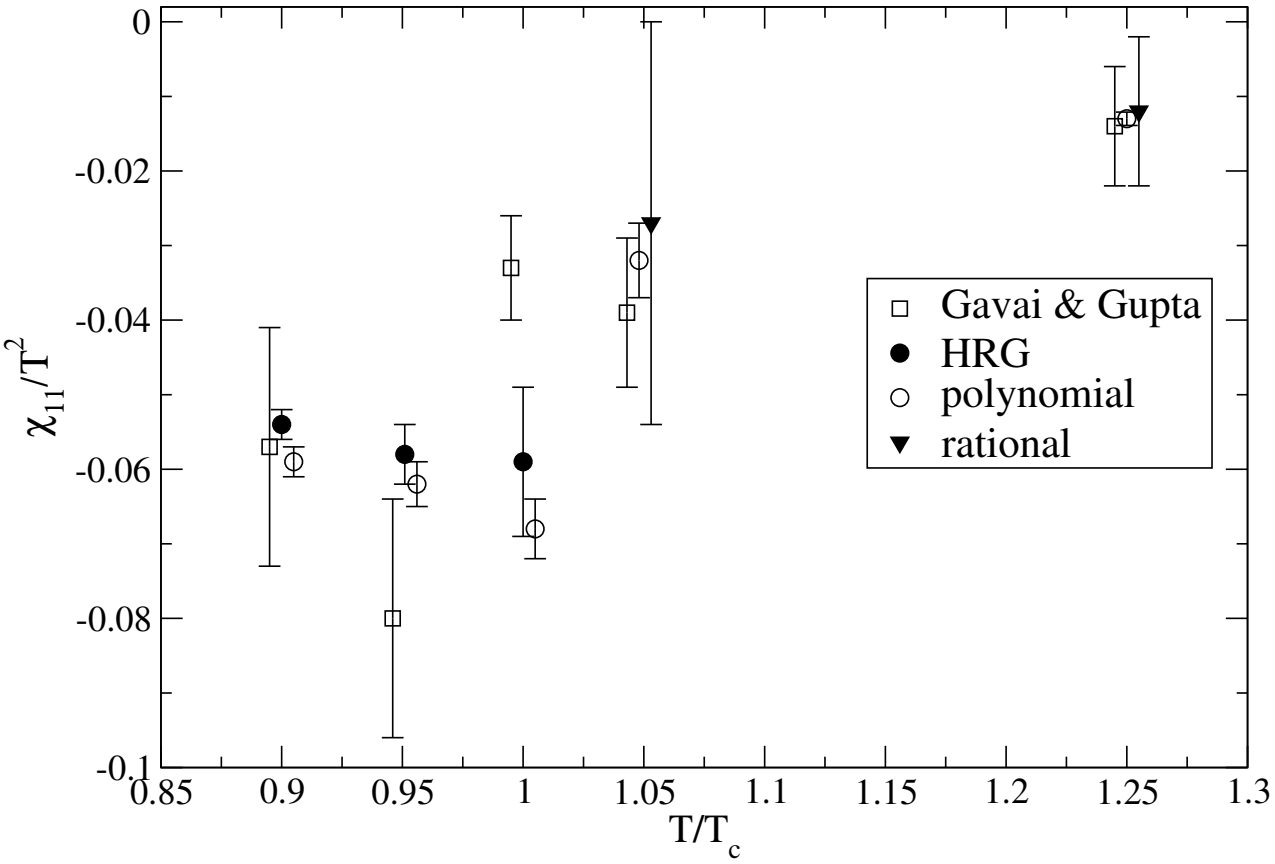
In the following we will show results for generalized suceptibilities and compare to results obtained by Gupta and Gavai.

Polynomial have been used also at $T < T_c$ (in a restricted region of small potentials) to check for systematics in analytic continuation.

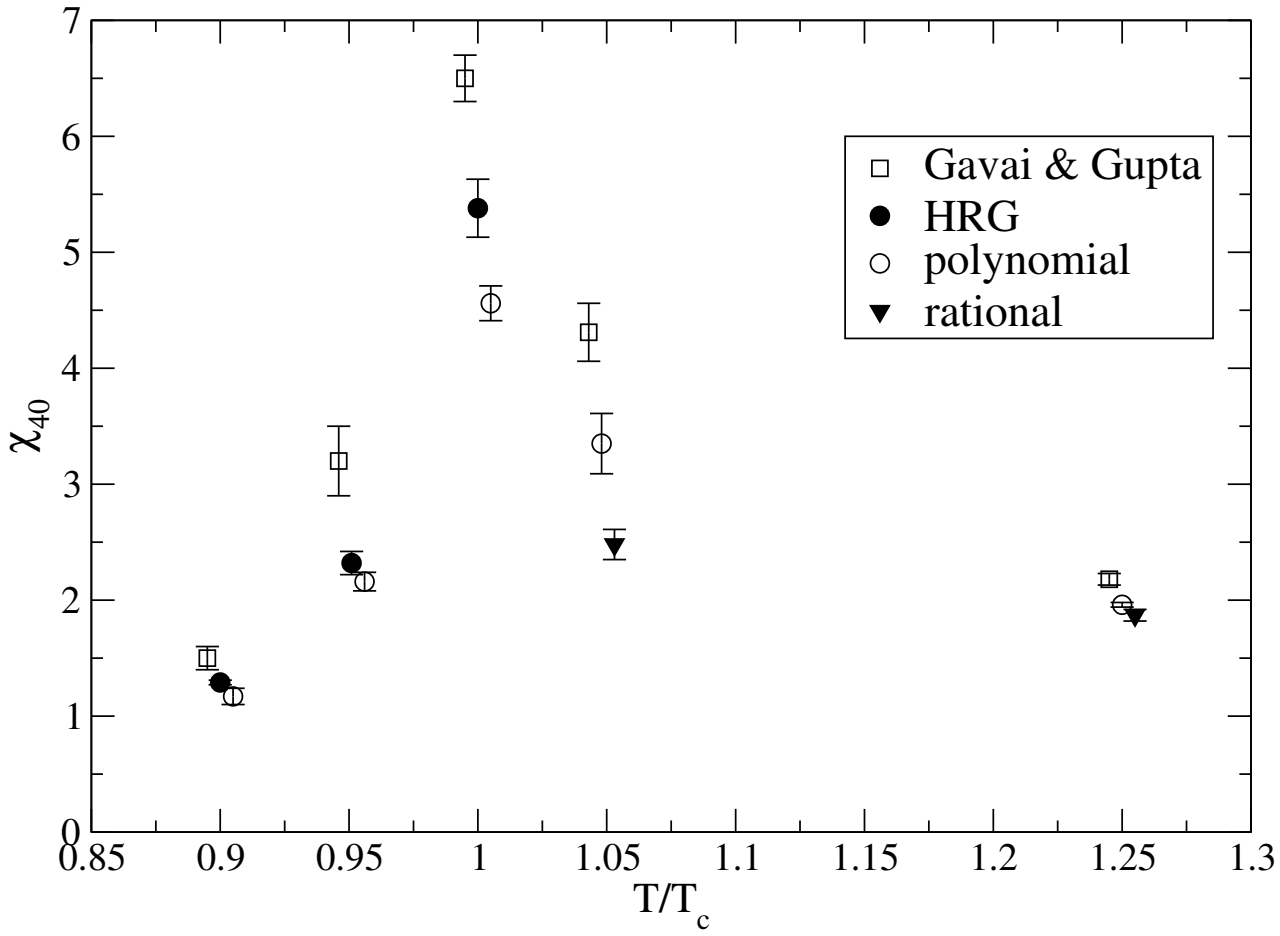
GENERALIZED SUSCEPTIBILITIES



χ_{20}


0
-0.02
-0.04
-0.06
-0.08
-0.1
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
1.3
χ_{11}/T^2
T/T_c
Gavai & Gupta
HRG
polynomial
rational

χ_{11}


Gavai & Gupta
HRG
polynomial
rational
χ_{40}
T/T_c
0
1
2
3
4
5
6
7
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
1.3

χ_{40}

Different extrapolations provide always consistent results for $\chi_{2,0}$ and $\chi_{1,1}$. A good agreement with Taylor expansion results can be observed as well, apart from the $T = T_c$ case.

For $\chi_{4,0}$ we observe a discrepancy between different interpolations for $T = T_c$ and $T = 1.048\, T_c$; the agreement with Taylor expansion results is lost at the same temperatures.

Not enough information is contained in our data to extract $\chi_{6,0}$: different extrapolations disagree or are at most marginally compatible in the whole range of temperatures.

An improvement in the determination of generalized susceptibilities could be obtained by combining analytic continuation with other techniques: for instance fixing lowest order terms in a polynomial expansion by the Taylor expansion method or by reweighting could lead to enhanced predictivity for analytic continuation.

5 – Analytic continuation of the average phase factor

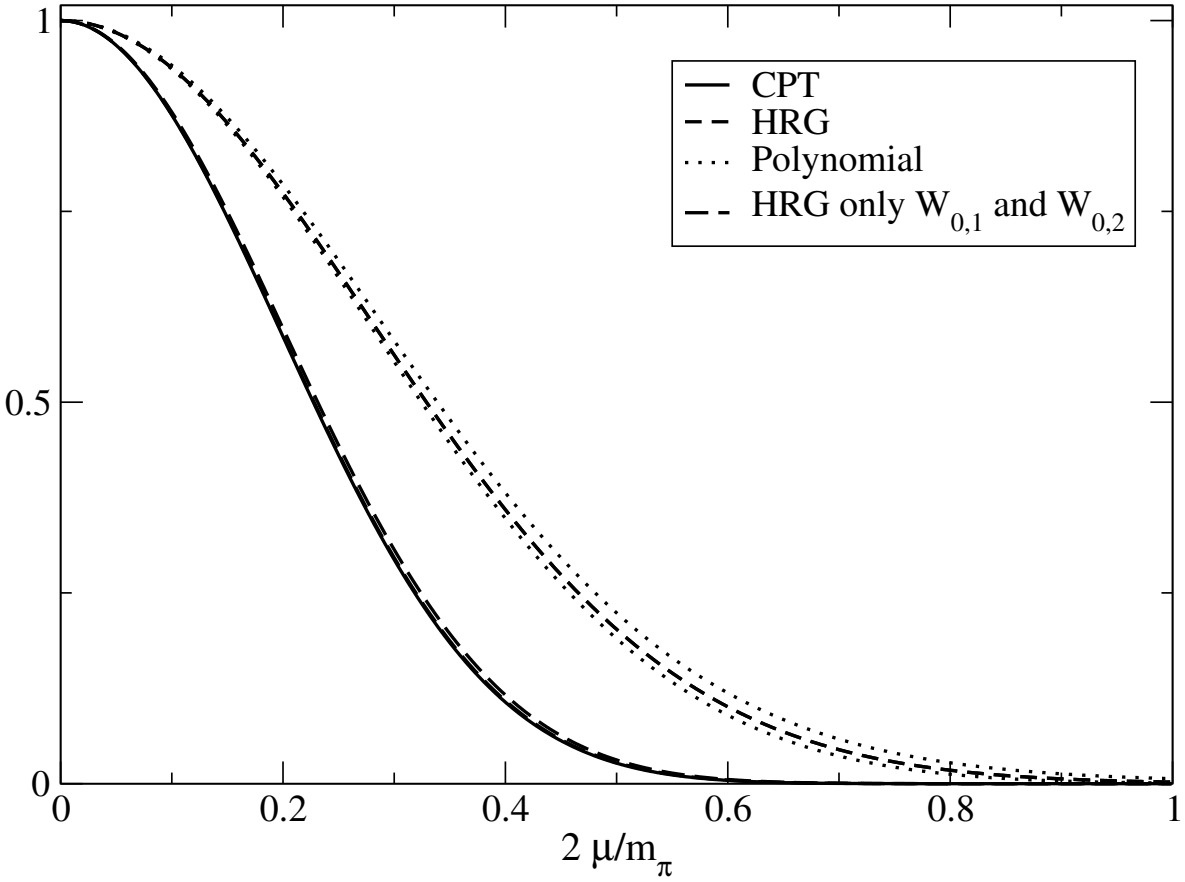
$$\langle e^{i2\theta} \rangle_\mu = \frac{Z(\mu,\mu)}{Z(\mu,-\mu)} = \exp\left(-\frac{F(\mu,\mu) - F(\mu,-\mu)}{T}\right)$$

We can make use of best interpolations at imaginary chemical potentials to get $\langle e^{i2\theta} \rangle_\mu$ by analytic continuation, the average phase being continuous at $\mu^2 = 0$ (K. Splittorff and J. J. M. Verbaarschot, 2007 ; K. Splittorff and B. Svetitsky, 2007)

- **Comparison of different interpolations (e.g. HRG inspired and polynomial below T_c) may give an idea of systematic effects.**
- **In the case of HRG inspired fits, we can actually distinguish the contribution of each particle species to the average phase factor!**

$$\langle e^{i2\theta} \rangle_\mu = \exp\left(\frac{V}{T}\sum_{h,l} W_{h,l}(\cosh(3h\frac{\mu}{T}) - \cosh(l\frac{\mu}{T}))\right)$$

Average phase factor, $T = 0.9T_c$

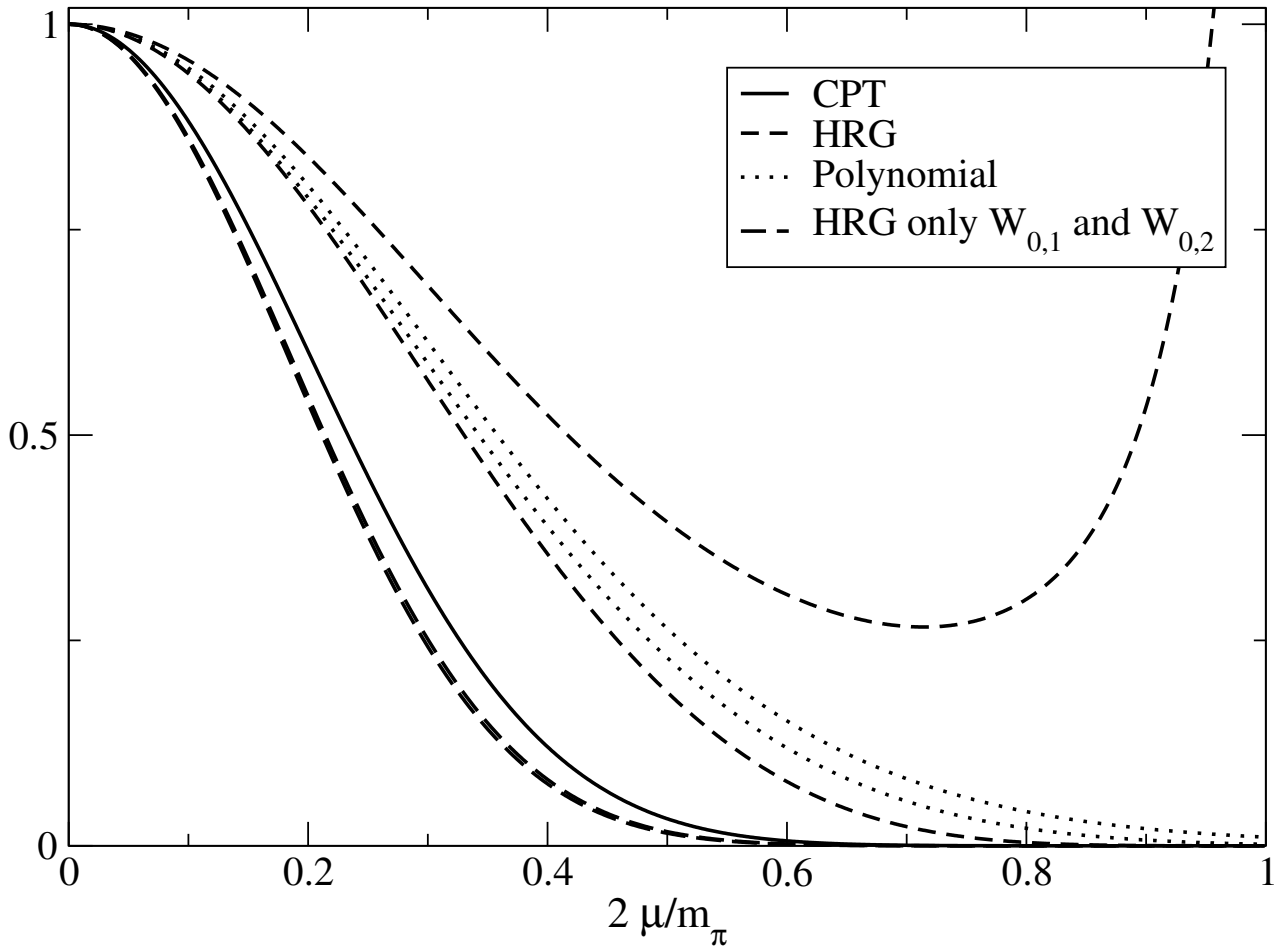


Different continuations (HRG and polynomial) in perfect agreement

Agreement with Chiral Perturbation Theory (Splittorff Verbaarschot, 2007) if only pion-like terms in the HRG fit are taken into account

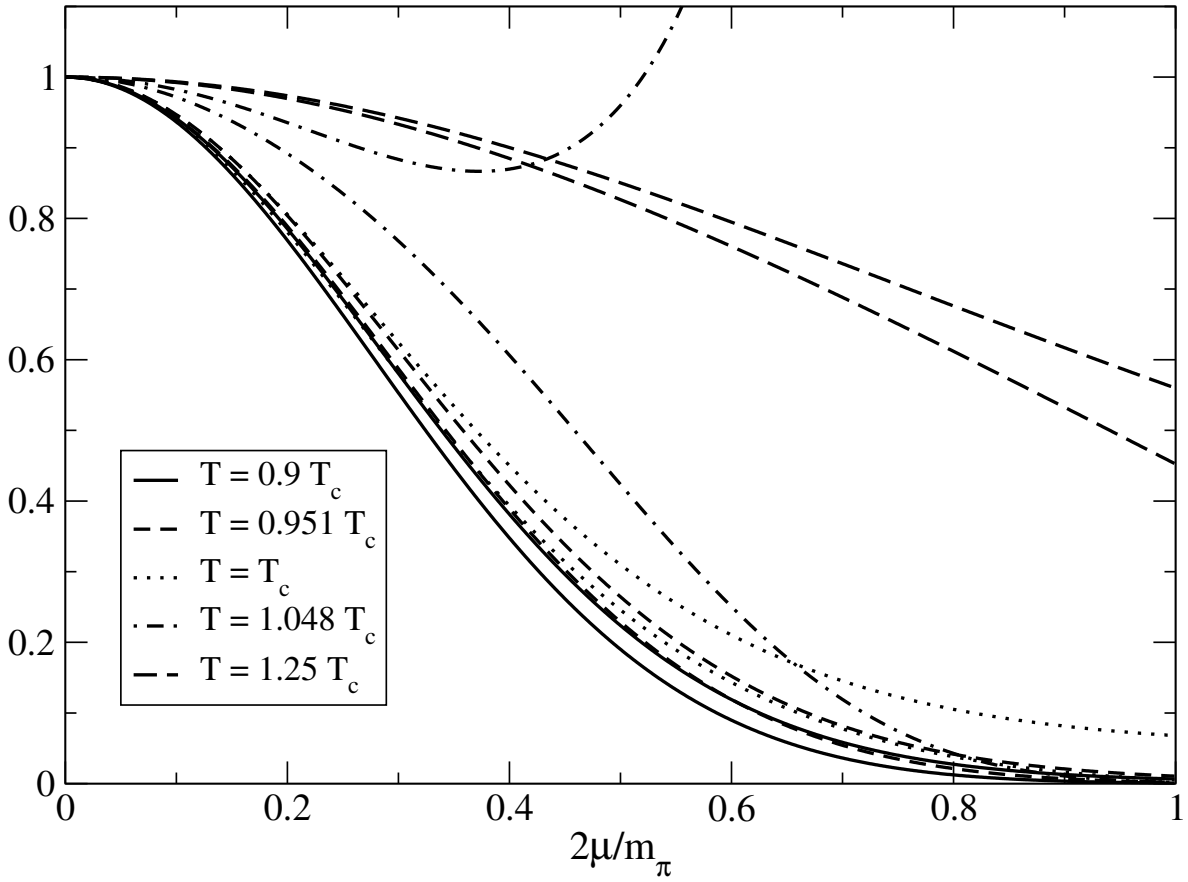
Baryons contribute to increase the average phase factor! (better sign problem?)

Average phase factor, $T = 0.951 T_c$



Same considerations as for $T = 0.95\, T_c$

Average phase factor, all T from polynomial fits



The sign problem improves at higher T.

6 – Conclusions and Perspectives

Tests of analytic contination: non-linear terms in μ^2 can be predicted for the critical line in QCD at finite isospin density.

The situation may be worse in real QCD (RW-line closer to the $\mu = 0$ axis)

Prediction from analytic contination in $N_f = 2$ QCD:

- **HRG model:** corrections starting from $T = 0.95 T_c$
- **Generalized susceptibilities:** not eough information to go beyond χ_{40}, room for considerable improvement by mixing with other methods.
- **Average phase factor:** Agreement with CPT below T_c if only pions are taken into account. Baryon contribution increases the average phase factor.