Cold dense matter without a Sign Problem: what can we learn?

Simon Hands
Swansea University

THOR meets THOR: exploring the interface of fundamental and effective approaches to extreme matter, Lisbon, 12 June 2018

Plan

- **When *isn't* there a Sign Problem?**

- **GN_{2+1}**
  - Friedel oscillations
  - medium modification of σ propagator
  - mesons and zero sound

  Fermi Liquid

- **NJL_{3+1}**
  - superfluid condensate and gap
  - isospin chemical potential

  BCS superfluid

- **NJL_{2+1}**
  - superfluid condensate
  - helicity modulus

  Thin film superfluid

- **Bilayer Graphene**
  - excitonic condensate
  - quasiparticle dispersion

  Strongly correlated superfluid

- **Summary**

When *isn't* there a Sign Problem?

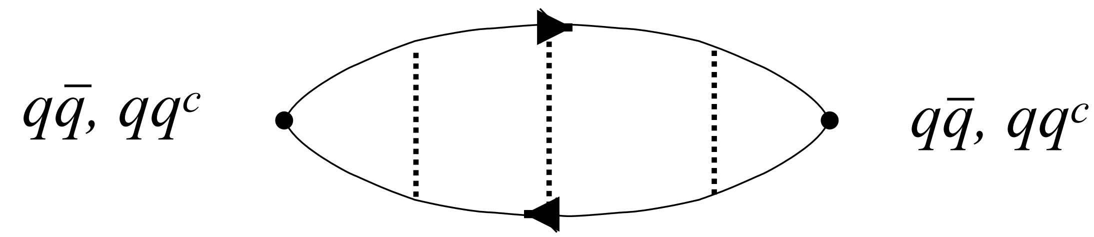
Whenever the fermion measure $\equiv \det(M^{\dagger}M)$

describes ***conjugate*** quarks $q^c, \bar{q}^c$ describes quarks $q, \bar{q}$

QCD simulations fail due to light qq^c bound states carrying non-zero baryon charge

2 cases where this isn't an issue

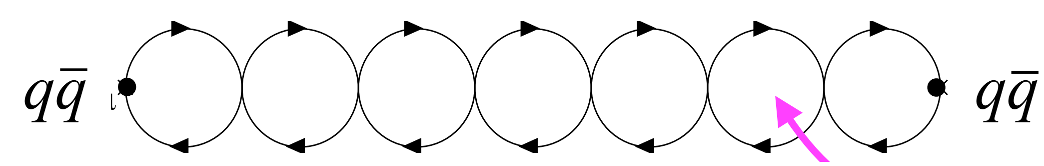
A: $q\bar{q}$ and qq^c states bind with different dynamics and are not degenerate



Generic channel binding ~ O(1/N)

eg. Gross-Neveu, NJL

B: Goldstone baryons are a feature, not a bug



Goldstone channel binding ~ O(1)

eg. QC_2D, isospin QCD, adjoint QCD,
6 in SU(4), **7** in G_2, bilayer graphene….

some models contain gauge invariant fermion states

Today we're mostly focussed on Case A

Gross-Neveu model in $2+1$ dimensions…

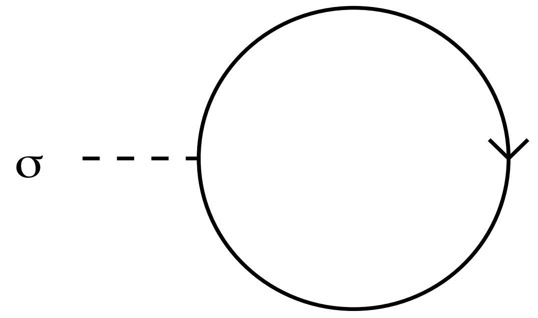
$$\mathcal{L} = \sum_{i=1}^{N_f} \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i - \frac{g^2}{2N_f}(\bar{\psi}_i\psi_i)^2,$$

…just about the simplest QFT with fermions
Can also write in terms of an auxiliary scalar σ:

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m + \frac{g}{\sqrt{N_f}}\sigma)\psi_i + \frac{1}{2}\sigma^2.$$

For $g^2 > g_c^2 \sim O(\Lambda^{-1})$ the ground state has a dynamically-generated fermion mass $\Sigma_0 = \frac{g}{\sqrt{N_f}}\langle\sigma\rangle \neq 0$ given in the $N_f \to \infty$ limit by the chiral *Gap Equation*

$$\Sigma_0 = g^2 \mathrm{tr} \int_p \frac{1}{ip\!\!\!/ + \Sigma_0}$$

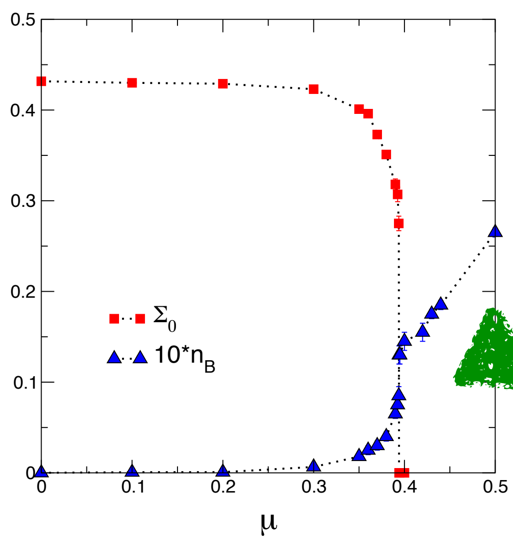


GN Thermodynamics

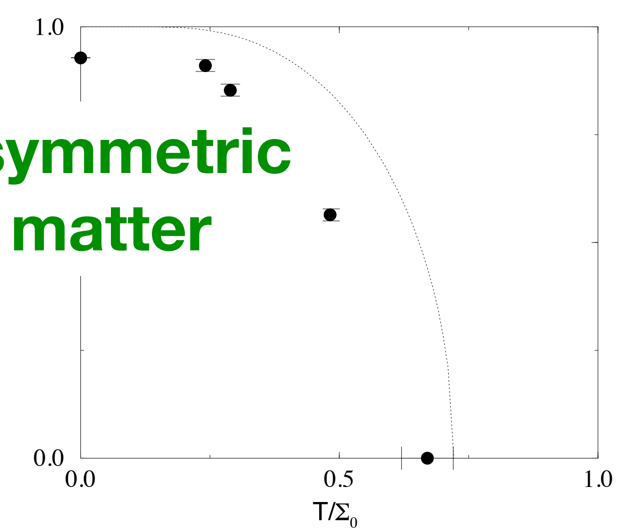
The large-N_f approach can also to be applied to $T, \mu \neq 0$ and predicts a chiral symmetry restoring phase transition:

$$T_c|_{\mu=0} = \frac{\Sigma_0}{2\ln 2}; \quad \mu_c|_{T=0} = \Sigma_0$$

Remarkably, lattice Monte Carlo simulations can be applied to $N_f < \infty$ even for $\mu \neq 0$ *Action is real!*



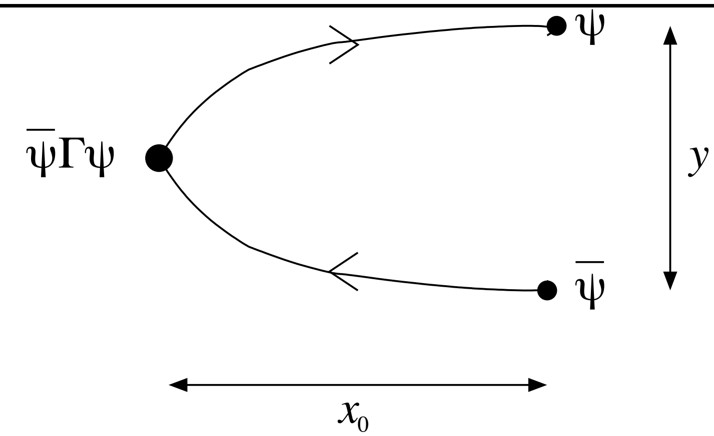
chirally symmetric quark matter



There is even evidence for a tricritical point at *small* $\frac{T}{\mu}$!

[J.B. Kogut and C.G. Strouthos PRD63(2001)054502]

Fermi Surface Phenomena



Consider $q\bar{q}$ "jawbone" diagram

$$C(\vec{y},x_0)=\sum_{\vec{x}}\mathrm{tr}\int_p\int_q\Gamma\frac{e^{ipx}}{i\not{p}+\mu\gamma_0+M}\Gamma\frac{e^{-iqx}e^{-i\vec{q}.\vec{y}}}{i\not{q}+\mu\gamma_0+M}$$

$\mu<\mu_c$:

$$C\propto\int_0^\infty pdpJ_0(py)e^{-2x_0\sqrt{p^2+M^2}}\sim\frac{M}{x_0}e^{-2Mx_0}\exp\left(-\frac{|\vec{y}|^2M}{4x_0}\right)$$

Gaussian width $O(\sqrt{x_0})$

$\mu>\mu_c$:

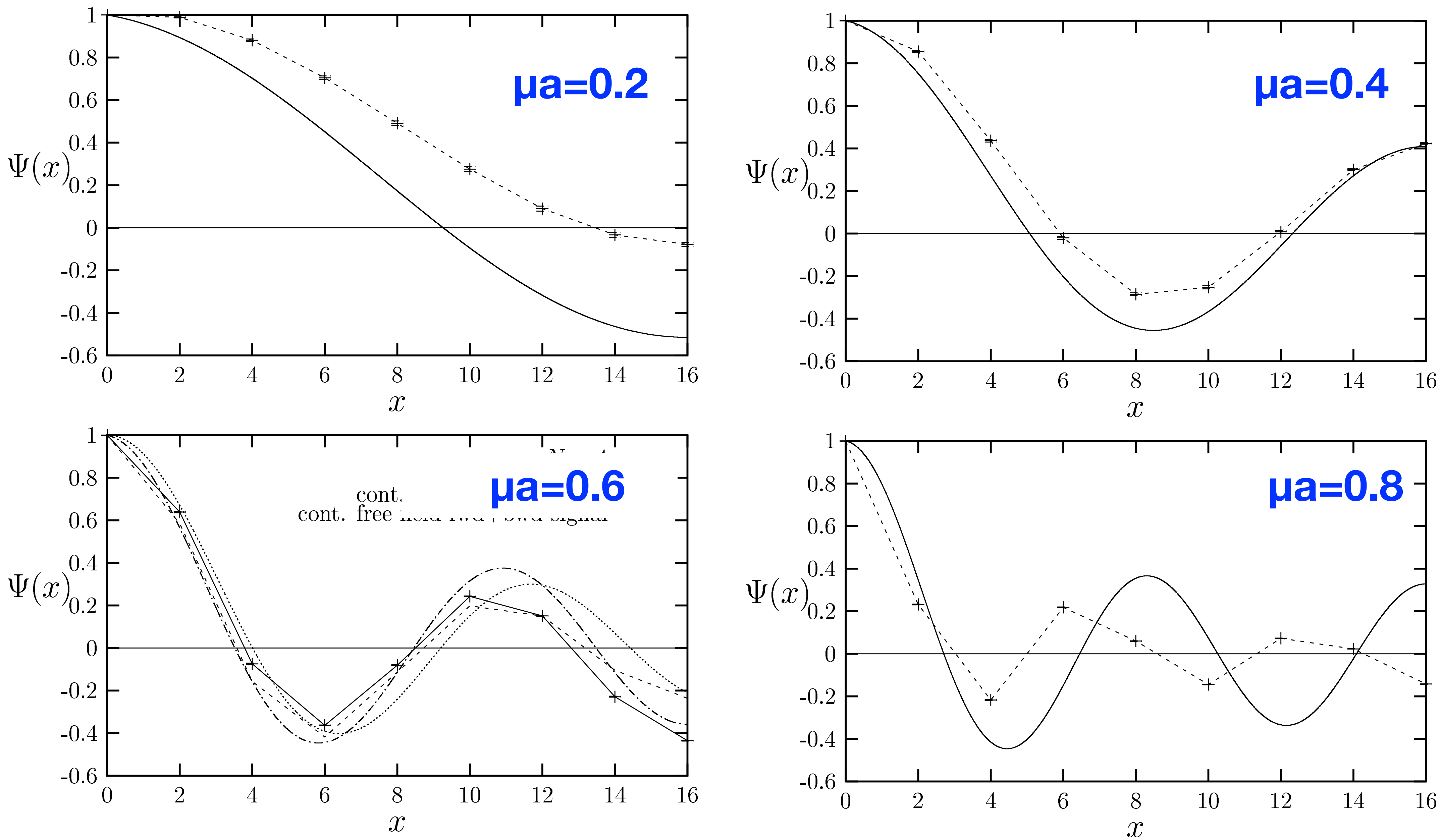
$$C\propto\int_\mu^\infty pdpJ_0(py)e^{-2px_0}\sim\frac{\mu}{x_0}e^{-2\mu x_0}J_0(\mu|\vec{y}|)\propto J_0(k_Fy)$$

Oscillatory profile; shape constant as $x_0\nearrow$

y dependence yields **Bethe-Salpeter wave function**

GN on $32^2\times48$

SJH, JB Kogut, CG Strouthos, TN Tran, PRD68 016005

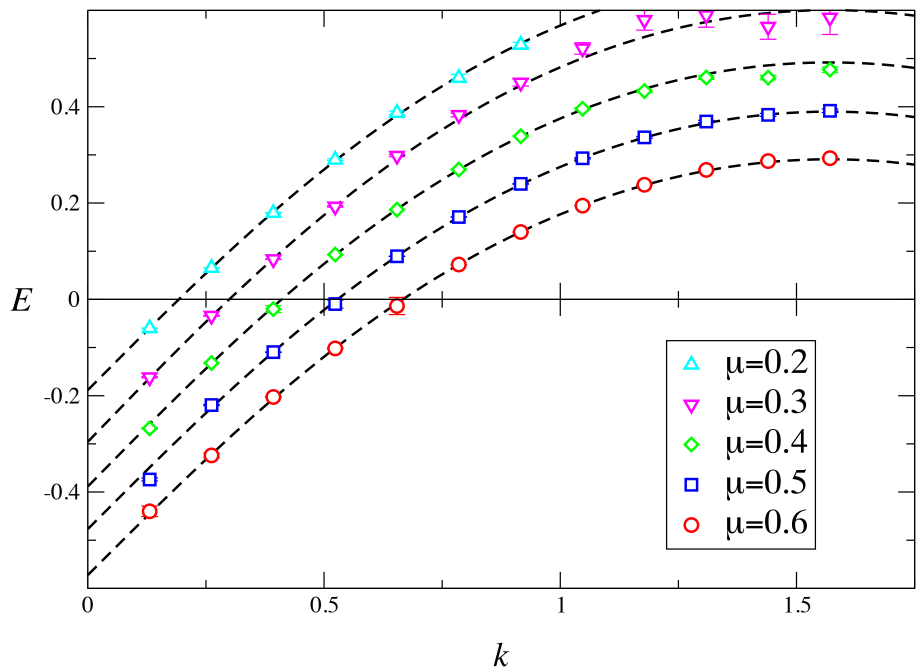


Oscillations develop as $\mu \nearrow$

Graphic evidence for existence of a sharp Fermi surface

Why does free-field theory prediction work so well?

Fermion Dispersion relation



μ	K_F	β_F	$K_F/\mu\beta_F$
0.2	0.190(1)	0.989(1)	0.962(5)
0.3	0.291(1)	1.018(1)	0.952(4)
0.4	0.389(1)	0.999(1)	0.973(1)
0.5	0.485(1)	0.980(1)	0.990(2)
0.6	0.584(3)	0.973(1)	1.001(2)

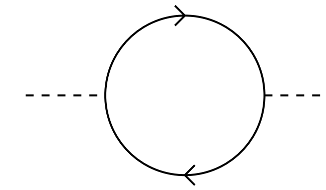
The fermion dispersion relation is fitted with

$$E(|\vec{k}|) = -E_0 + D\sinh^{-1}(\sin|\vec{k}|)$$

yielding the Fermi liquid parameters

$$K_F = \frac{E_0}{D}; \qquad \beta_F = D\frac{\cosh E_0}{\cosh K_F}$$

σ Propagator in Quark Matter $(N_f \to \infty)$

Given by $D_\sigma^{-1}(k;\mu) = 1 - \Pi(k;\mu) =$ 1 – 

Static Limit $k_0 = 0$: $D_\sigma^{-1} = \frac{g^2}{\pi}(\mu - \mu_c)$

Complete screening for $r > 0$ $\Leftrightarrow$ Debye mass $M_D = \infty$

Explains free-field Friedel oscillations

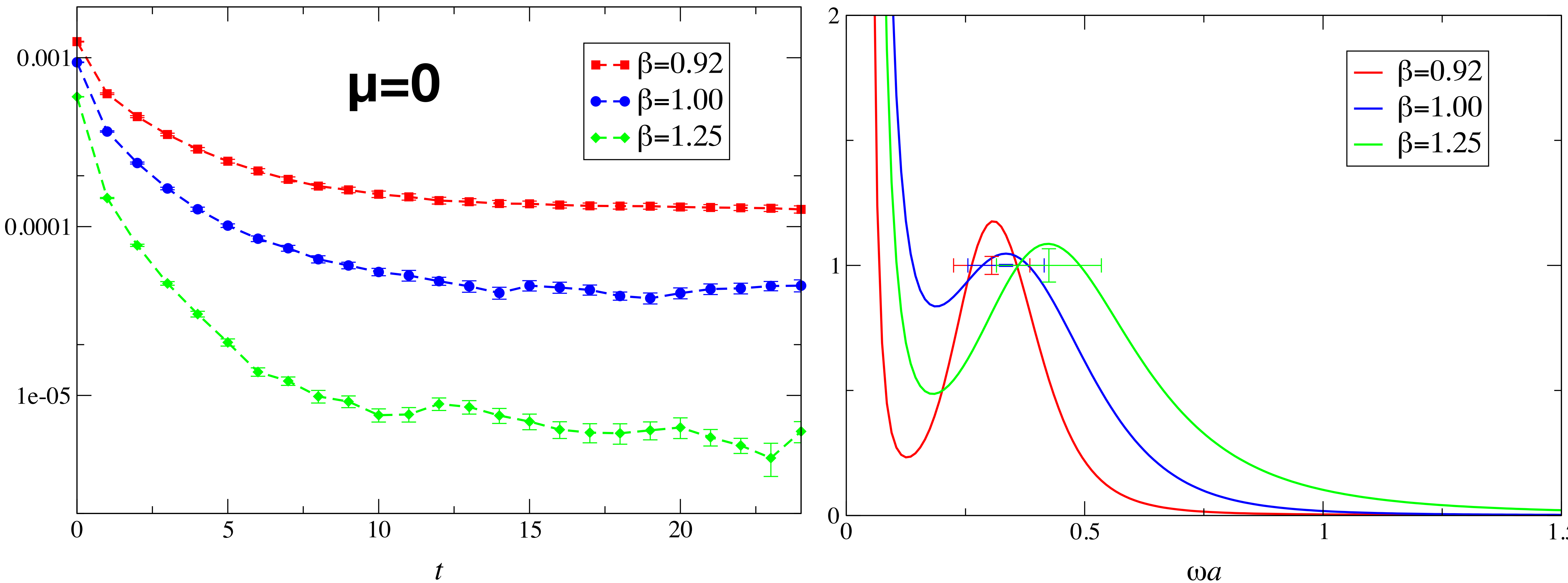
Zero Momentum Limit $\vec{k} = \vec{0}$: $D_\sigma^{-1} = \frac{g^2}{4\pi\mu}[M_\sigma^2 + k_0^2]$

Conventional boson of mass $M_\sigma = 2\sqrt{\mu(\mu - \mu_c)}$

Stable because decay into $q\bar{q}$ requires energy 2μ

and is Pauli-blocked. $\Leftrightarrow$ Plasma frequency $\omega_P = M_\sigma$

Numerical Results with $N_f = 4$

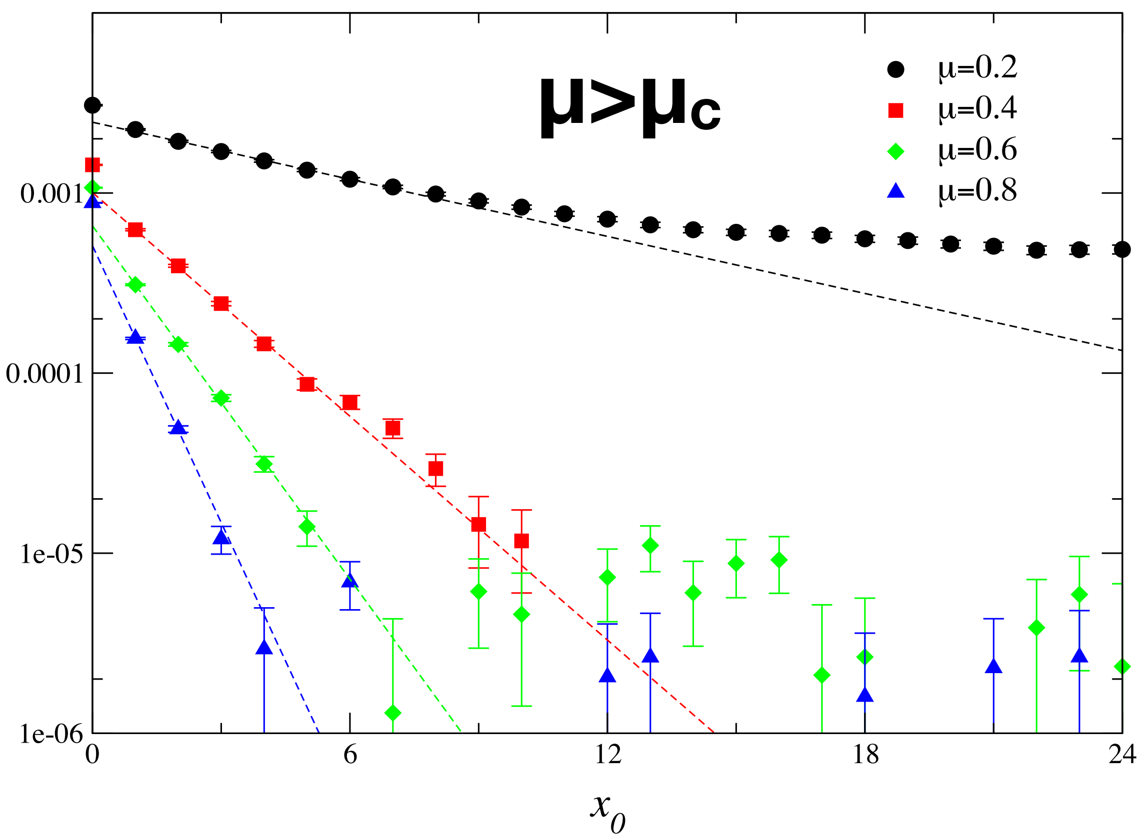


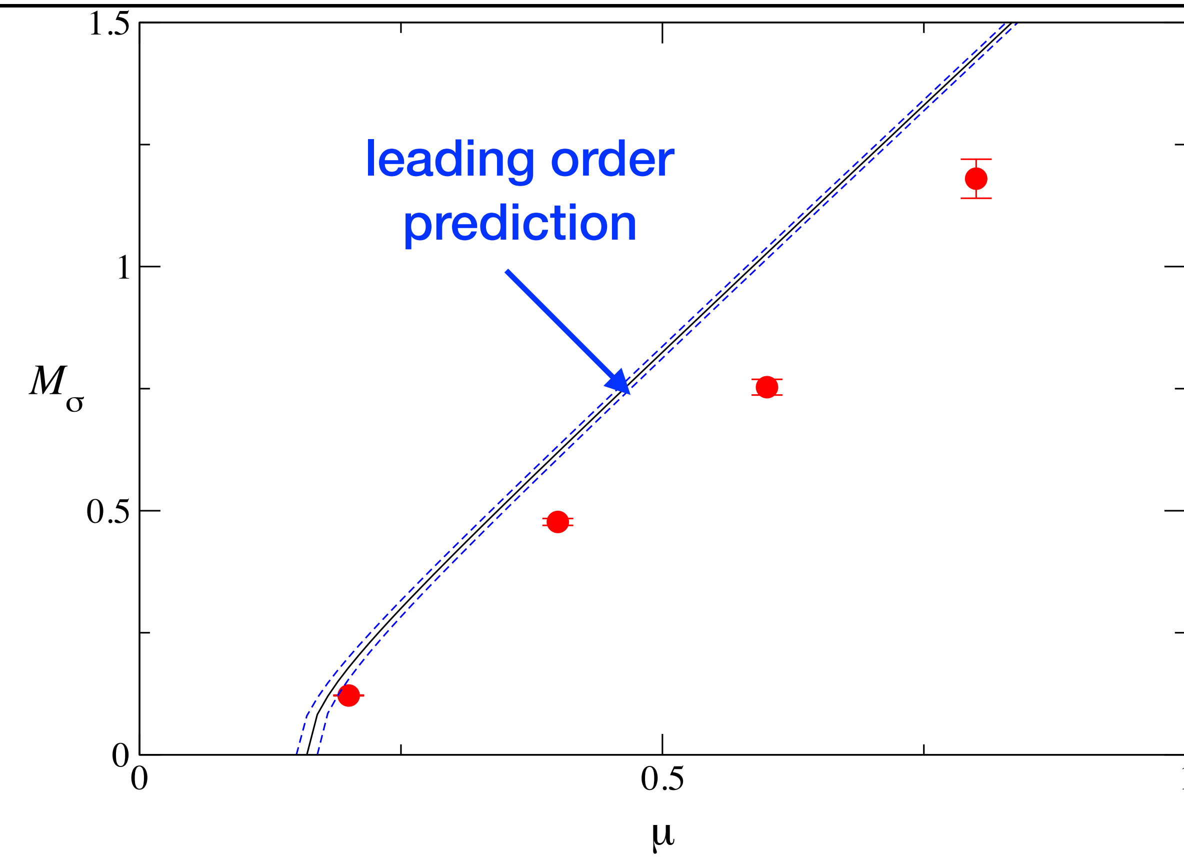
CR Allton, JE Clowser, SJH, JB Kogut, CG Strouthos PRD66 094511

In the bulk chirally symmetric phase ($g < g_c, \mu = T = 0$), the σ correlator does not resemble that of a bound state, but rather a resonance with width Γ increasing as $g \searrow 0$

ie. $D_\sigma^{-1} \propto (k + \Gamma) \Rightarrow \rho_\sigma(\omega) \propto \frac{\Gamma\omega}{\omega^2+\Gamma^2}$

Numerical Results with $N_f = 4$

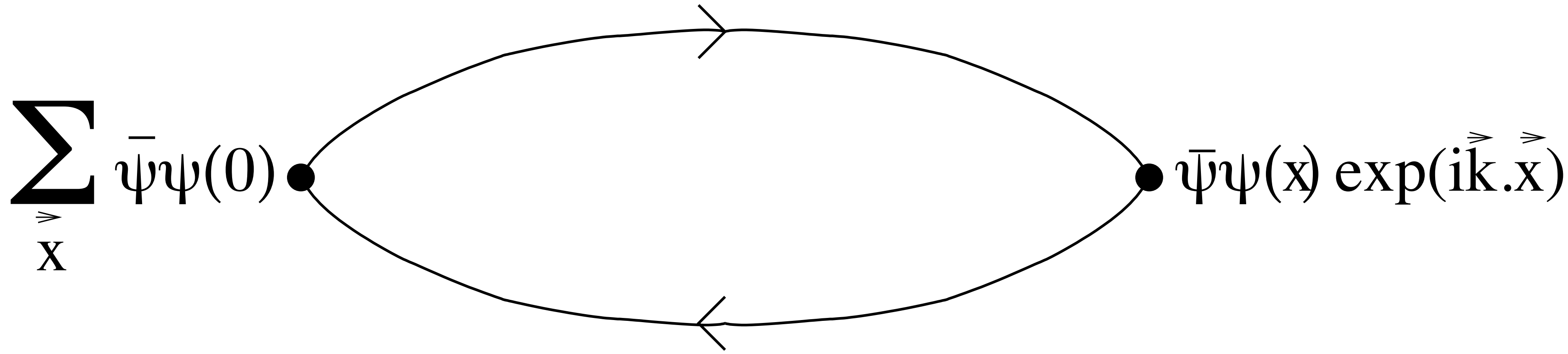




In contrast with behaviour in the chirally-symmetric bulk phase, in quark matter the σ exhibits a sharply-defined pole at $M_\sigma(\mu)$ consistent with $O(1/N_f)$ corrections to the leading order result $M_\sigma = 2\sqrt{\mu(\mu-\mu_c)}$ with $\mu_c a \approx 0.16$

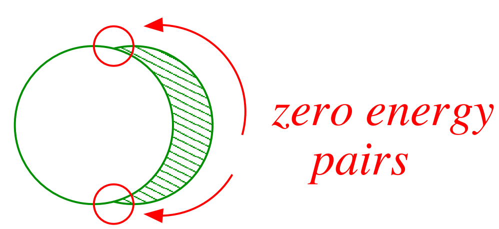
Note σ tightly bound for $\frac{\mu-\mu_c}{\mu} \ll 1$

Meson Correlation Functions



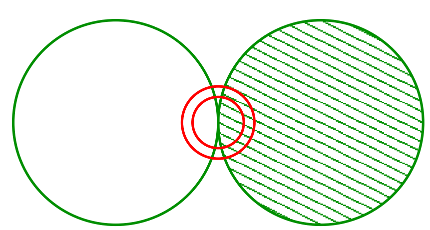
For $\vec{k} \neq 0$ can always excite a particle-hole pair with almost zero energy $\Rightarrow$ algebraic decay of correlation functions

$|\vec{k}| \ll \mu$



$\Rightarrow C \sim \frac{1}{x_0^2}$

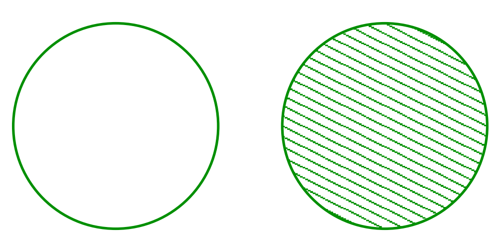
$|\vec{k}| = 2\mu$



Overhauser instability

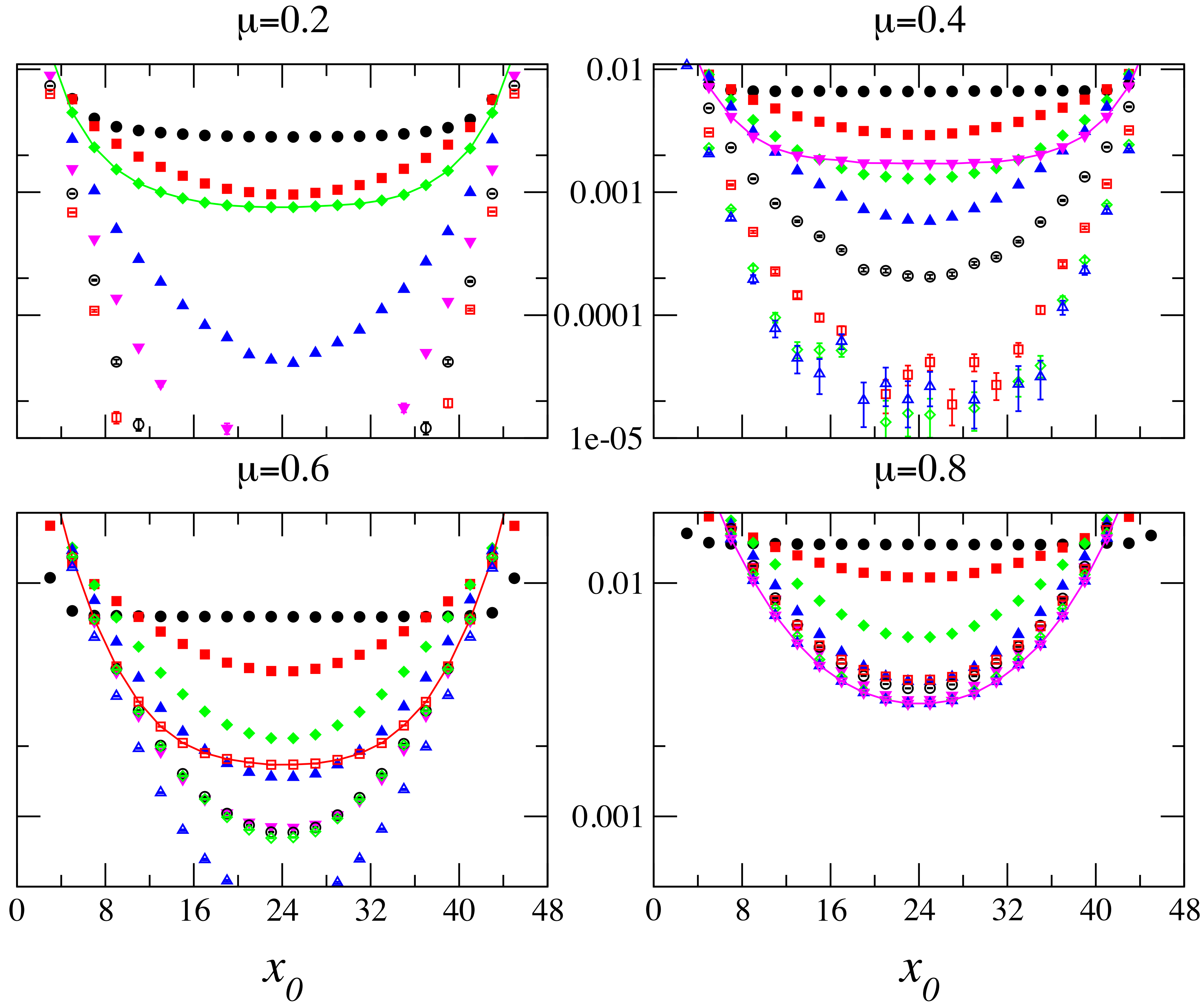
$\Rightarrow C \sim \frac{1}{x_0^{3/2}}$

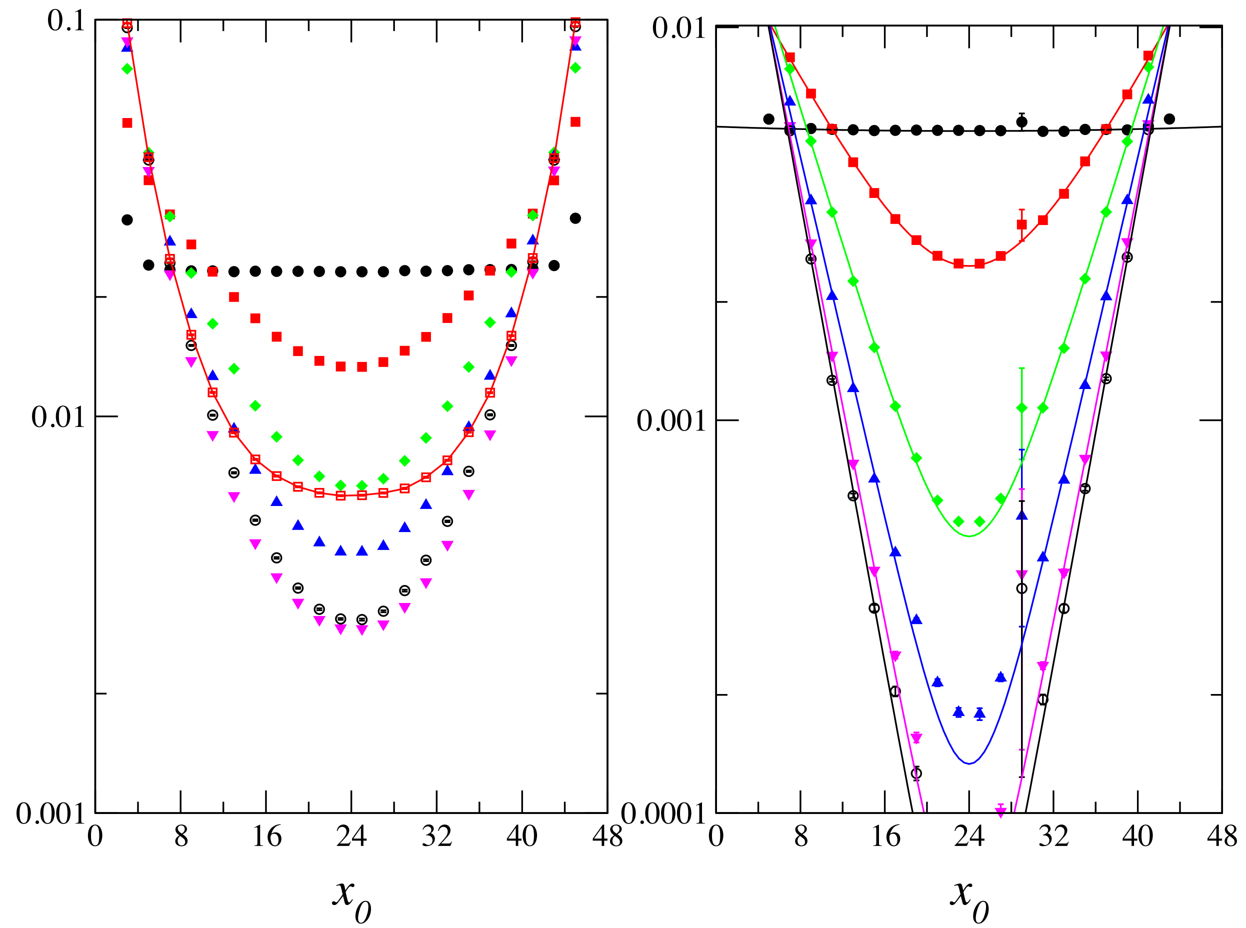
$|\vec{k}| > 2\mu$



$\Rightarrow C \sim \frac{e^{-(|\vec{k}|-2\mu)x_0}}{x_0^{3/2}}$

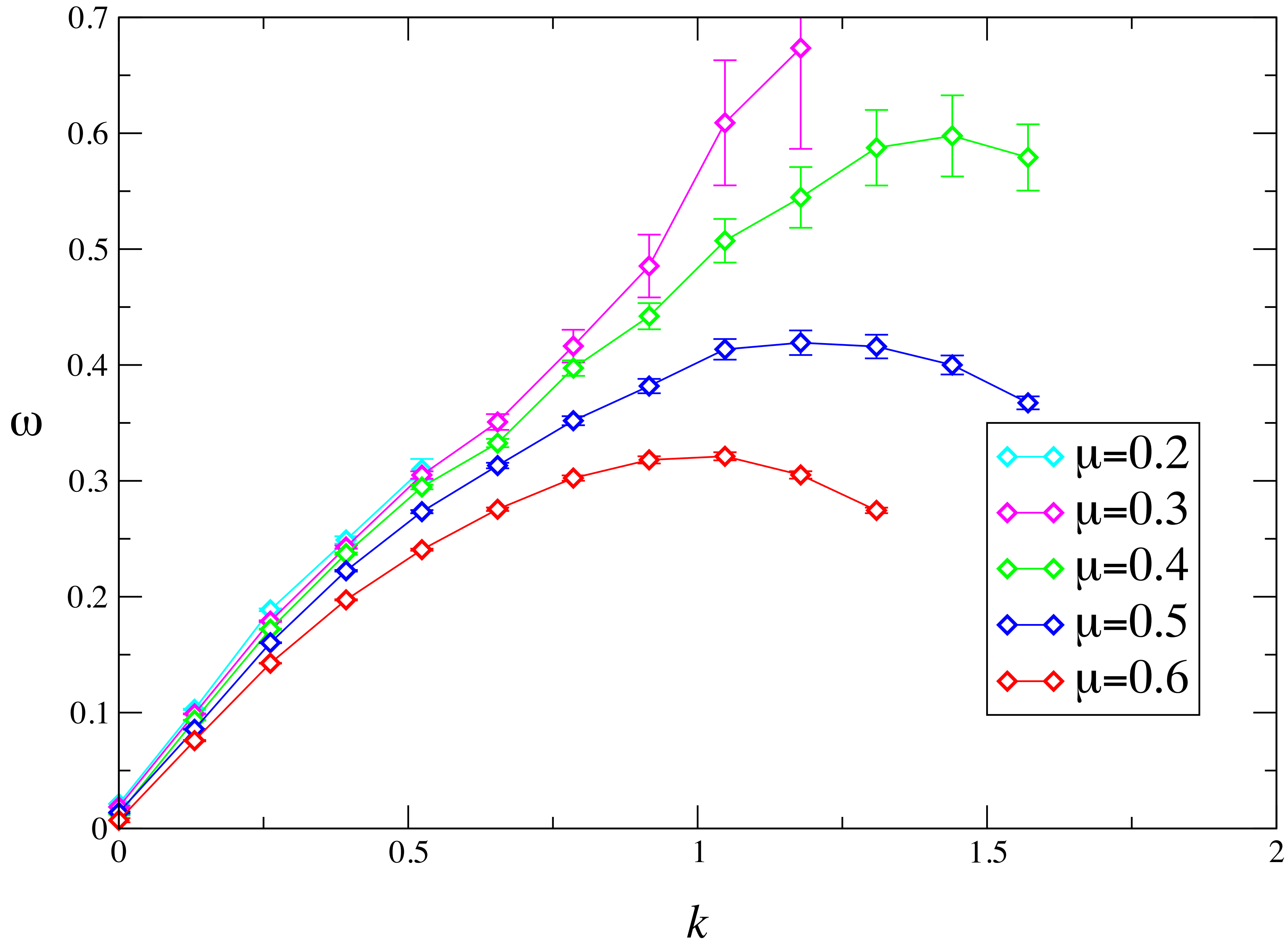
Plots of $C_{\gamma_5}(\vec{k}, x_0)$ show special behaviour for $|\vec{k}| \approx 2\mu$





eg. in the spin-1 channel at $\mu a = 0.6$, $C_{\gamma_\perp}$ (left) looks algebraic as predicted by free field theory, but $C_{\gamma_{||}}$ (right) decays exponentially.

The interpolating operator for $C_{\gamma_{||}}$ in terms of continuum fermions is $\bar{q}(\gamma_0 \otimes \tau_2)q$
ie. with same quantum numbers as baryon charge density

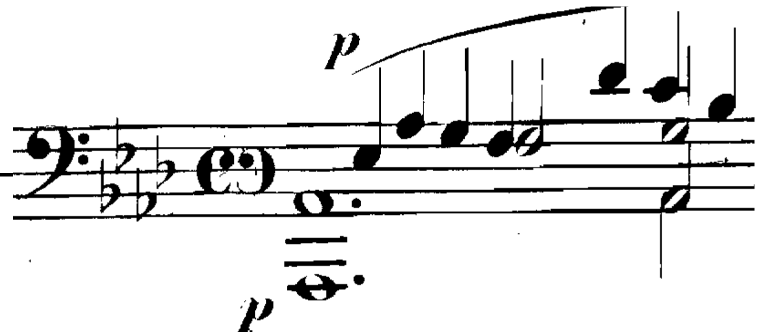


Dispersion relation $E(|\vec{k}|)$ extracted from $C_{\gamma_{||}}$

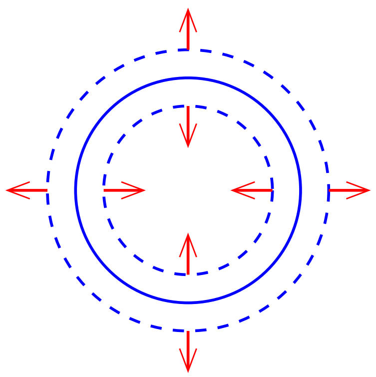
A massless vector excitation?

longitudinal

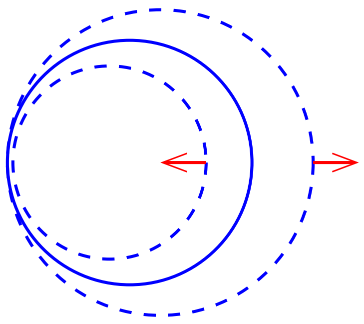
Sounds Unfamiliar?



In the Fermi liquid framework a possible explanation is a *collective excitation* thought to become important as $T \to 0$: *Zero Sound*



Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \simeq \frac{1}{\sqrt{2}} \frac{k_F}{\mu}$



ZERO sound is a propagating distortion of the Fermi surface: velocity β_0 must be determined self-consistently

Fermi Liquid Behaviour

Landau (1958) Baym & Chin (1976)

Basic idea: dominant low energy excitations are *quasiparticles* carrying same quantum numbers as fundamental particles

Quasiparticle energy: $\varepsilon_{\vec{k}}$ Width: $\sim (\varepsilon_{\vec{k}} - \mu)^2$

Equilibrium distribution: $n_{\vec{k}} = \left(\exp(\frac{\varepsilon_{\vec{k}} - \mu}{T}) + 1\right)^{-1}$

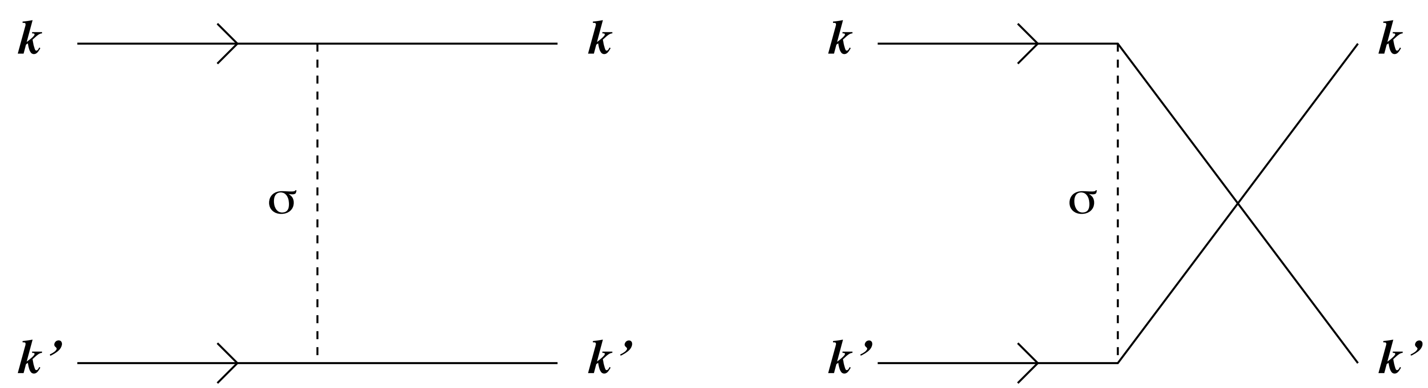
For T→0 $\varepsilon_{\vec{k}} \simeq \mu + \beta_F(|\vec{k}| - k_F)$

The heart of Landau's approach is the variation of $\varepsilon_{\vec{k}}$ under small departures from equilibrium:

$$\delta\varepsilon_{\vec{k}} = \int \frac{d^2\vec{k}'}{(2\pi)^2} \mathcal{F}_{\vec{k},\vec{k}'} \delta n_{\vec{k}'}$$

The Fermi Liquid Interaction is related to the 2-particle forward scattering amplitude

$$\mathcal{F}_{\vec{k},\sigma,\vec{k}',\sigma'} = -\mathcal{M}_{\vec{k},\sigma,\vec{k}',\sigma'}$$



Direct

attractive

vanishes in chiral limit

Exchange

repulsive

naturally $O(1/N_f)$

$$\begin{aligned}\mathcal{F}_{\vec{k},\vec{k}'} &= \frac{g^2}{4N_f}\left[1 - \frac{\vec{k}.\vec{k}'}{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}\right] D_\sigma(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}, \vec{k} - \vec{k}') \\ &= \frac{\pi\mu}{N_f M_\sigma^2(\mu)}(1 - \cos\theta)\end{aligned}$$

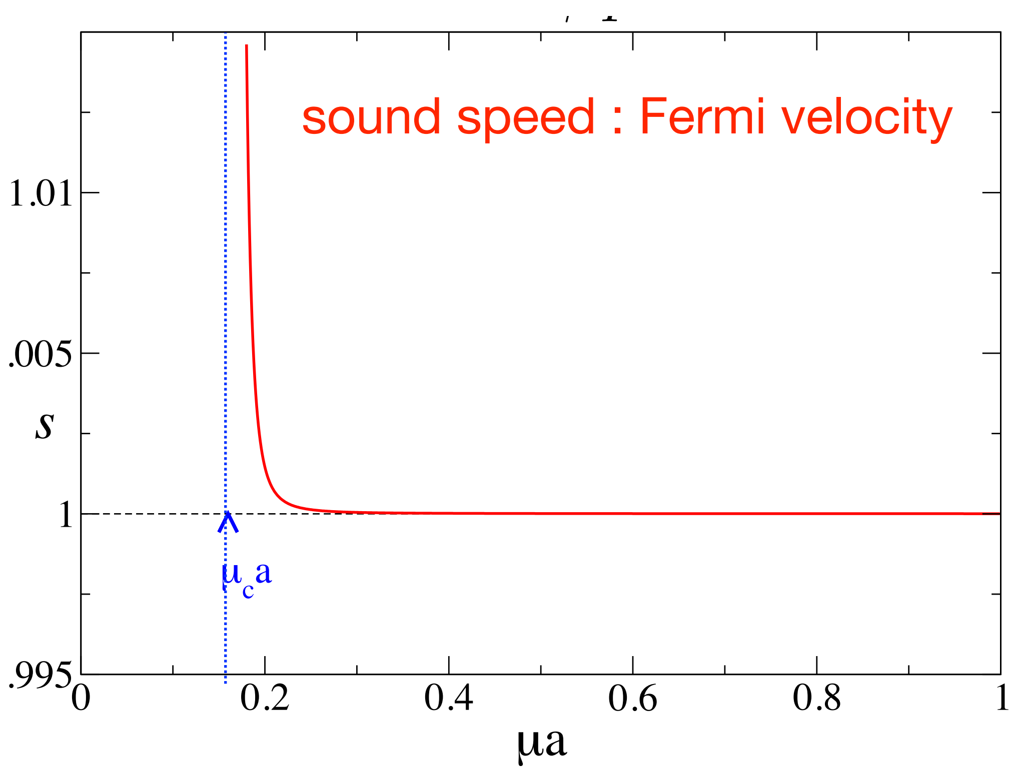
Since at Fermi surface $\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} \simeq 0$

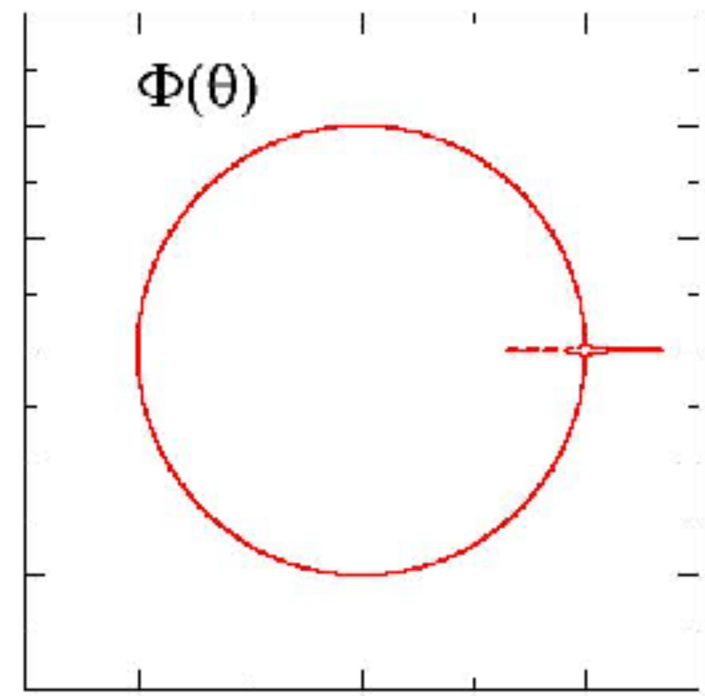
we can take the static limit of D_σ.

Boltzmann equation in collisionless limit:

$$\frac{s - \cos\theta}{\cos\theta}\Phi(\theta) = \frac{\mu\mathfrak{g}}{4\pi^2}\oint_{\theta'}\mathcal{F}_{\theta,\theta'}\Phi(\theta') = G\int\frac{d\theta'}{2\pi}[R - \cos(\theta-\theta')]\Phi(\theta')$$

for GN model $G \simeq \frac{\mathfrak{g}\mu}{8N_f(\mu-\mu_c)}$, $R = \frac{2+G}{2-G}$, $s \equiv \frac{\beta_0}{\beta_F}$.





A solution with $s > 1$ exists for almost all $\mu > \mu_c$

$\Phi(\theta)$ highly peaked in the forward direction

SJH, CG Strouthos PRD70 056006

The NJL Model

Effective description of soft pions interacting with nucleons/constituent quarks

$$\begin{aligned}\mathcal{L}_{NJL} &= \bar\psi(\partial\!\!\!/ + m + \mu\gamma_0)\psi - \frac{g^2}{2}[(\bar\psi\psi)^2 - (\bar\psi\gamma_5\vec\tau\psi)^2] \\ &\sim \bar\psi(\partial\!\!\!/ + m + \mu\gamma_0 + \sigma + i\gamma_5\vec\pi.\vec\tau)\psi + \frac{2}{g^2}(\sigma^2 + \vec\pi.\vec\pi)\end{aligned}$$

Introduce isopsin indices so full global symmetry is SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$

Dynamical χSB for $g^2 > g_c^2 \Rightarrow$ isotriplet Goldstone $\vec\pi$

Scalar isoscalar diquark $\psi^{tr}C\gamma_5\otimes\tau_2\otimes A^{color}\psi$ breaks U(1)$_B$

$\Rightarrow$ diquark condensation signals high density ground state is superfluid

The NJL model informs phenomenology of colour superconductivity

M.G. Alford, K. Rajagopal and F. Wilczek, Nucl.Phys.B537(1999)443

Model is renormalisable in 2+1d so GN analysis holds

In 3+1d, an explicit cutoff is required. We follow the large-N_f (Hartree) approach of Klevansky (1992) and match lattice parameters to low energy phenomenology:

Phenomenological Observables fitted	Lattice Parameters extracted
$\Sigma_0 = 400$MeV	$ma = 0.006$
$f_\pi = 93$MeV	$1/g^2 = 0.495$
$m_\pi = 138$MeV	$a^{-1} = 720$MeV

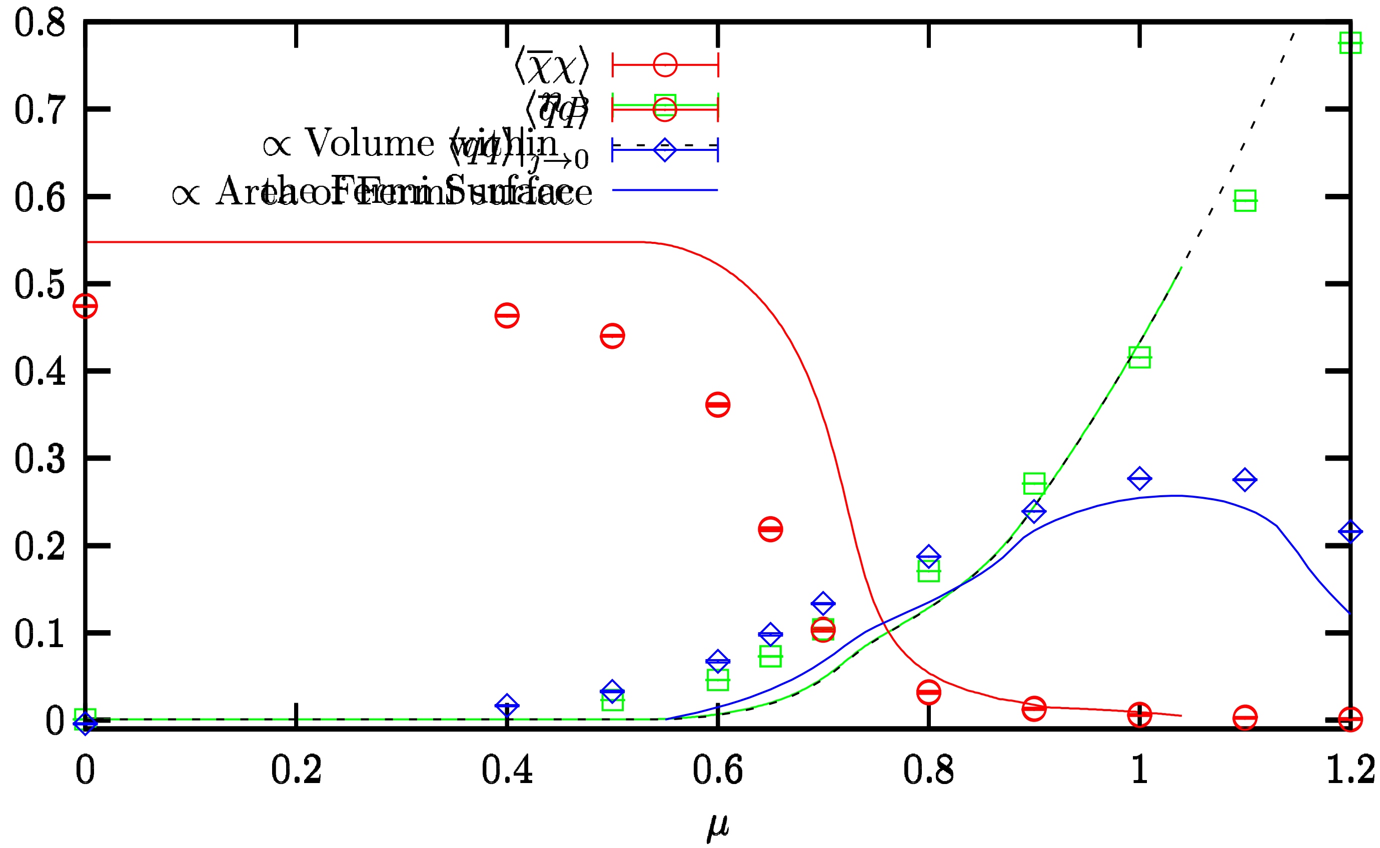
Barely a field theory!

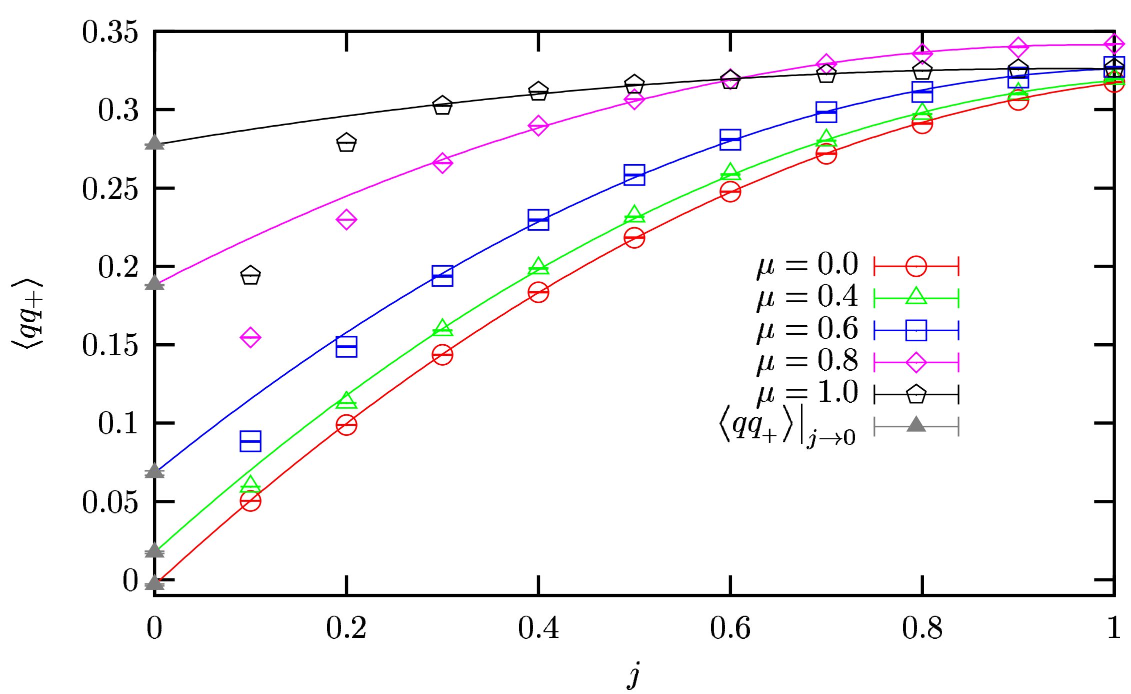
The lattice regularisation preserves

SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$

SJH, DN Walters PRD69 076011

Equation of State and Diquark Condensation





Add source $j[\psi^{tr}\psi + \bar{\psi}\bar{\psi}^{tr}]$

Diquark condensate estimated by taking $j \to 0$

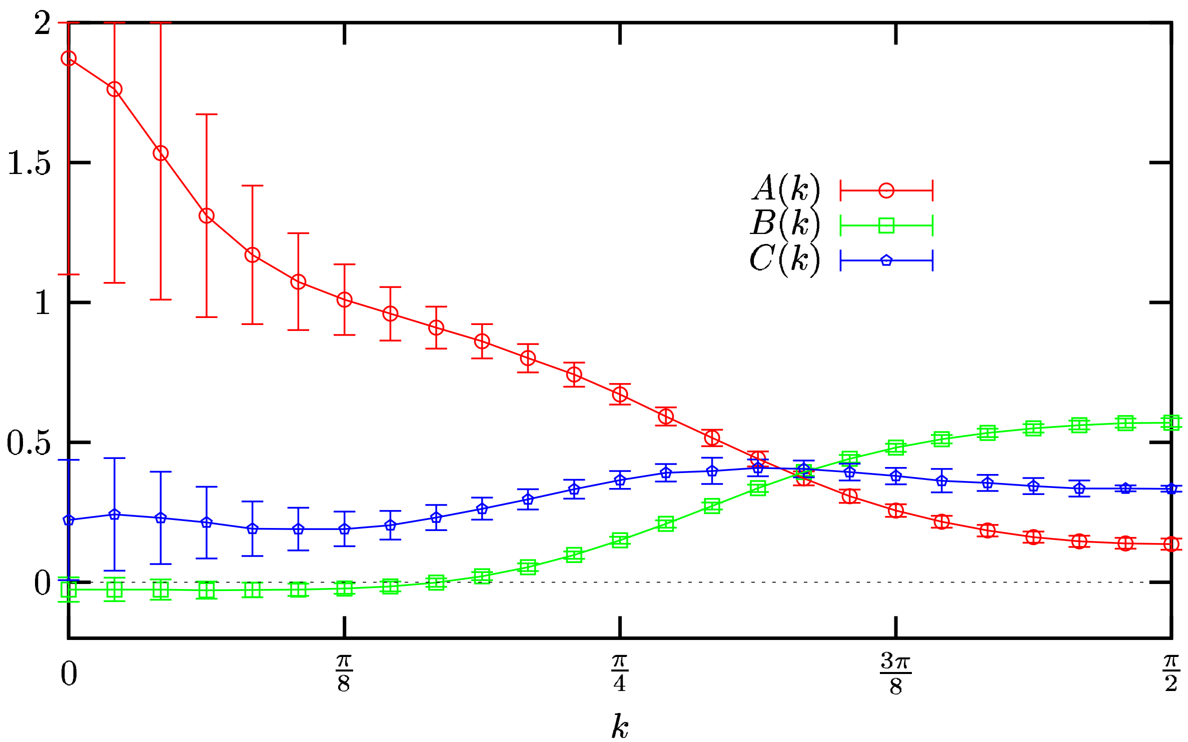
Our fits exclude $j \leq 0.2$

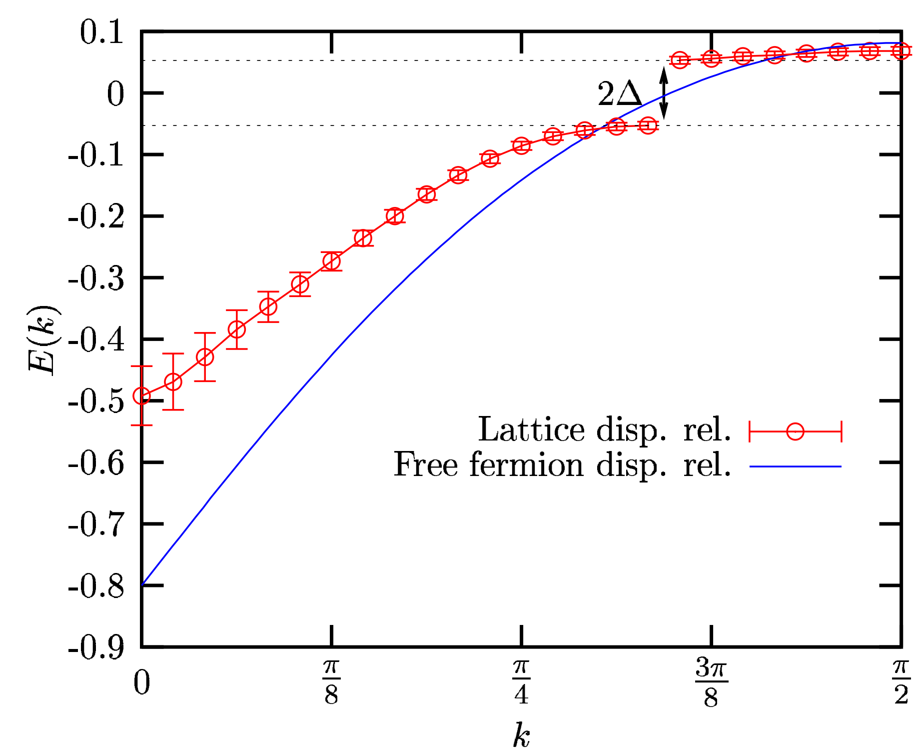
The Superfluid Gap

Quasiparticle propagator:

$$\begin{aligned}\langle\psi_u(0)\bar{\psi}_u(t)\rangle &= Ae^{-Et} + Be^{-E(L_t-t)}\\ \langle\psi_u(0)\psi_d(t)\rangle &= C(e^{-Et} - e^{-E(L_t-t)})\end{aligned}$$

Results from $96 \times 12^2 \times L_t$, $\mu a = 0.8$ extrapolated to $L_t \to \infty$ (ie. $T \to 0$) then $j \to 0$





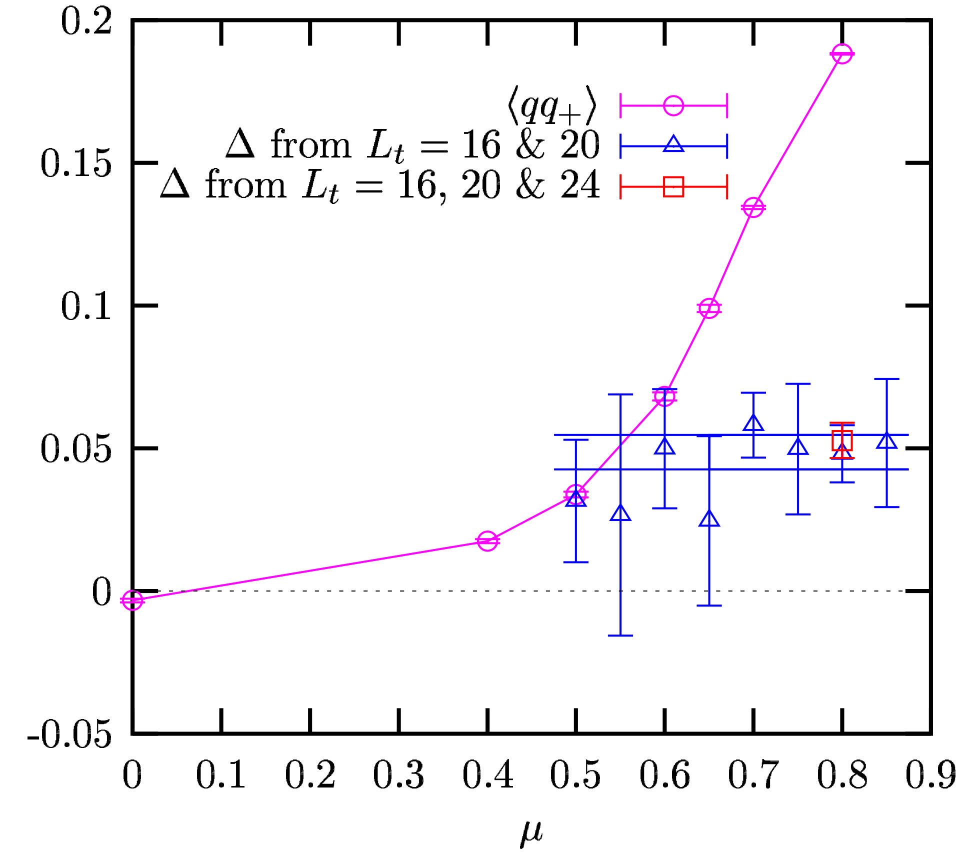
The gap at the Fermi surface signals superfluidity

- Near transition, $\Delta \sim$ const, $\langle \psi\psi \rangle \sim \Delta\mu^2$
- $\Delta/\Sigma_0 \simeq 0.15 \Rightarrow \Delta \simeq 60$MeV

in agreement with self-consistent approaches

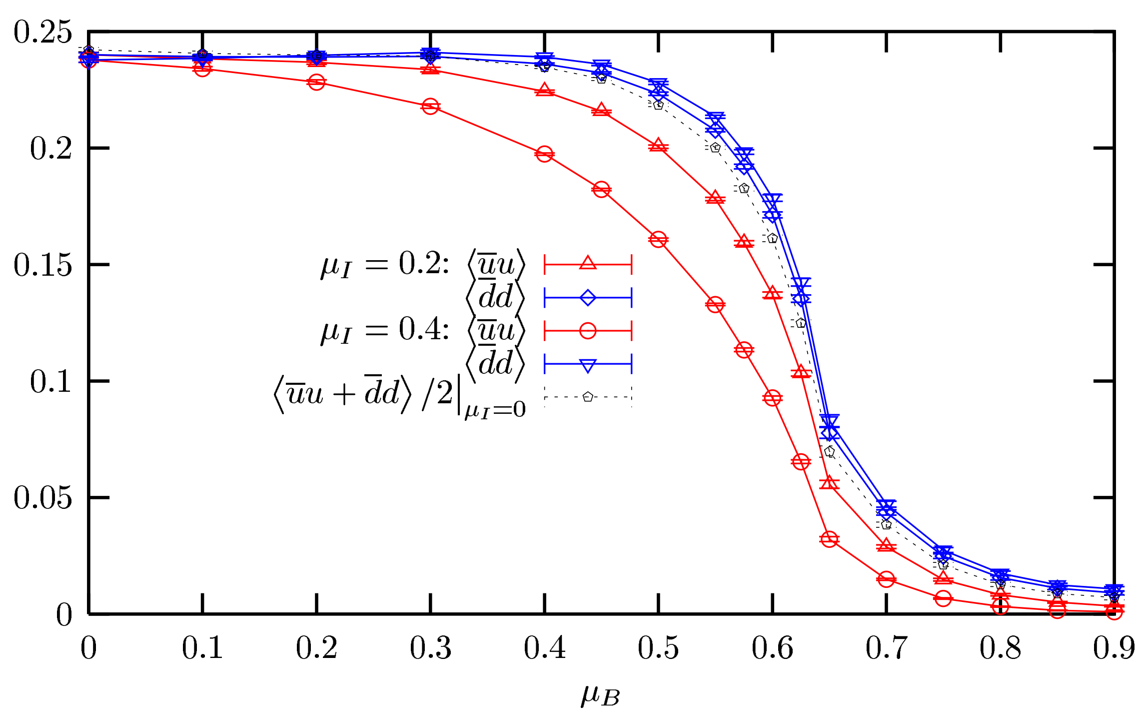
- $\Delta/T_c = 1.764$ (BCS) $\Rightarrow L_{tc} \sim 35$

explains why $j \to 0$ limit is problematic



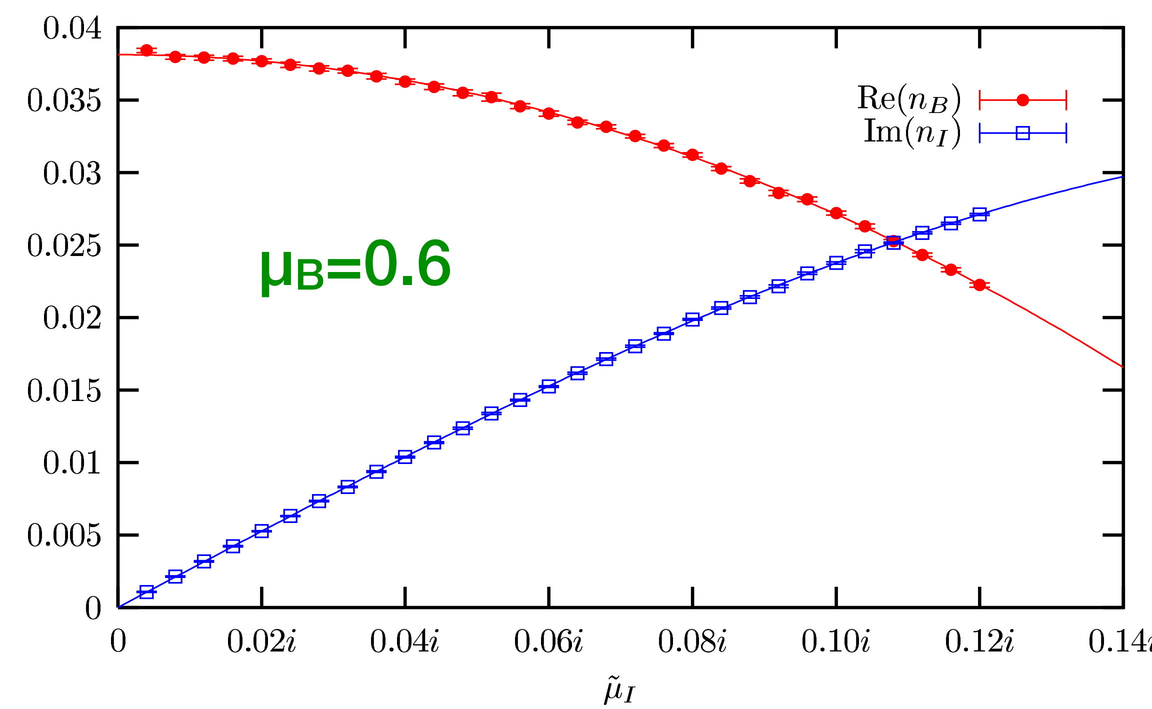
Study of $\mu_I = (\mu_u - \mu_d) \neq 0$,

reintroduces a sign problem!



partially quenched study of ⟨ūu⟩ vs ⟨d̄d⟩

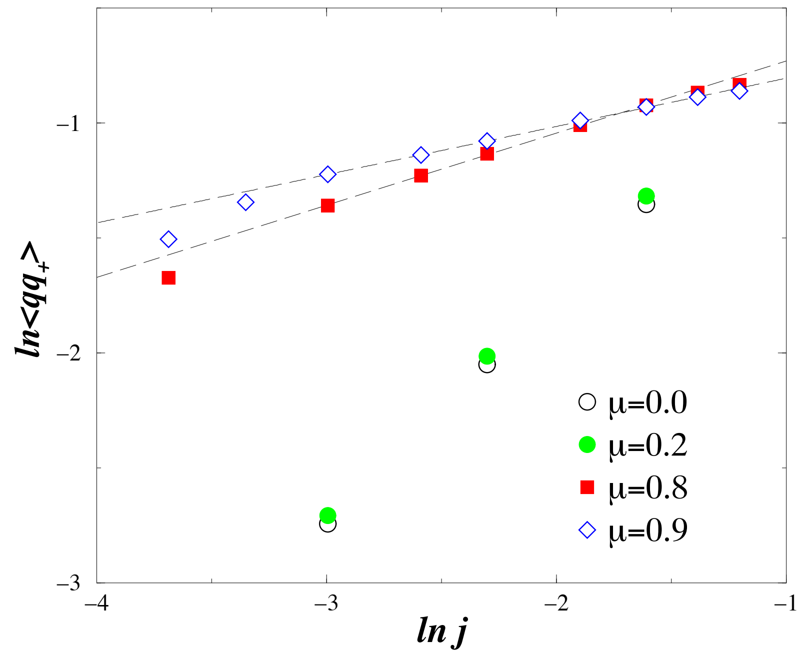
SJH, DN Walters NPhys.Proc.Suppl. **140** 532

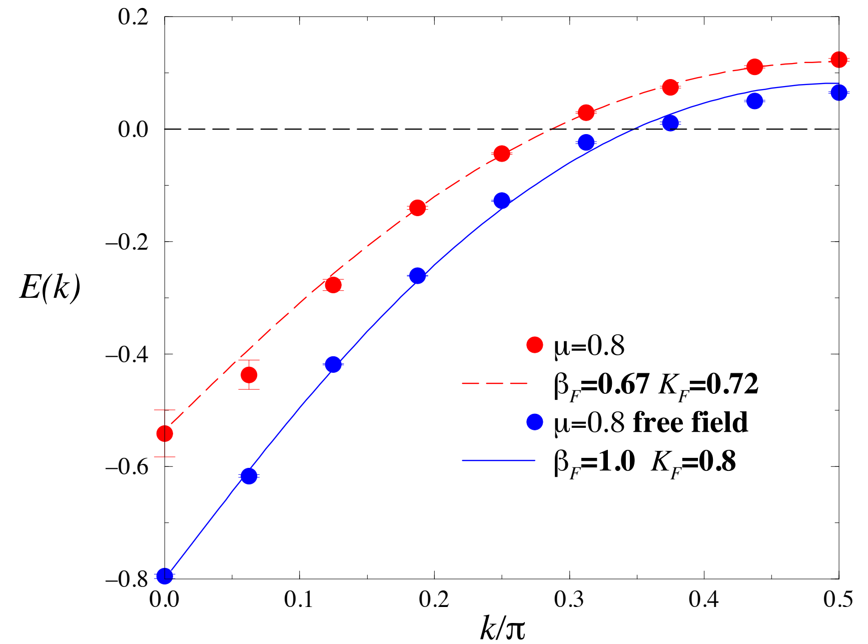


baryon and isospin densities via imaginary μ_I

NJL Model in 2+1d

SJH, B Lucini, SE Morrison PRL86 753; PRD65 036004





Condensate vanishes as

$$\langle\psi\psi\rangle \propto j^{\frac{1}{\delta}}$$

No gap at Fermi surface

High density phase $\mu > \mu_c$ is *critical*, rather like the low-T phase of the 2d XY model   Kosterlitz & Thouless (1973)

$\delta = \delta(\mu) \simeq 3 - 5$ Cf. 2d XY model $\delta \geq 15$

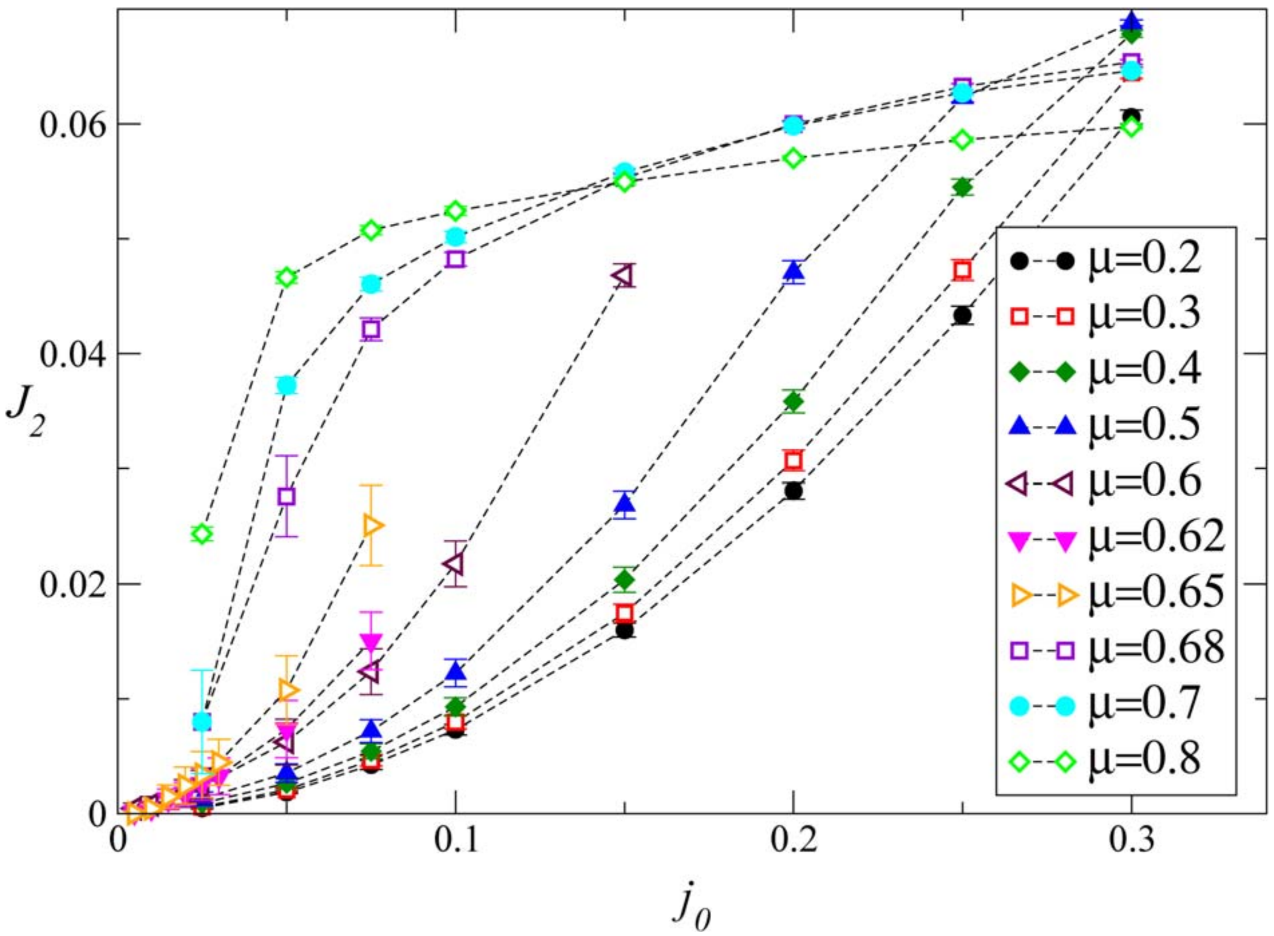
New universality class due to massless fermions

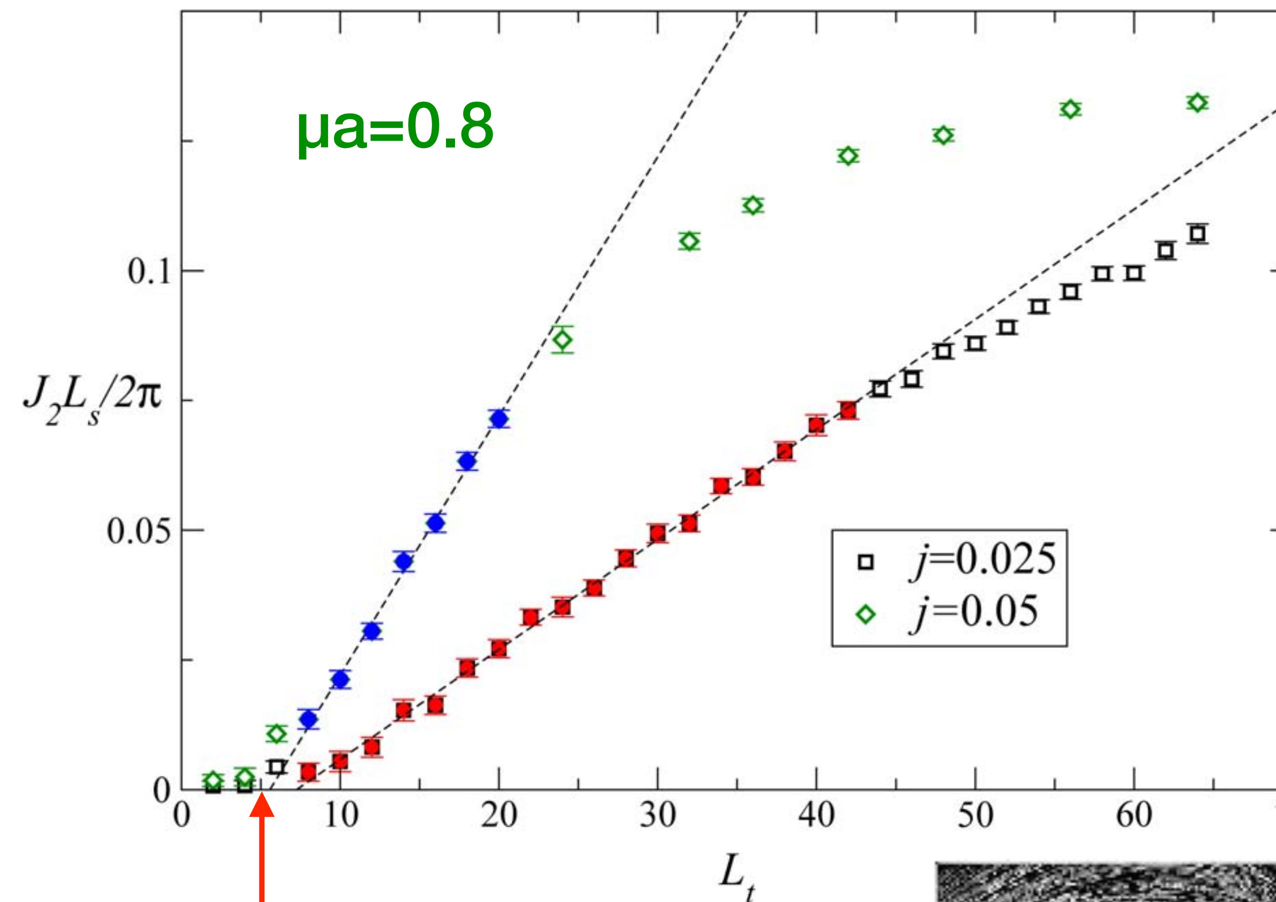
No long-range ordering, but phase coherence

$$\langle\psi\psi(0)\psi\psi(r)\rangle \propto r^{-\eta(\mu)} \Rightarrow$$ *Thin Film Superfluidity*

Use a twisted source $j(x) = j_0 e^{i\theta(x)}$ with θ periodic so $\nabla\theta=2\pi/L$

Expect $\vec{J}_s = \langle\bar{\psi}\vec{\gamma}\psi\rangle = \frac{2\pi}{L}\Upsilon$ as $j_0 \to 0$ where Υ is the *helicity modulus*





For $L_s\to\infty$ with μa=0.8

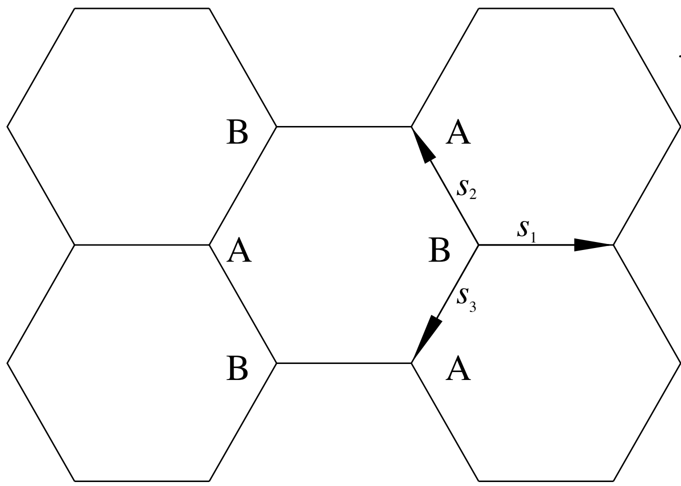
Υ/Σ=0.200(2)

SJH, AS Sehra PLB637 229

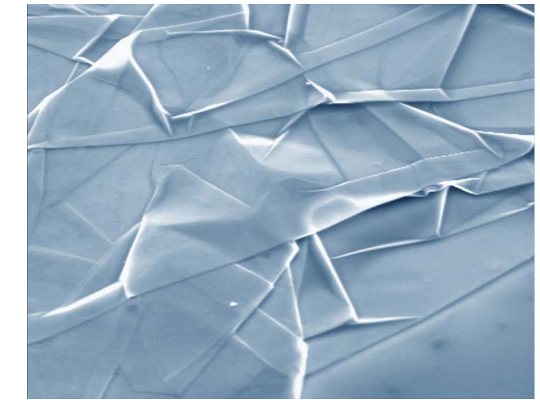
KT estimate for vortex unbinding $T_c = \frac{\pi}{2}\Upsilon$

supports superfluidity hypothesis

Relativity in Graphene



$$H = -t \sum_{\mathbf{r}\in \mathbf{B}} \sum_{i=1}^{3} b^\dagger(\mathbf{r}) a(\mathbf{r}+\mathbf{s}_i) + a^\dagger(\mathbf{r}+\mathbf{s}_i) b(\mathbf{r})$$

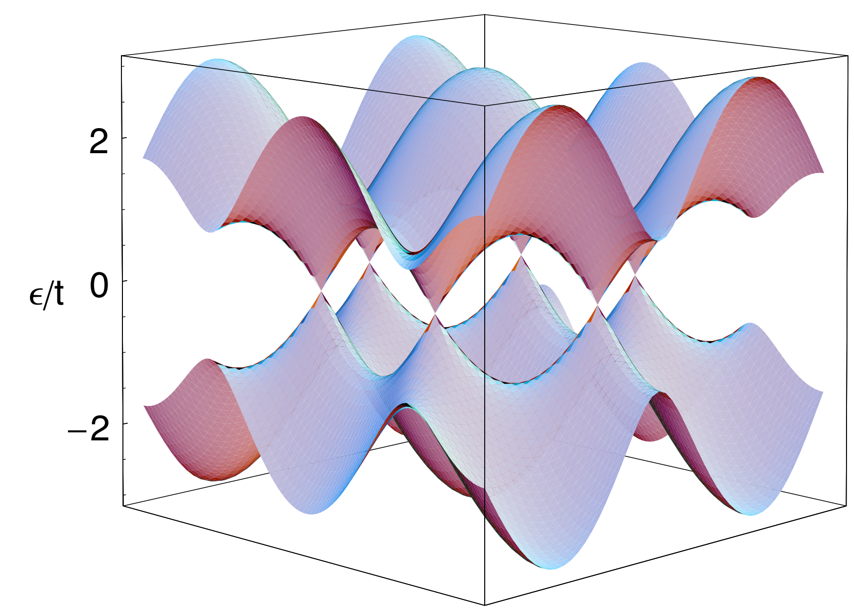


"tight-binding" Hamiltonian

describes hopping of electrons in π-orbitals
from A to B sublattices and *vice versa*

Define modified operators $a_\pm(\vec{p}) = a(\vec{K}_\pm + \vec{p})$

yielding a "4-spinor" $\Psi = (b_+, a_+, a_-, b_-)^{tr}$



$$H \simeq v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \begin{pmatrix} & p_y + ip_x & & \\ p_y - ip_x & & & \\ & & & -p_y - ip_x \\ & & -p_y + ip_x & \end{pmatrix} \Psi(\vec{p})$$

$$= v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \vec{\alpha}.\vec{p}\, \Psi(\vec{p})$$

⇒ low-energy massless fermions

with velocity $v_F = \frac{3}{2} t l \approx \frac{1}{300} c$

For monolayer graphene the number of flavors $N_f = 2$

(2 C atoms/cell × 2 Dirac points/zone × 2 spins = 2 flavors × 4 spinor)

Bilayer effective theory

W Armour, SJH, CG Strouthos PRD87 065010

$$\mathcal{L} = (\bar{\psi}, \bar{\phi}) \begin{pmatrix} D[A;\mu] + m & ij \\ -ij & D[A;-\mu] - m \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} + \frac{1}{2g^2} A^2$$

$$\equiv \bar{\Psi} \mathcal{M} \Psi. + \frac{1}{2g^2} A^2$$

Bias voltage μ couples to layer fields ψ, φ with opposite sign

(Cf. isospin chemical potential in QCD)

Intra-layer $(\psi\psi)$ and inter-layer $(\psi\phi)$ interactions have same strength

"Gap parameters" m, j are IR regulators

"Covariant" derivative $D^\dagger[A;\mu] = -D[A;-\mu].$ inherited from gauge theory

☞ $\det \mathcal{M} = \det[(D+m)^\dagger (D+m) + j^2] > 0$ **No sign problem!**

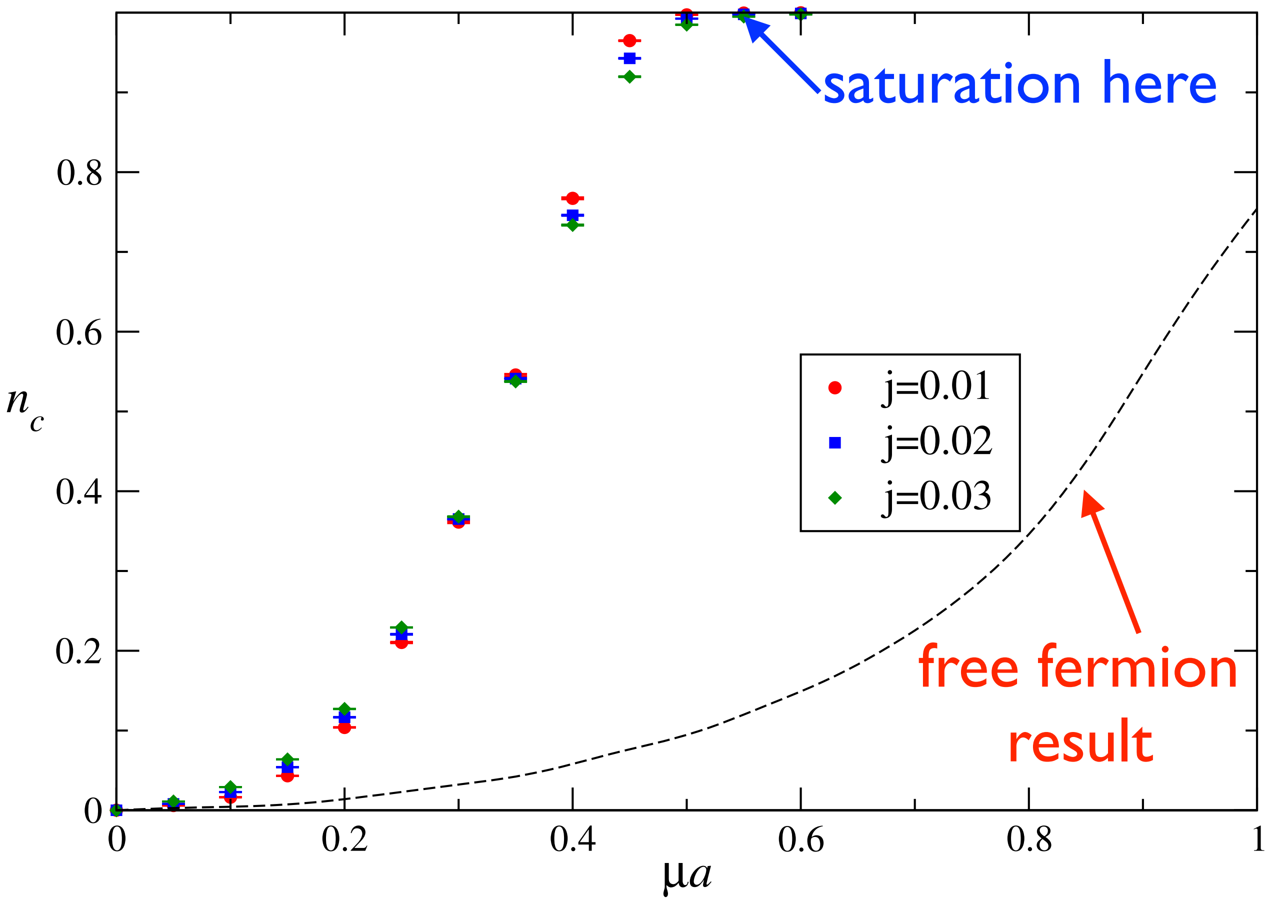
Case B

lattice sizes 32^3, 48^3

$(g^2a)^{-1} = 0.4 \Rightarrow$ close to QCP on chirally symmetric side

Carrier Density

$$n_c \equiv \frac{\partial \ln Z}{\partial \mu} = \langle \bar{\psi} D_0 \psi \rangle - \langle \bar{\phi} D_0 \phi \rangle.$$



Observe premature **saturation** (ie. one fermion per site) at $\mu a \approx 0.5$

(other lattice models typically saturate at $\mu a \gtrsim 1$)

$\Rightarrow \quad \mu a_t \approx E_F a_t < k_F a_s$

no discernable onset $\mu_o > 0$

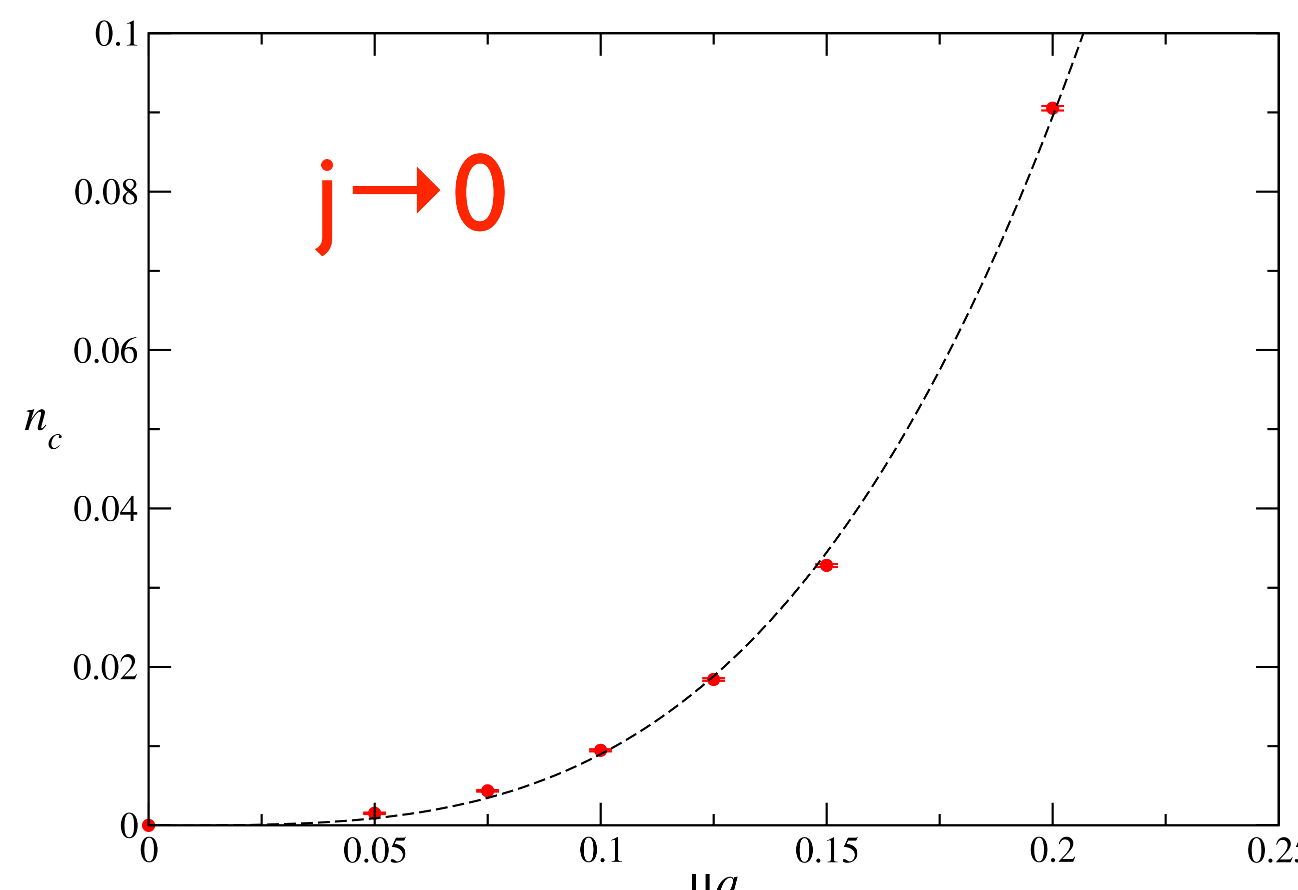
$$n_c^{free}(\mu) << n_c^{free}(k_F) \approx n_c(\mu)$$

Fit small-μ data:

$n_c(j{=}0) \propto \mu^{3.32(1)}$

Cf. free-field

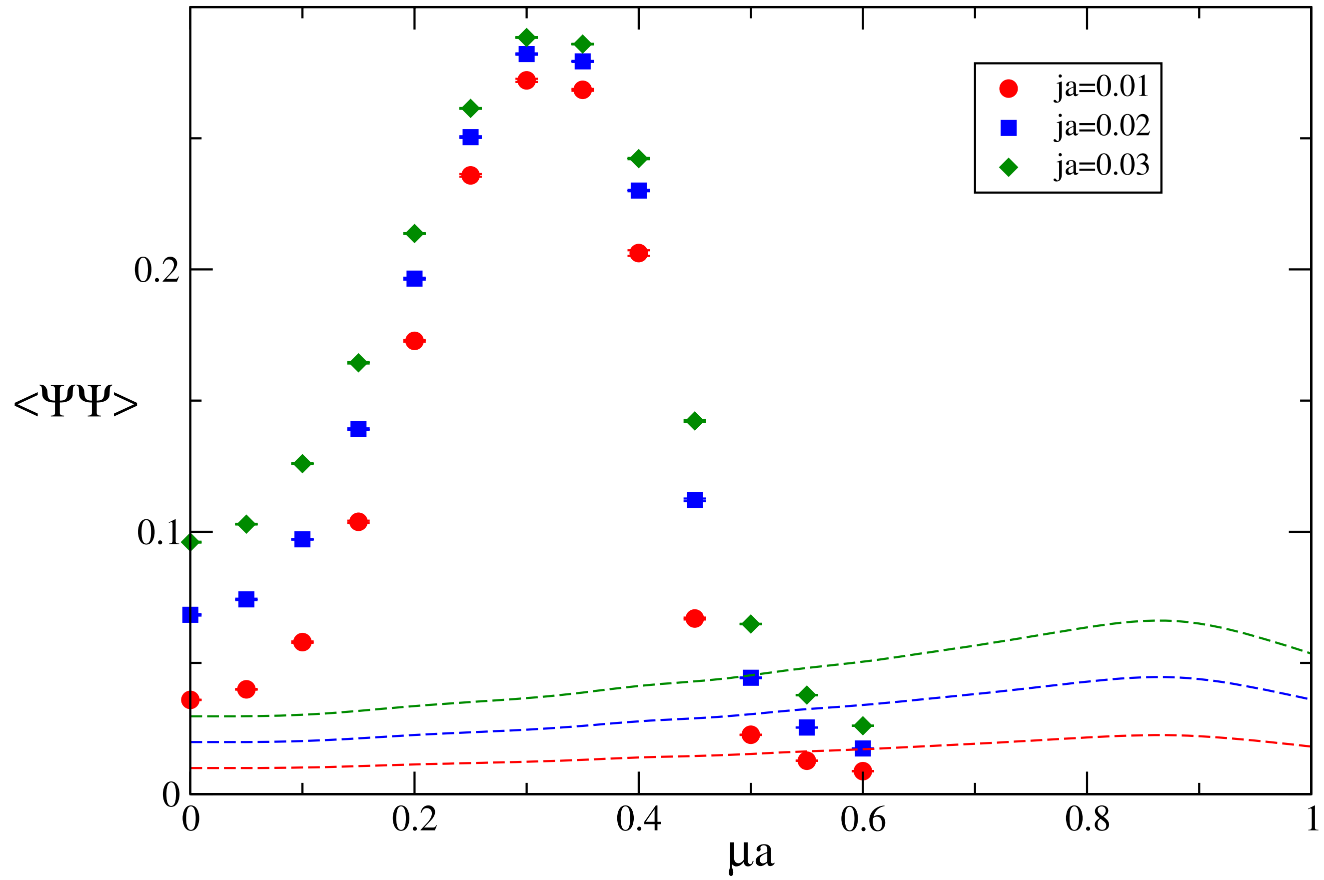
$n_c^{free} \propto \mu^d \propto \mu^2$



NB $n_c \propto k_F^2$ (Luttinger's theorem)

Exciton Condensate

$$\langle \Psi\Psi \rangle \equiv \frac{\partial \ln Z}{\partial j} = i\langle \bar{\psi}\phi - \bar{\phi}\psi \rangle$$

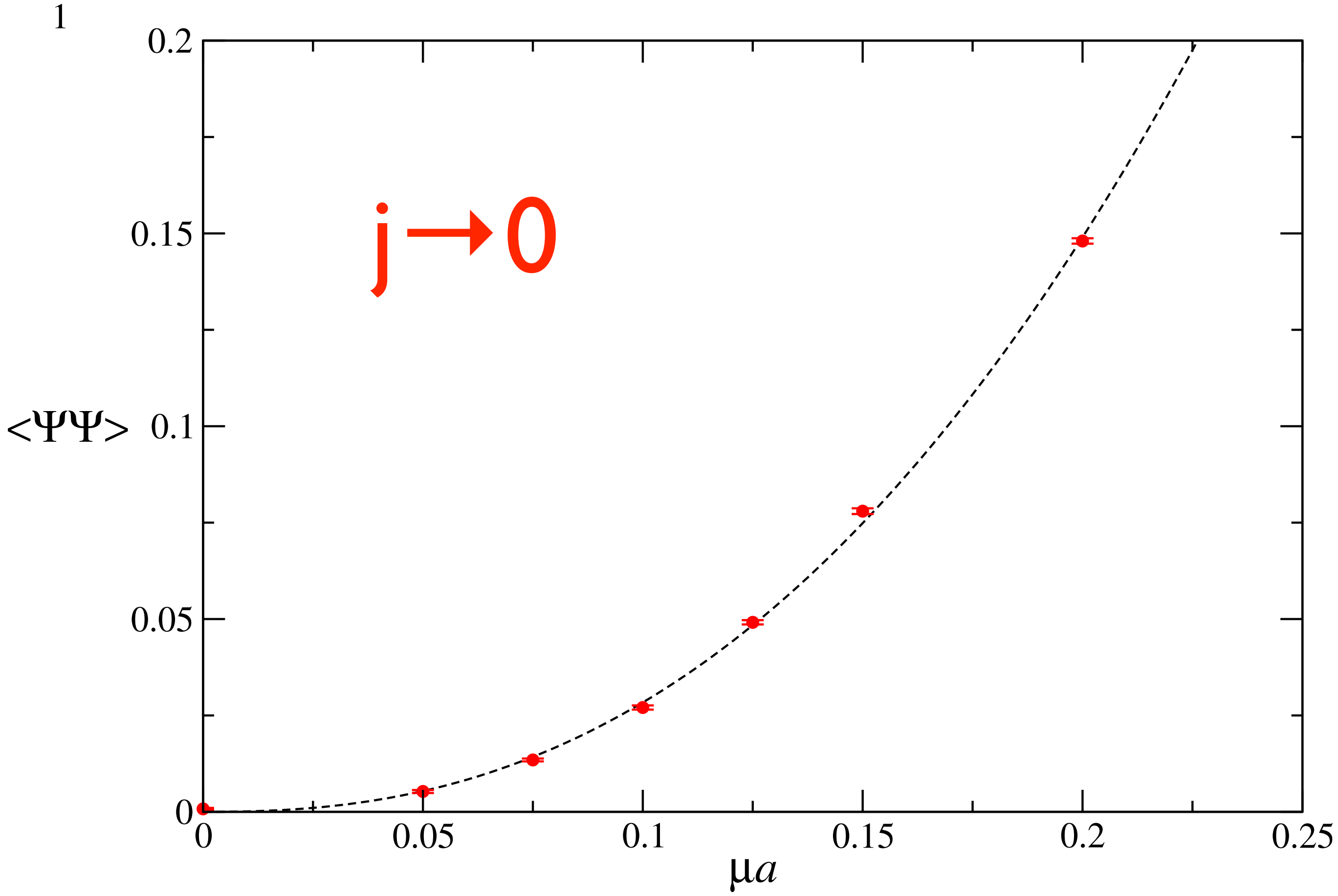


rapid rise with μ to exceed free-field value; then peak at $\mu a \approx 0.3$; then fall to zero at saturation

Exciton (ie superfluid) condensation, with no discernable onset $\mu_o > 0$

Fit small-μ data:

$$\langle \Psi\Psi(j=0) \rangle \propto \mu^{2.39(2)}$$

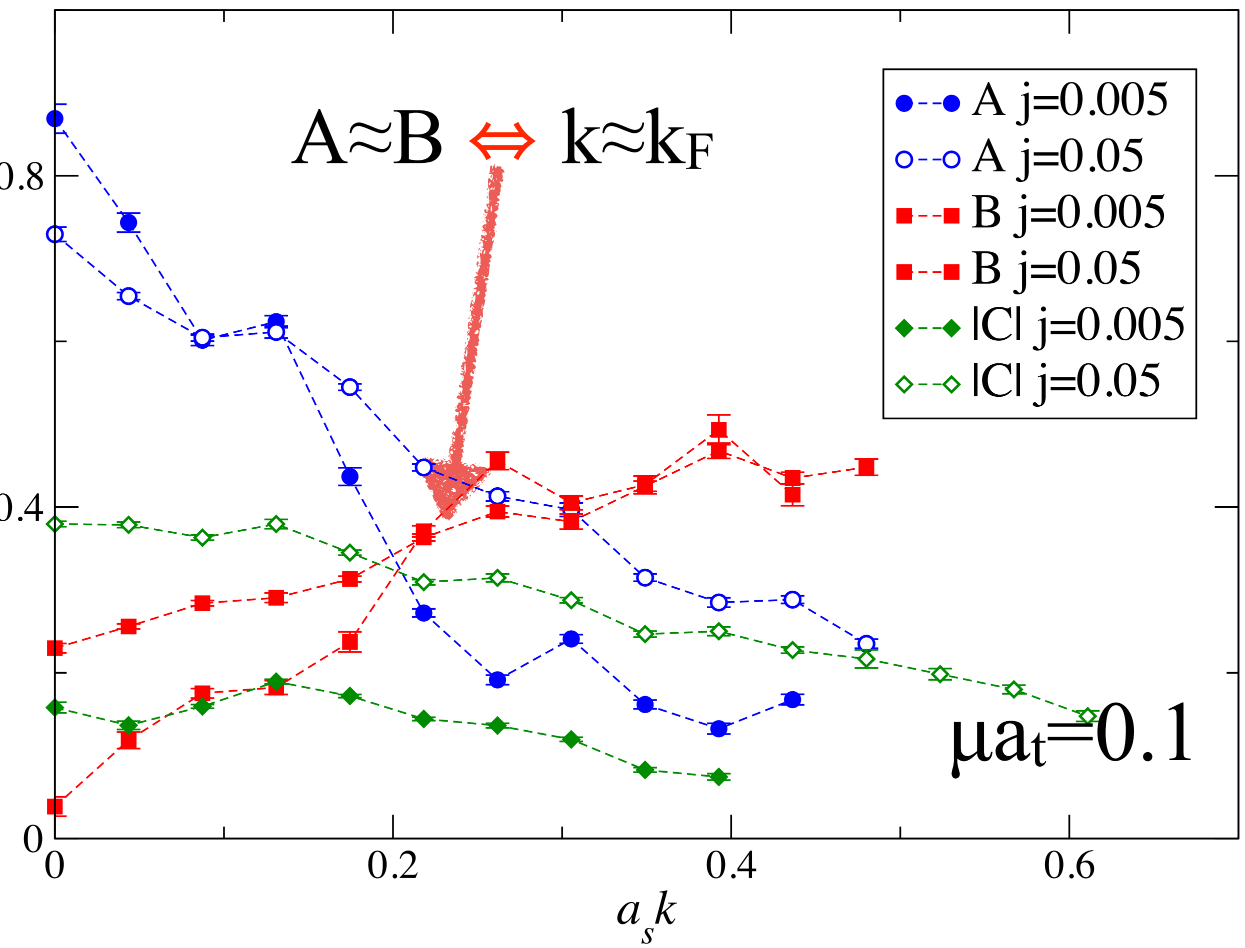


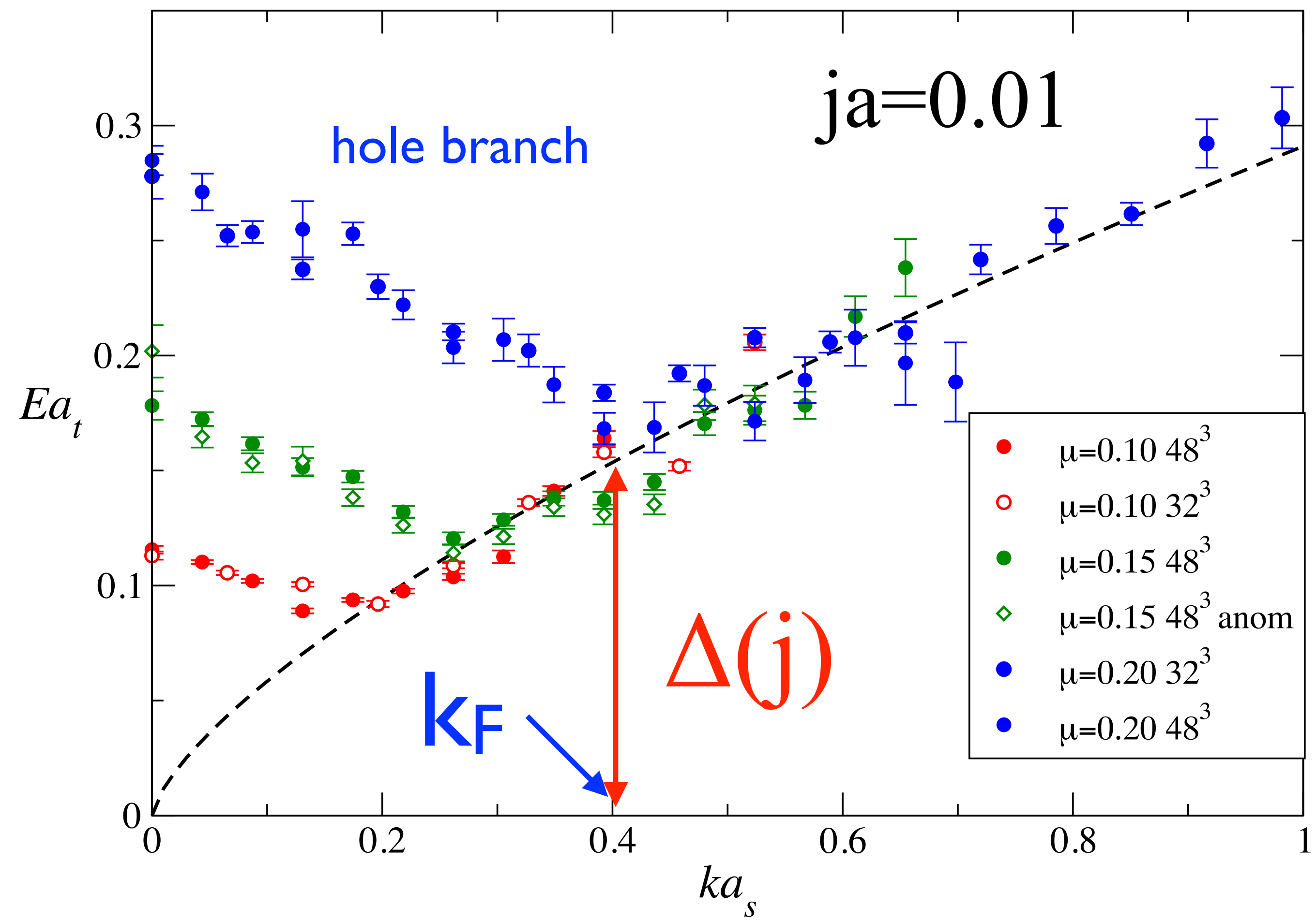
Cf. weak BCS pairing

$$\langle \Psi\Psi \rangle \propto \Delta\mu^{d-1} \propto \mu \ ?$$

Quasiparticle Dispersion

$<\Psi(k)\overline{\Psi}(k)>\sim e^{-E(k)t}$





Normal $$C_N(\vec{k},t) = \langle\psi(\vec{k},t)\bar{\psi}(\vec{k},t)\rangle = Ae^{-E_Nt} + Be^{-E_N(L_t-t)};$$

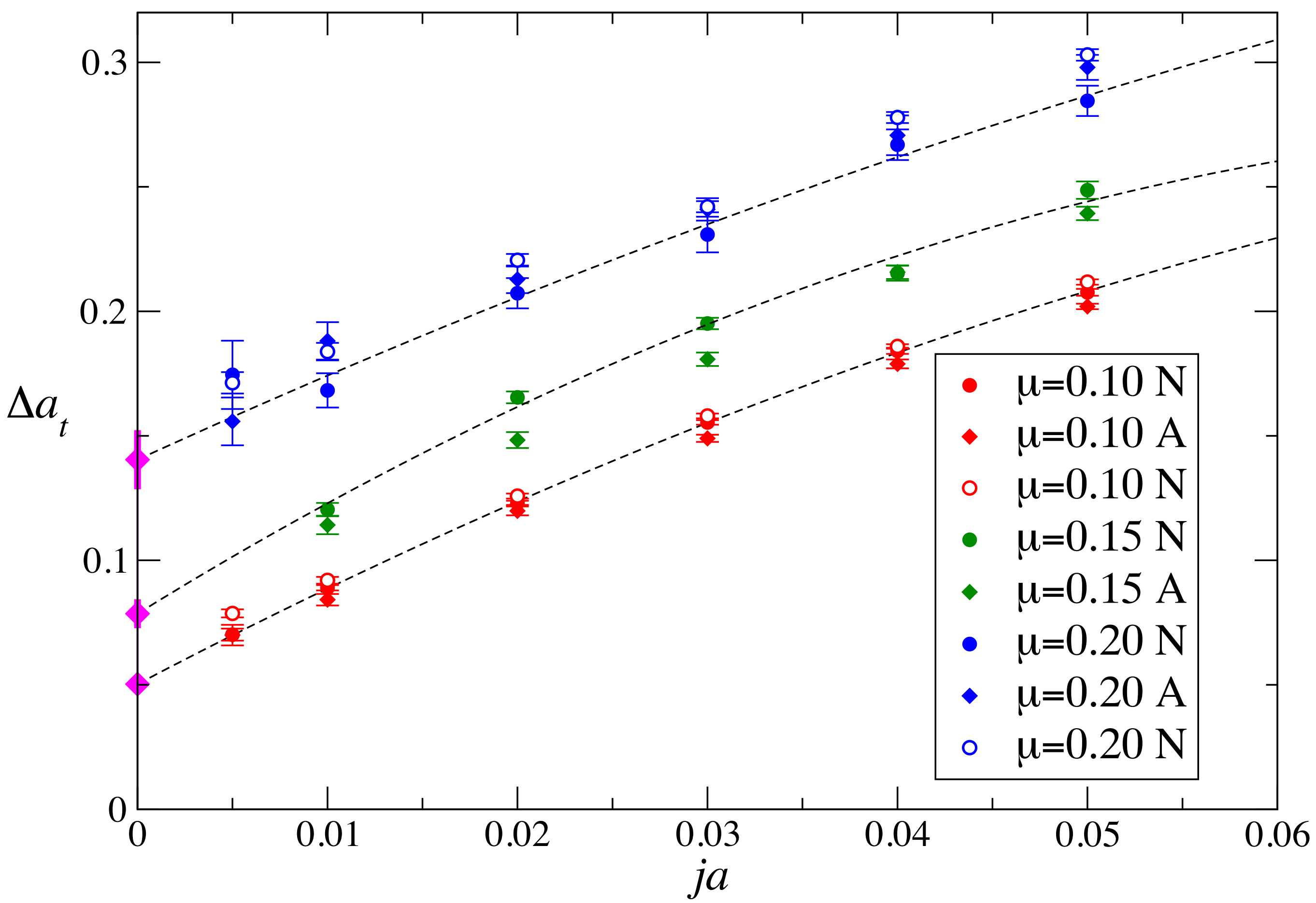
Anomalous $$C_A(\vec{k},t) = \langle\psi(\vec{k},t)\bar{\phi}(\vec{k},t)\rangle = C[e^{-E_At} - e^{-E_A(L_t-t)}].$$

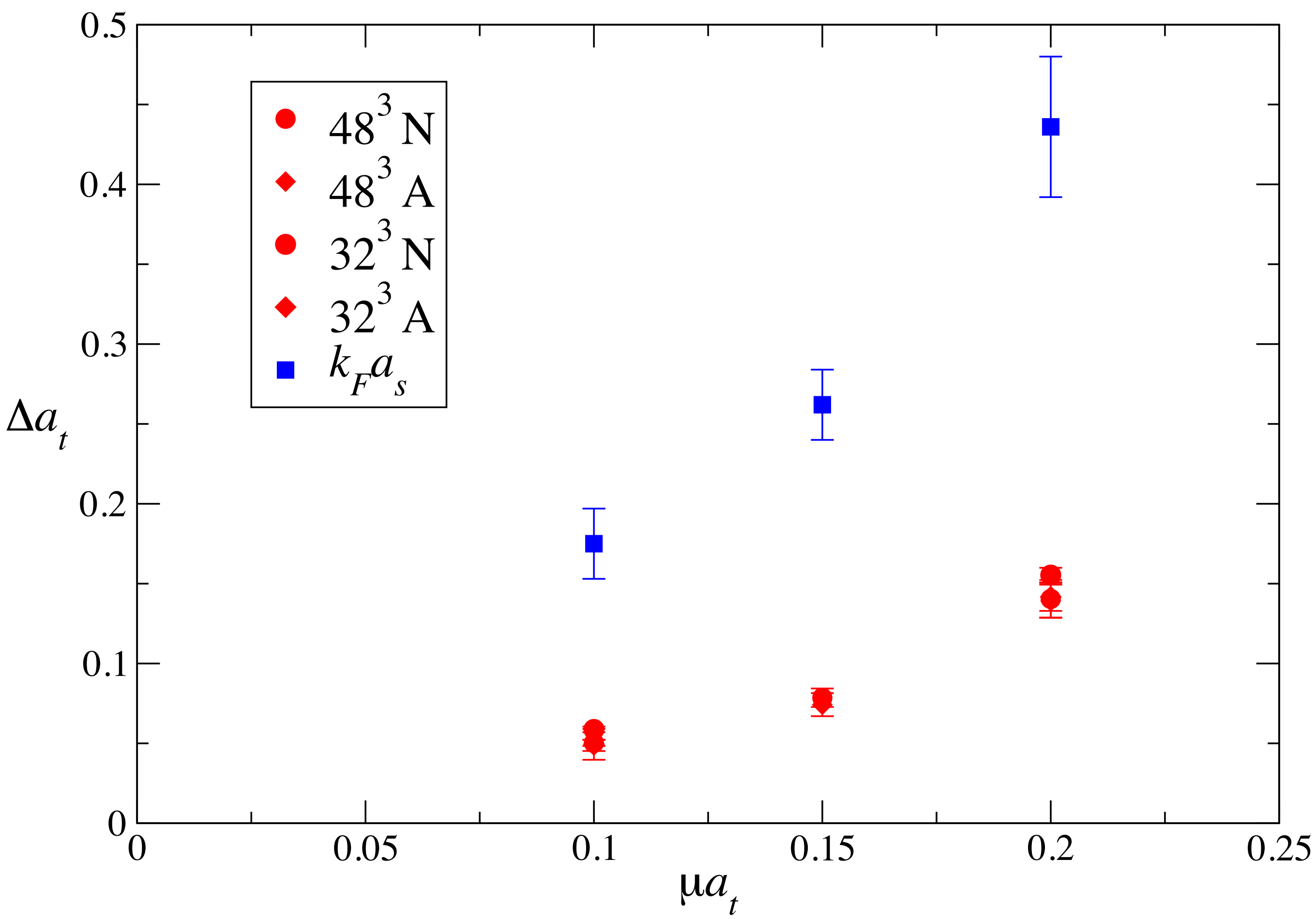
Amplitudes A, B, C show crossover from holes to particles

Dispersions E(k) show k_F varying with μ with $k_Fa_s > \mu a_t$

W.Armour, SJH & C.G. Strouthos PRB92 235143

And the gap Δ ?....





Again, consistent with a gapped Fermi surface with $\Delta/\mu = O(1)$

Both Δ and k_F scale superlinearly with μ

This is a *much* more strongly correlated system than the GN model!

Summary

Simple models support rich behaviour once $\mu \neq 0$
which can be exposed with orthodox simulation techniques

- in-medium modification of interactions
- Friedel oscillations
- sound
- Fermi surface pairing
- thin-film superfluidity
- strongly-correlated superfluidity

Left hanging:

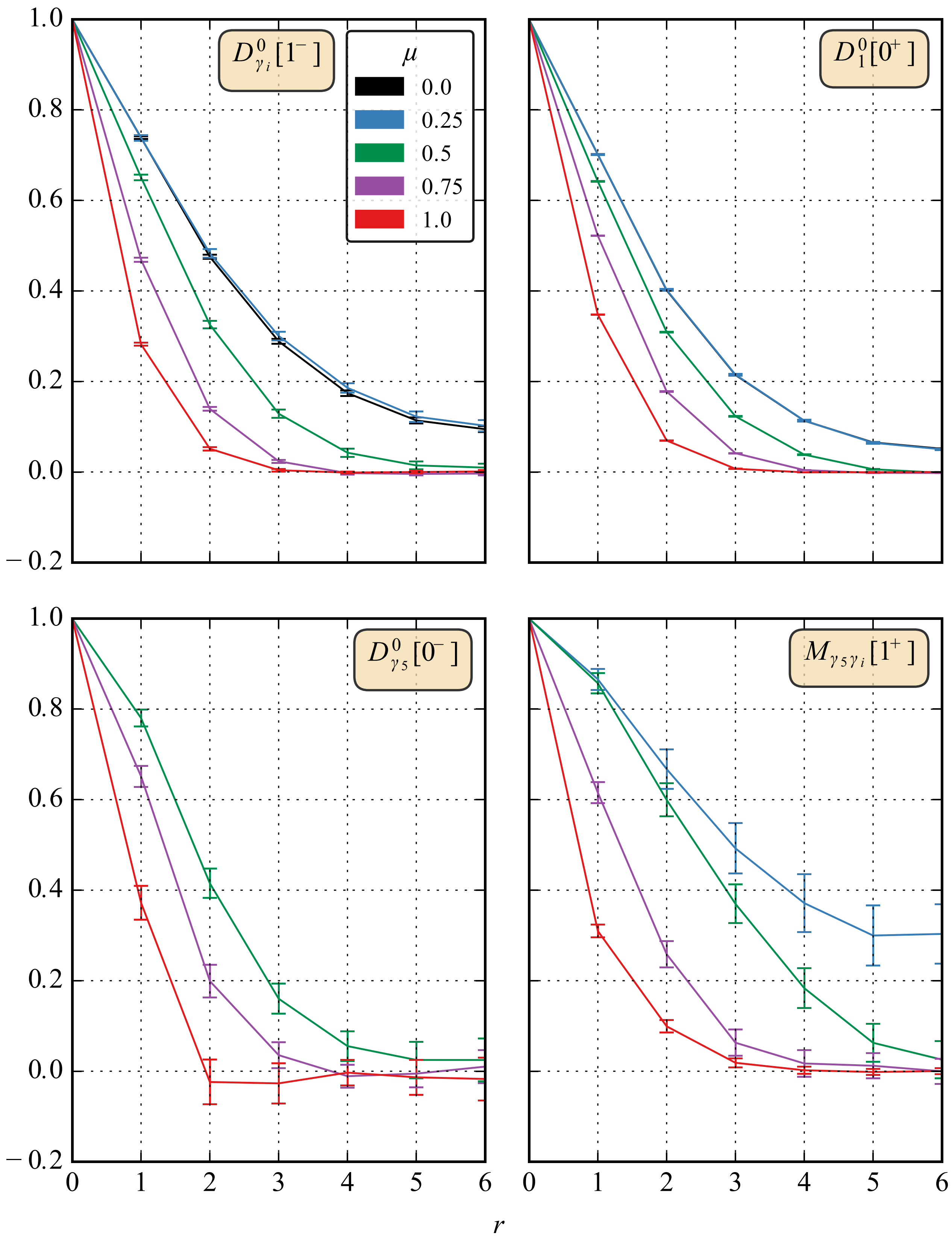
how can we identify a Fermi surface in a gauge theory?

what extra physics does the Sign Problem “buy” for us?
superconductivity through pairing?

There is life beyond the Sign Problem!

Hadron Wavefunctions in Two Color QC$_2$D

$$\Psi(\vec{r},\tau) = \int \mathrm{d}^3\vec{x}\langle 0|\bar{\psi}(\vec{x},\tau)\psi(\vec{x}+\vec{r},\tau)|H\rangle.$$

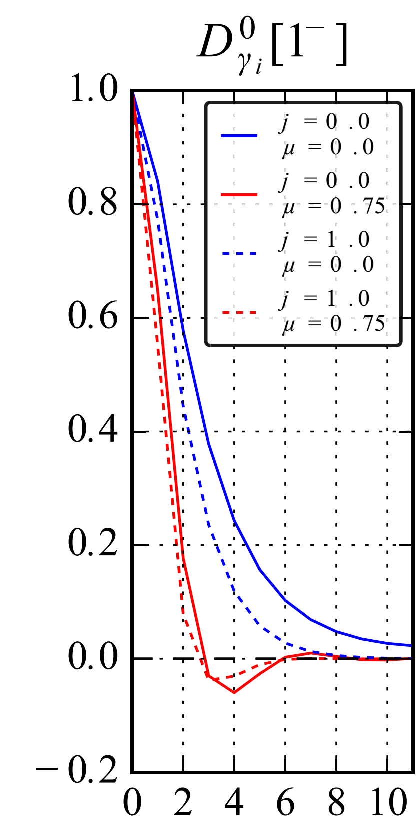


both meson and diquark channels

no Friedel oscillations, indicating a blurred Fermi surface?

⇔

superfluid gap $\Delta>0$?



free field results

A Amato, P Giudice & SJH, EPJA51 39