# THOR meets THOR

exploring the interface of fundamental and effective approaches to extreme matter

COST THOR Working Group I & II & GDRI Meeting

COST Action CA15213

Instituto Superior Técnico, Universidade de Lisboa

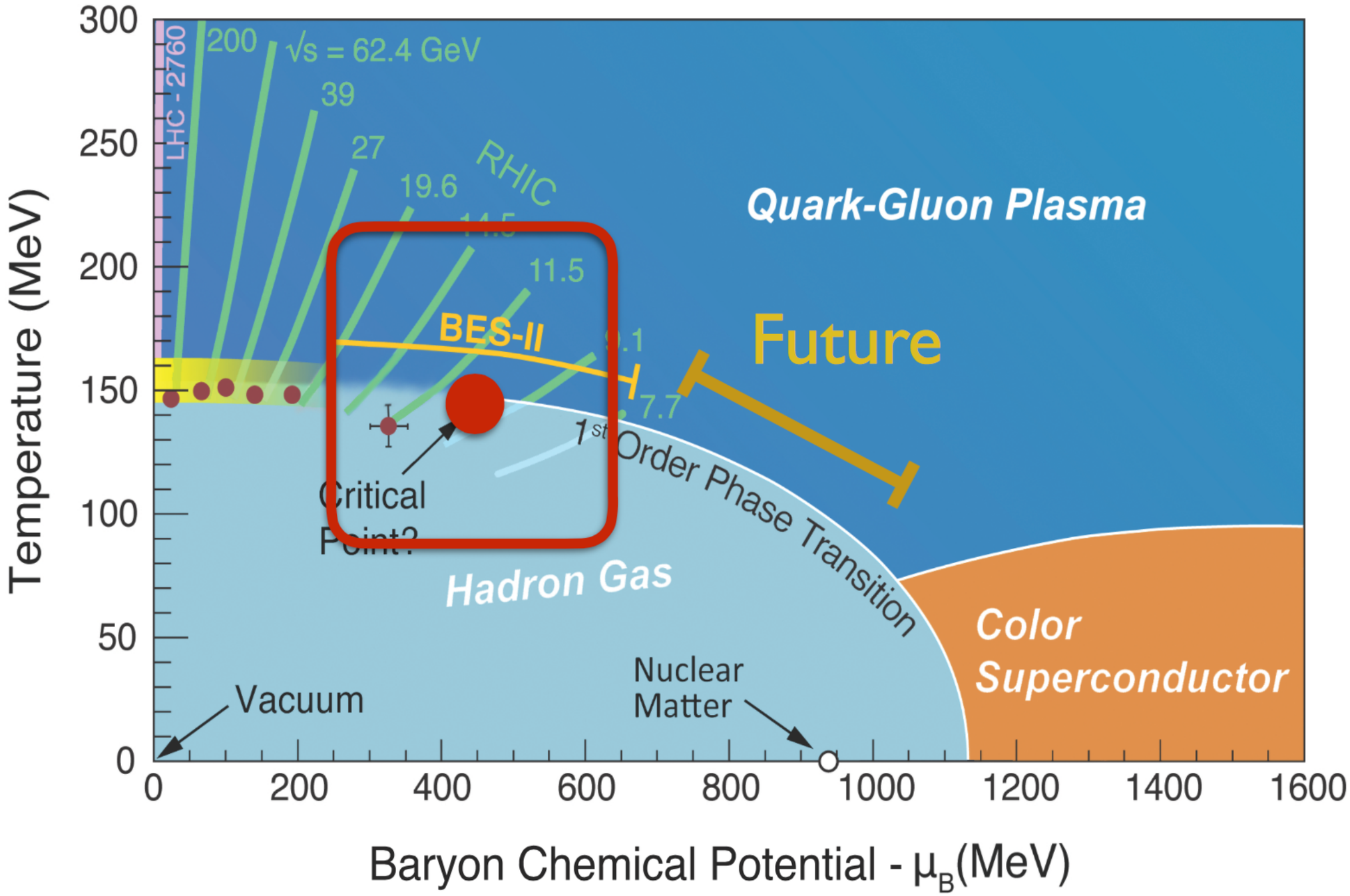
Monday June 11 - Thursday June 14 2018

# Canonical partition functions, virial expansion and the critical point(s) of QCD

Vitaly G.Bornyakov(IHEP&FEFU&ITEP),Vladimir A. Goy(FEFU),Maria Paola Lombardo(INFN) and Atsushi Nakamura (FEFU&RIKEN&Osaka U.)

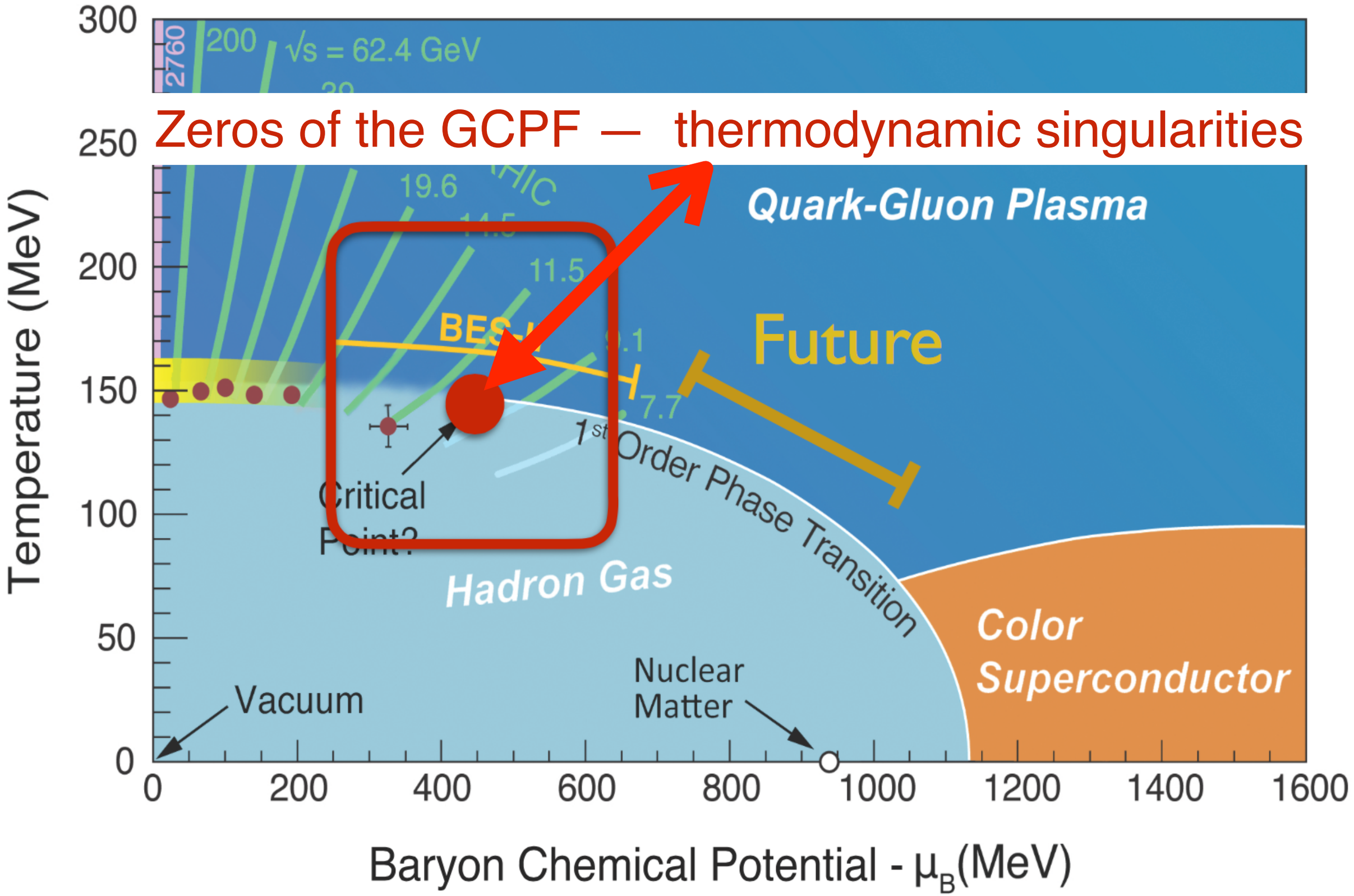
# The critical point of QCD -

(and there are more in the complex $\mu_B$ plane )




Yi Yin plenary talk at QM2018

Zeros of the GCPF — thermodynamic singularities




Yi Yin plenary talk at QM2018

# Searching for the QCD critical point

Talks by:
G. Aarts, S. Hands, L. von Smekal

Series expansions are useful tools to study the phase diagram in the region which is not directly accessible to MonteCarlo simulations due to the sign problem.

Typical parameters for the expansion are the the chemical potential $\mu$ and the fugacity $e^{\mu/T}$, while the observables to be expanded include the Grand Canonical Partition Function or the associated thermodynamic quantities, derivatives of the GCPF

$$Z(\mu) = \sum_{k=-3n_s^3}^{3n_s^3} < b_i > e^{k\mu T} = \sum_{k=-3n_s^3}^{3n_s^3} Q_k f^k$$

GCFP
on the lattice
Glasgow-style

I.M. Barbour,N. Behilil, E. Dagotto, F. Karsch, A.Moreo, M.Stone, H. Wyld, Nuclear Physics **B275** (FS17)(1986) 296.

$$\frac{p(\mu/T)}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu}{T}\right)^n$$

$$p = (T/V) \ln \mathcal{Z}$$

Pressure
on the lattice Bielefeld-Swansea -style

C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and C. Schmidt, Phys. Rev. **D66**, 074507 (2002); **D68**, 014507 (2003); C. R. Allton, M. Doering, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, **D71**, 054508 (2005).

# Exploring QCD at finite density

Talks by:
G. Aarts, S. Hands, L. von Smekal

Series expansions are useful tools to study the phase diagram in the region which is not directly accessible to MonteCarlo simulations due to the sign problem.

Typical parameters for the expansion are the the chemical potential $\mu$ and the fugacity $e^{\mu/T}$, while the observables to be expanded include the Grand Canonical Partition Function or the associated thermodynamic quantities, derivatives of the GCPF

Compute $\mathcal{Z} = <\mathrm{Det}>$. This is a polynomial in complex $e^{3\mu/T}$ whose zeros can be computed.

GCFP
on the lattice
Glasgow-style

I.M. Barbour,N. Behilil, E. Dagotto, F. Karsch, A.Moreo, M.Stone, H. Wyld, Nuclear Physics **B275** (FS17)(1986) 296.

Compute $c_n$ at $\mu = 0$
Study the radius of convergence of the Taylor series in $\mu/T$

Gavai-Gupta et al. 2003—

Pressure
on the lattice Bielefeld-Swansea -style

C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and C. Schmidt, Phys. Rev. **D66**, 074507 (2002); **D68**, 014507 (2003); C. R. Allton, M. Doering, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, and K. Redlich, **D71**, 054508 (2005).

# Exploring QCD at finite density

Talks by:
G. Aarts, S. Hands, L. von Smekal

Series expansions are useful tools to study the phase diagram in the region which is not directly accessible to MonteCarlo simulations due to the sign problem.

Typical parameters for the expansion are the the chemical potential $\mu$ and the fugacity $e^{\mu/T}$, while the observables to be expanded include the Grand Canonical Partition Function or the associated thermodynamic quantities, derivatives of the GCPF

Compute $\mathcal{Z} = <\mathrm{Det}>$. This is a polynomial in complex $e^{3\mu/T}$ whose zeros can be computed.

**Issue: reweighting?**

Compute $c_n$ at $\mu = 0$
Study the radius of convergence of the Taylor series in $\mu/T$

**Issue: higher orders?**

# Alternative Strategy: the Canonical Approach

Beating the Sign problem by Canonical Approach
(Hasenfraz-Toussant,1992)

**Calculation in pure $\mu_I$ regions, where no sign problem.**

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(T, \theta \equiv \frac{Im\,\mu}{T})$$

**Then calculate Z in real $\mu$ regions.**

$$Z(\xi, T) = \sum_n Z_n(T)\,\xi^n$$

$$\xi \equiv e^{\mu/T}$$

Recent developments:

A. Nakamura, Mod. Phys. Lett. A **22** (2007) 473; V. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V. I. Zakharov, Nucl. Phys. A **956** (2016) 809.
V. G. Bornyakov, D. L. Boyda, V. A. Goy, A. V. Molochkov, A. Nakamura, A. A. Nikolaev and V. I. Zakharov, Phys. Rev. D **95** (2017) no.9, 094506. V. G. Bornyakov *et al.*, arXiv:1712.02830 [hep-lat].

# Alternative Strategy for GCPF: the Canonical Approach

Beating the Sign problem by Canonical Approach
(Hasenfraz-Toussant,1992)

**Calculation in pure $\mu_I$ regions, where no sign problem.**

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(T, \theta \equiv \frac{Im\,\mu}{T}) = e^{-\int_0^{\mu_I/T} n^n(x)dx}$$

**Then calculate Z in real $\mu$ regions.**

$$Z(\xi, T) = \sum_n Z_n(T)\,\xi^n$$

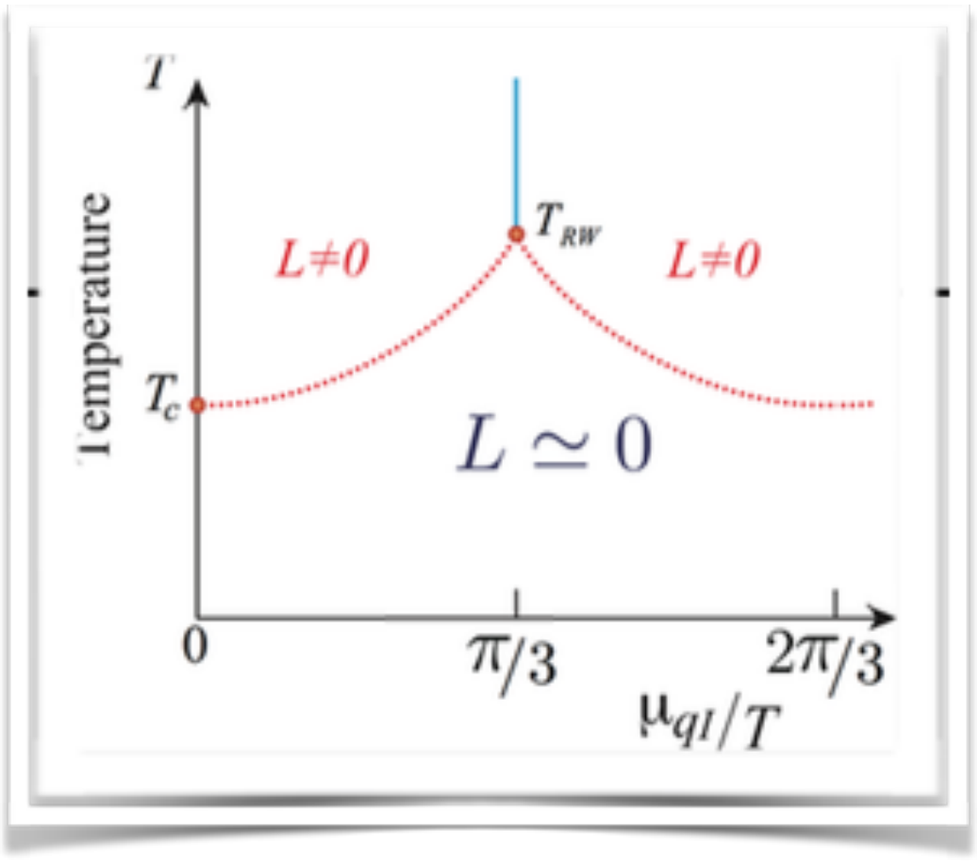
$$\xi \equiv e^{\mu/T}$$

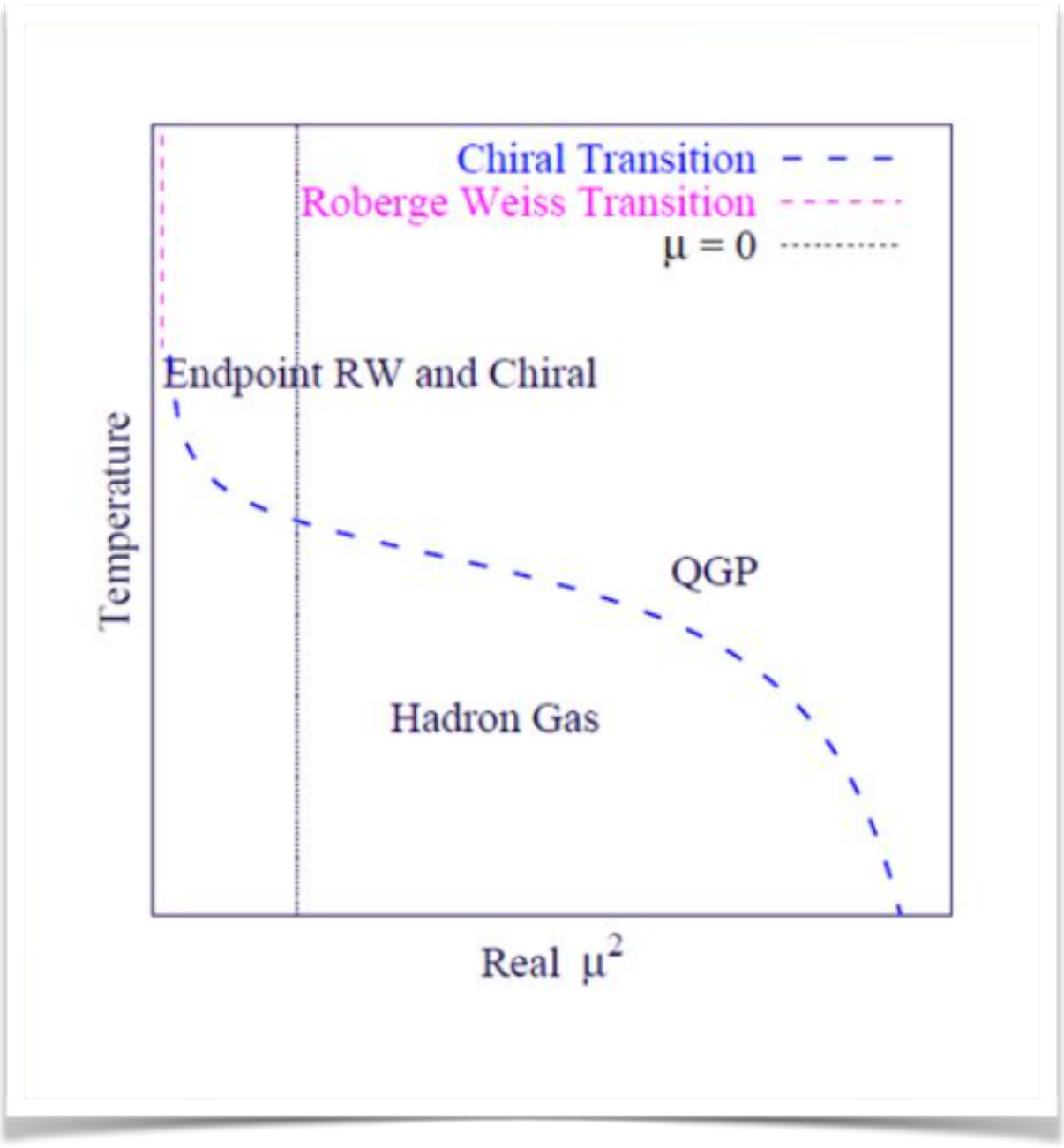
## Recent developments:

A. Nakamura, Mod. Phys. Lett. A **22** (2007) 473; V. Bornyakov, D. Boyda, V. Goy, A. Molochkov, A. Nakamura, A. Nikolaev and V. I. Zakharov, Nucl. Phys. A **956** (2016) 809.
V. G. Bornyakov, D. L. Boyda, V. A. Goy, A. V. Molochkov, A. Nakamura, A. A. Nikolaev and V. I. Zakharov, Phys. Rev. D **95** (2017) no.9, 094506. V. G. Bornyakov *et al.*, arXiv:1712.02830 [hep-lat].

# Phases of QCD at purely imaginary $\mu$

Roberge-Weiss 81



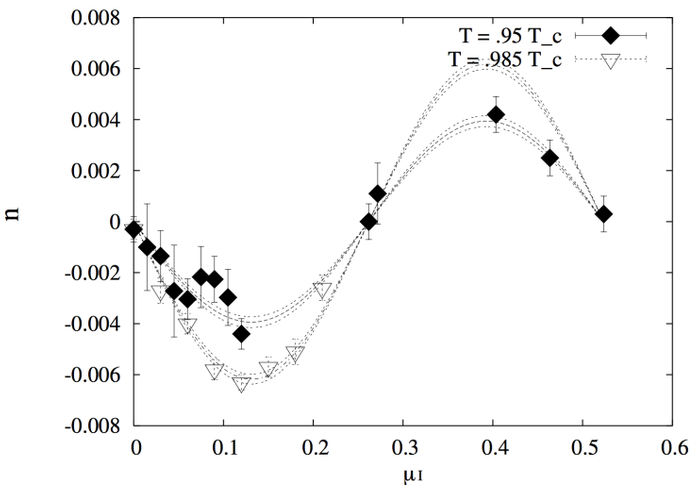

i.e. $\mu^2 < 0$

# Pressure and its derivatives from imaginary $\mu$

- Strategy: analytically continue from imaginary $\mu$

MpL 00 de Forcrand Philipsen 02 d'Elia MpL 02

- Observation: optimal parametrization depends on T

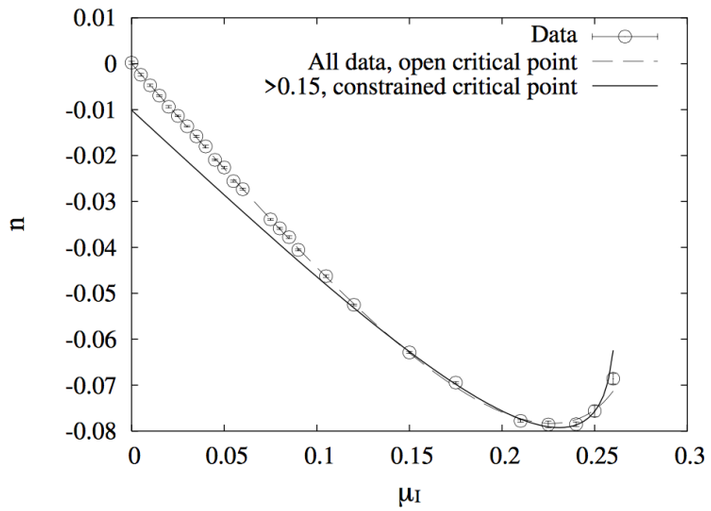



D'Elia, MpL 04

Below Tc:

**Fourier**

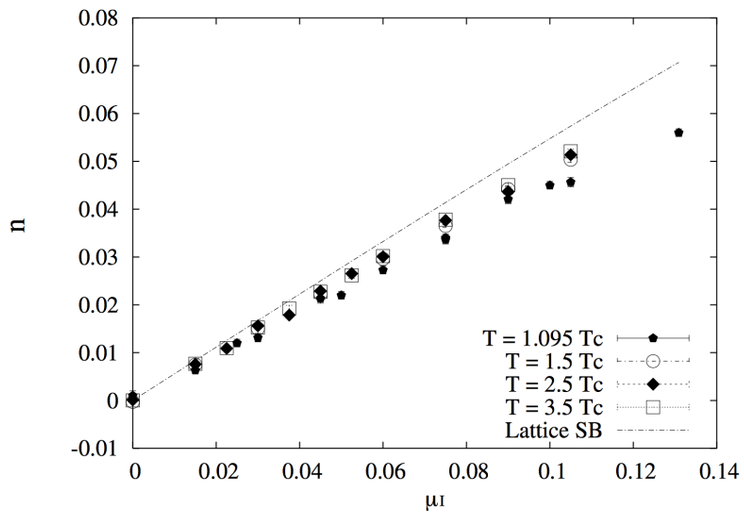
**Virial expansion natural**




D'Elia, di Renzo, MpL 05

$T_c < T < T_{RW}$

Singular




D'Elia, MpL 04

$T > T_{RW}$

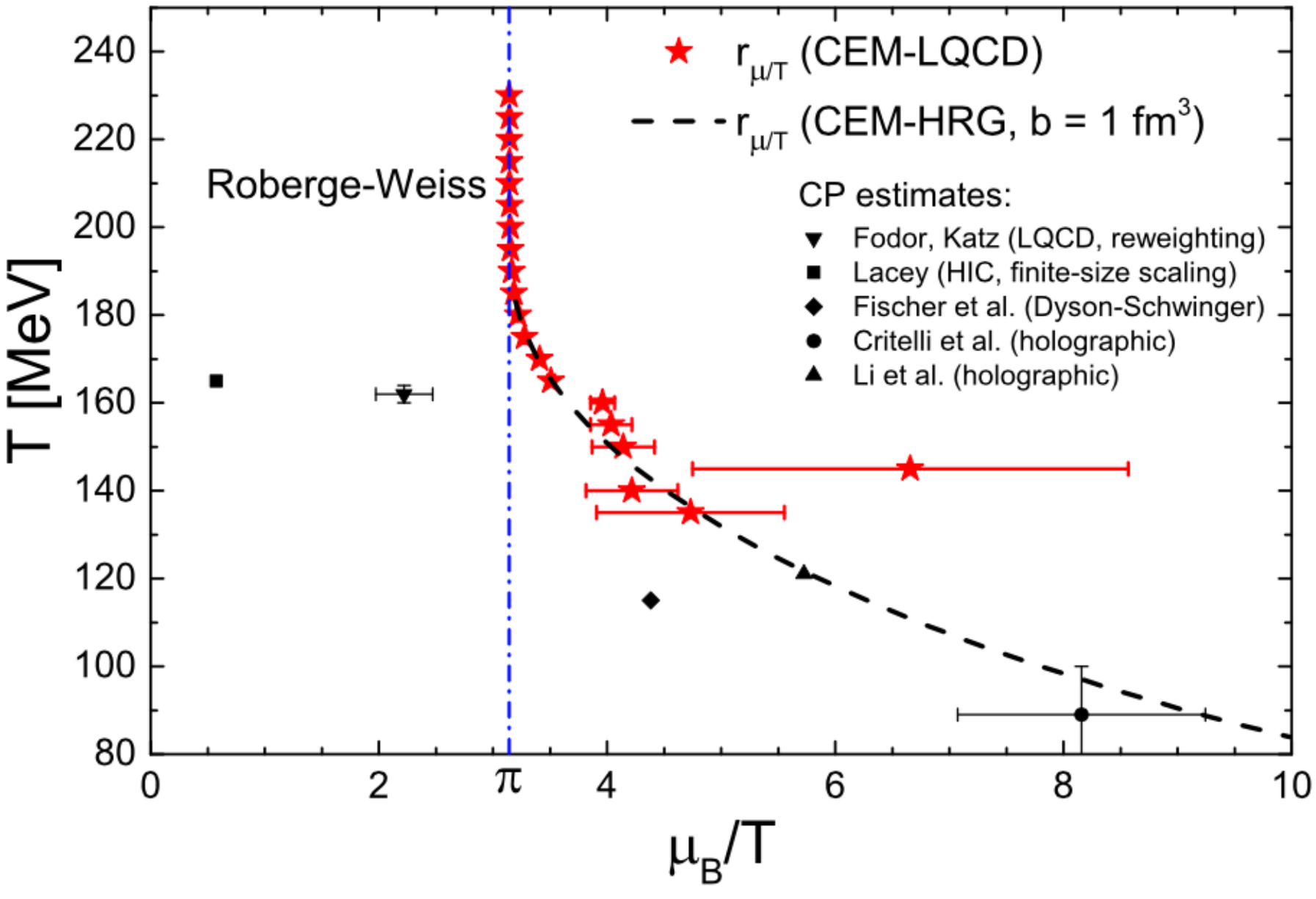
Taylor

see M.D'Elia plenary review at QM2018

# Virial expansion and Cluster Model Expansion

V. Vovchenko, J. Steinheimer, O. Philipsen, and H. Stoecker (2017), arXiv:1711.01261.

$$\frac{\rho_B}{T^3} = \frac{\partial(p/T^4)}{\partial(\mu_B/T)} = \sum_{k=1}^{\infty} b_k(T)\,\sinh\left(\frac{k\,\mu_B}{T}\right) \qquad b_k^{\mathrm{SB}} = \frac{(-1)^{k+1}}{k}\,\frac{4\left[3+4\,(\pi k)^2\right]}{27\,(\pi k)^2}.$$



## Summary of introduction

Interplay of virial expansion, canonical expansion, radius of convergence at real and imaginary chemical potential

Virial expansion within CME constraints the CEP

## Plan

Virial expansion, canonical PF and GCPF in 1dQCD

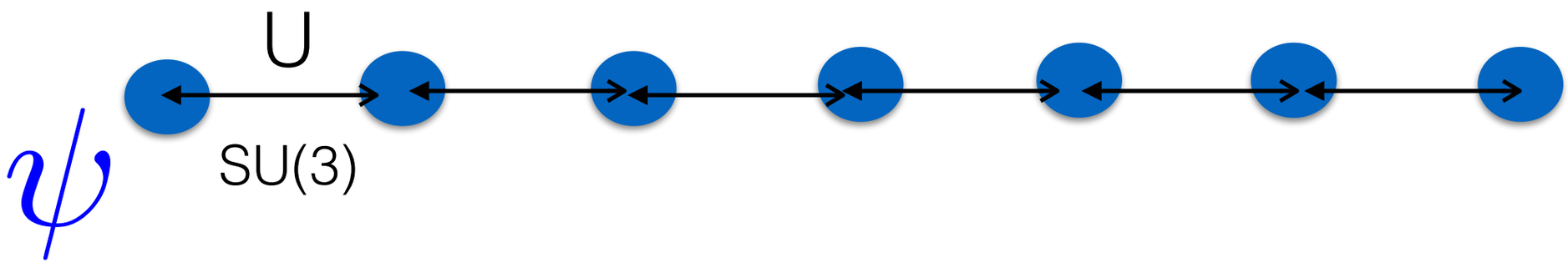
Check CME in QCD

# A simple model : QCD in 1d

One dimensional QCD is a an interesting, exactly solvable model. There is no spontaneous symmetry breaking, but there are baryons. Its partition function is formally the same as the one obtained in 4d QCD at strong coupling, once an explicit mass term is identified with the 4d dynamically generated mass. For any temperature $T$ and varying $\mu$ there is a crossover to a baryon rich phase, which turns into a first order transition at zero temperature.
The partition function reads

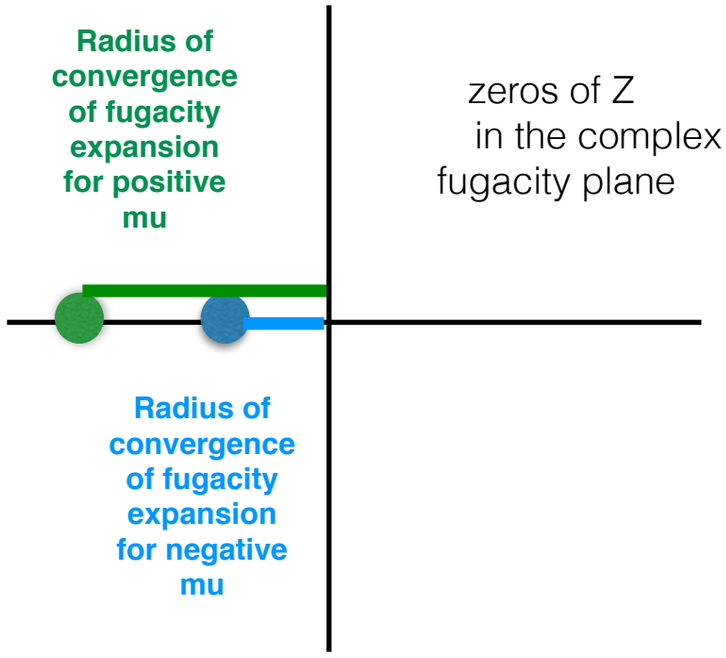
$$\mathcal{Z}(\mu,T) = 2\cosh(\mu/T) + \sinh(4m'/T)/\sinh(m'/T) = \sum_{n=-1,0,1} A_n \xi^n$$

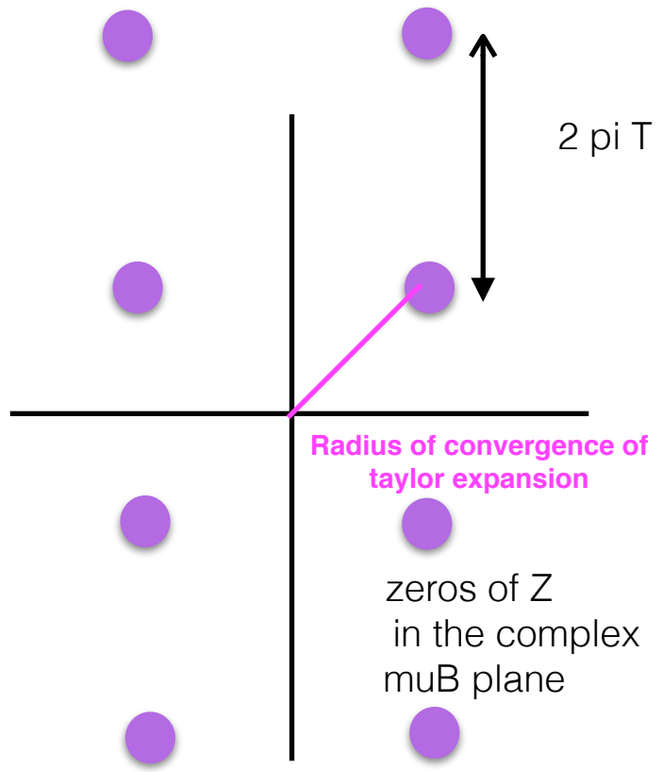
$A_0 = \sinh(4m'/T)/\sinh(m'/T) \equiv A, A_1 = A_{-1} = 1$, $\mu$ is understood to be $\mu_B$, $m' = sh^{-1}m$, with $m$ the quark mass.



# Zeros of the GCPF and radius of convergence in QCD 1d

**Convergence in fugacity**




**Convergence in $\mu/T$**

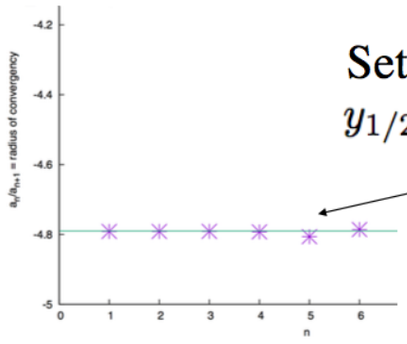
$4 < A < \infty \rightarrow y_1 = 1/y_2$ are two negative real roots.

$$\Re e^{\mu/T} = (-A + / - \sqrt{A^2 - 4})/2 \equiv y_{1/2}$$

$$\mu = \pm T(log|y_1|) + i(2k+1)T\pi.$$

The fugacity series converges for:

$$1/y_1 < e^{\mu/T} < y_1,$$




Setting A=5.

$y_{1/2} = -4.79, -0.208.$

Partial sums reveal the radius of convergence

$$n^n(\mu_I) = \sum_1^n a_k \sin(k\mu_I/T)$$

Analytic continuation

$$n^n(\mu) = \sum_1^n a_k \sinh(\mu/Tk)$$

Increasing odd n

Increasing even n

n=3
n=4
n=5
n=6
n=7
n=8
n=9
exact
convergence limits

$n(\mu)$

$e^{\mu/T}$

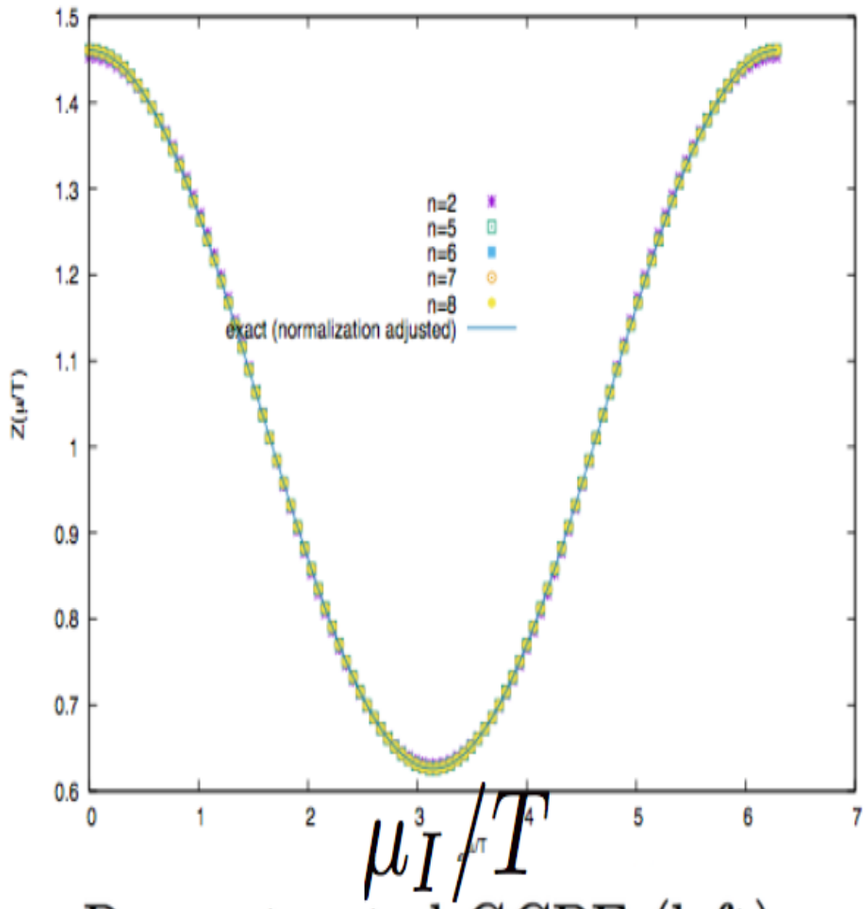
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6

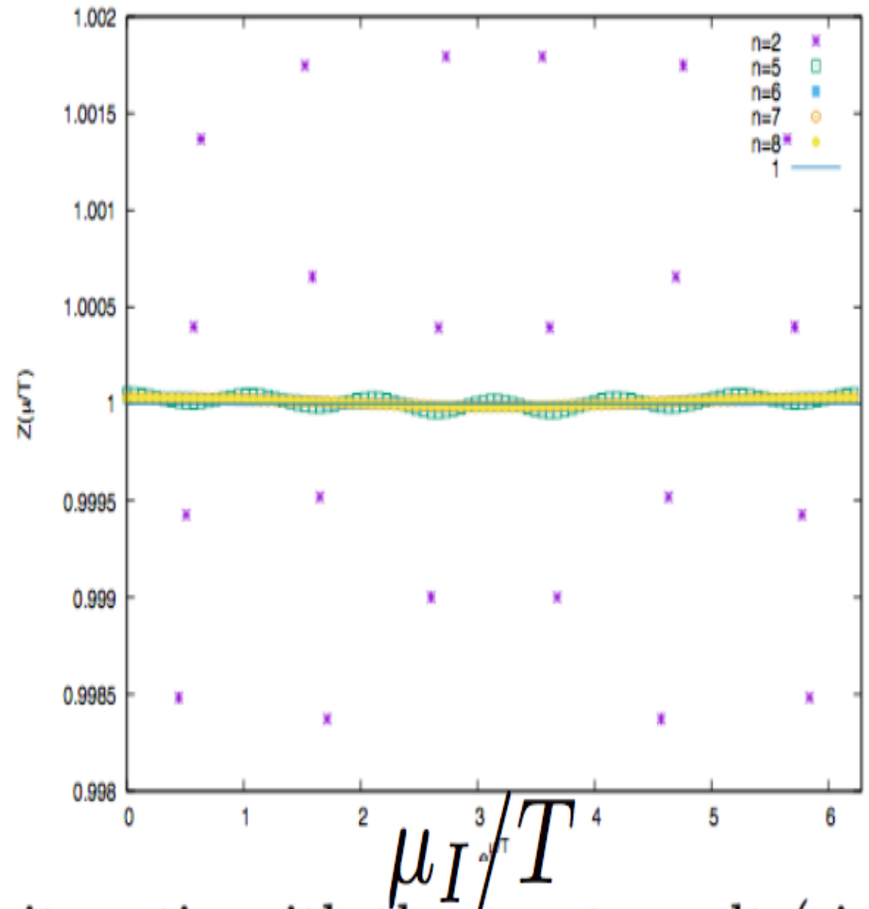
# The GCPF from the particle number: inherits the same radius of convergence

First,

for $\mu_I$

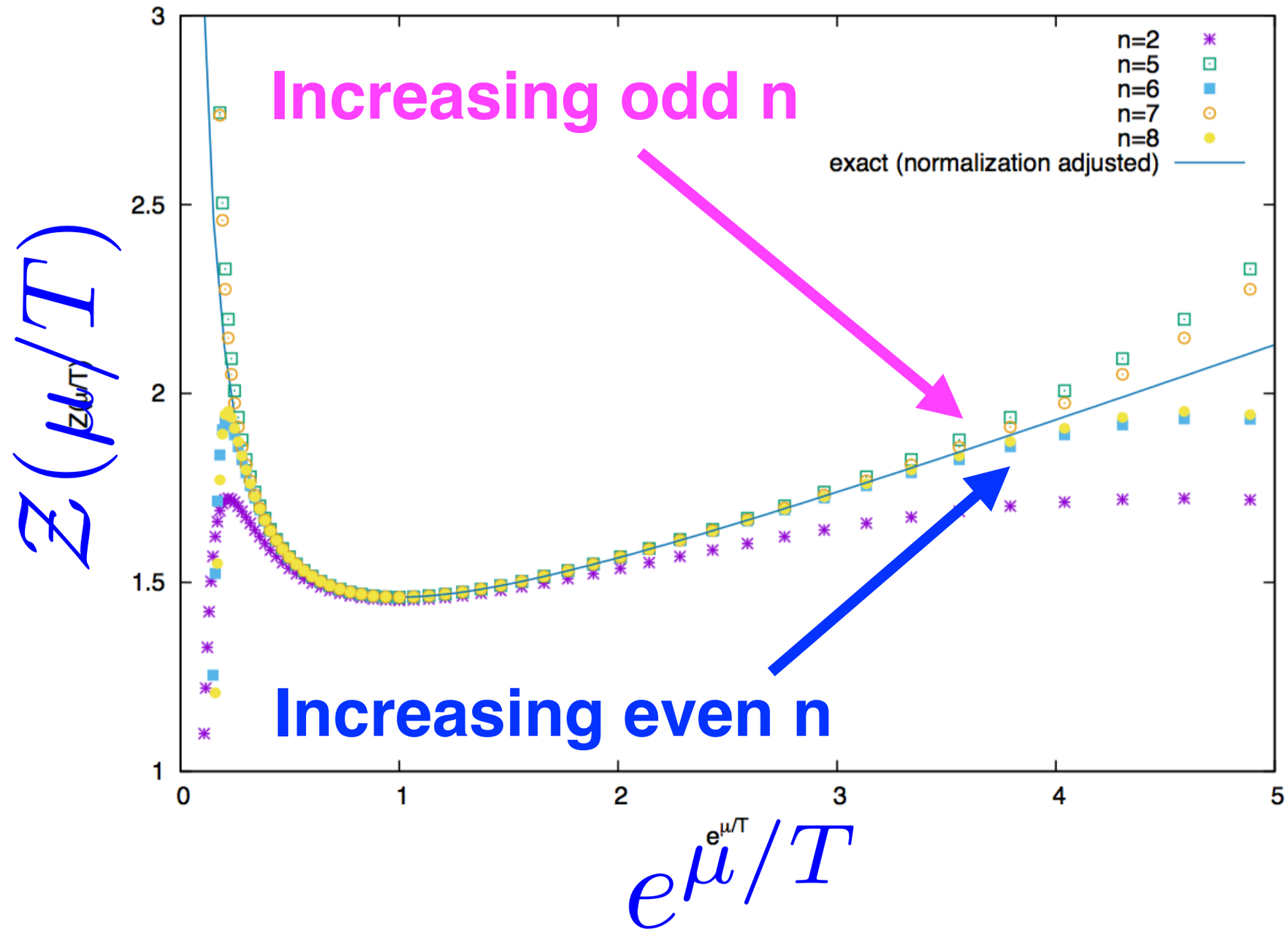
$$\mathcal{Z}^n(\mu_I/T) = e^{-\int_0^{\mu_I/T} n^n(x)\,dx}$$



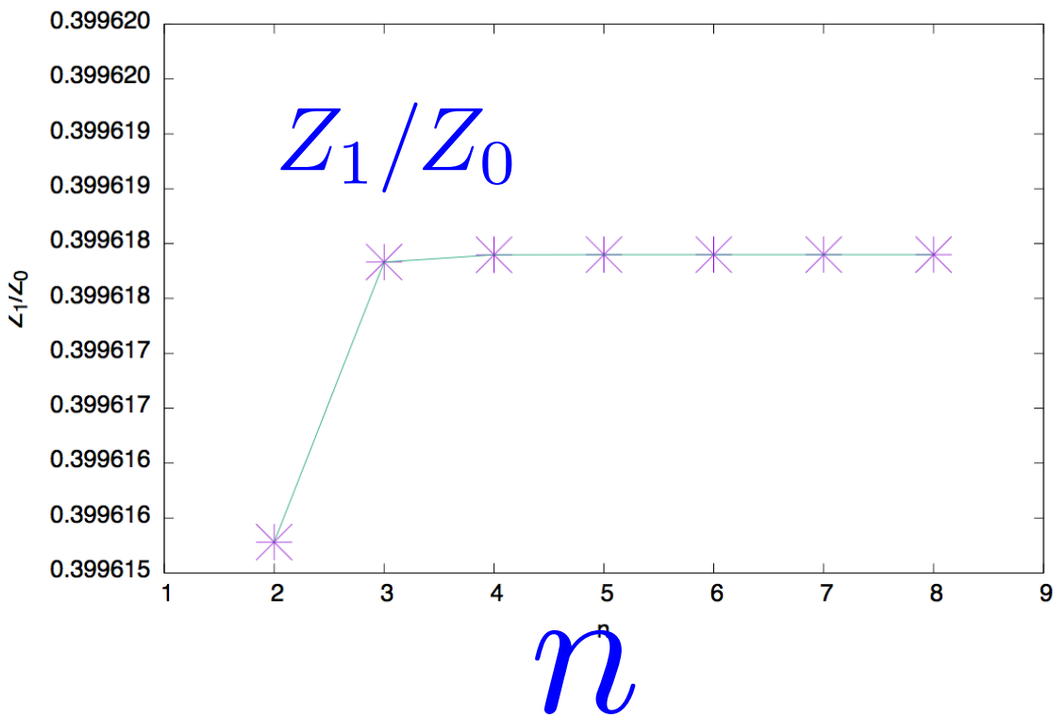



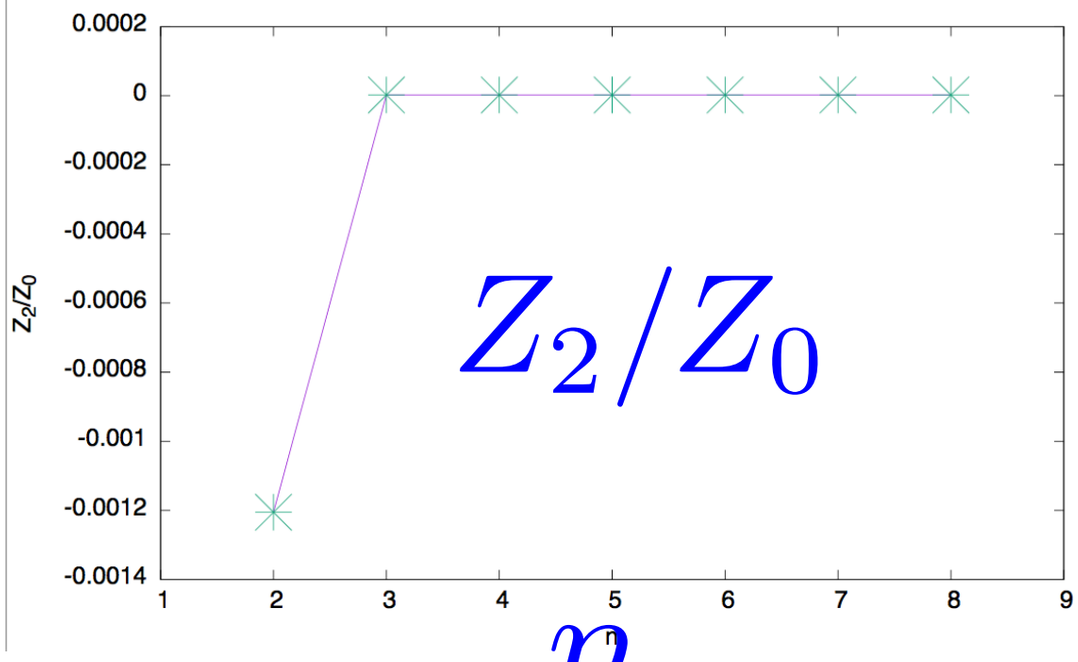

Reconstructed GCPF (left), and its ratio with the exact result (right)

# The GCPF from the particle number: inherits the same radius of convergence



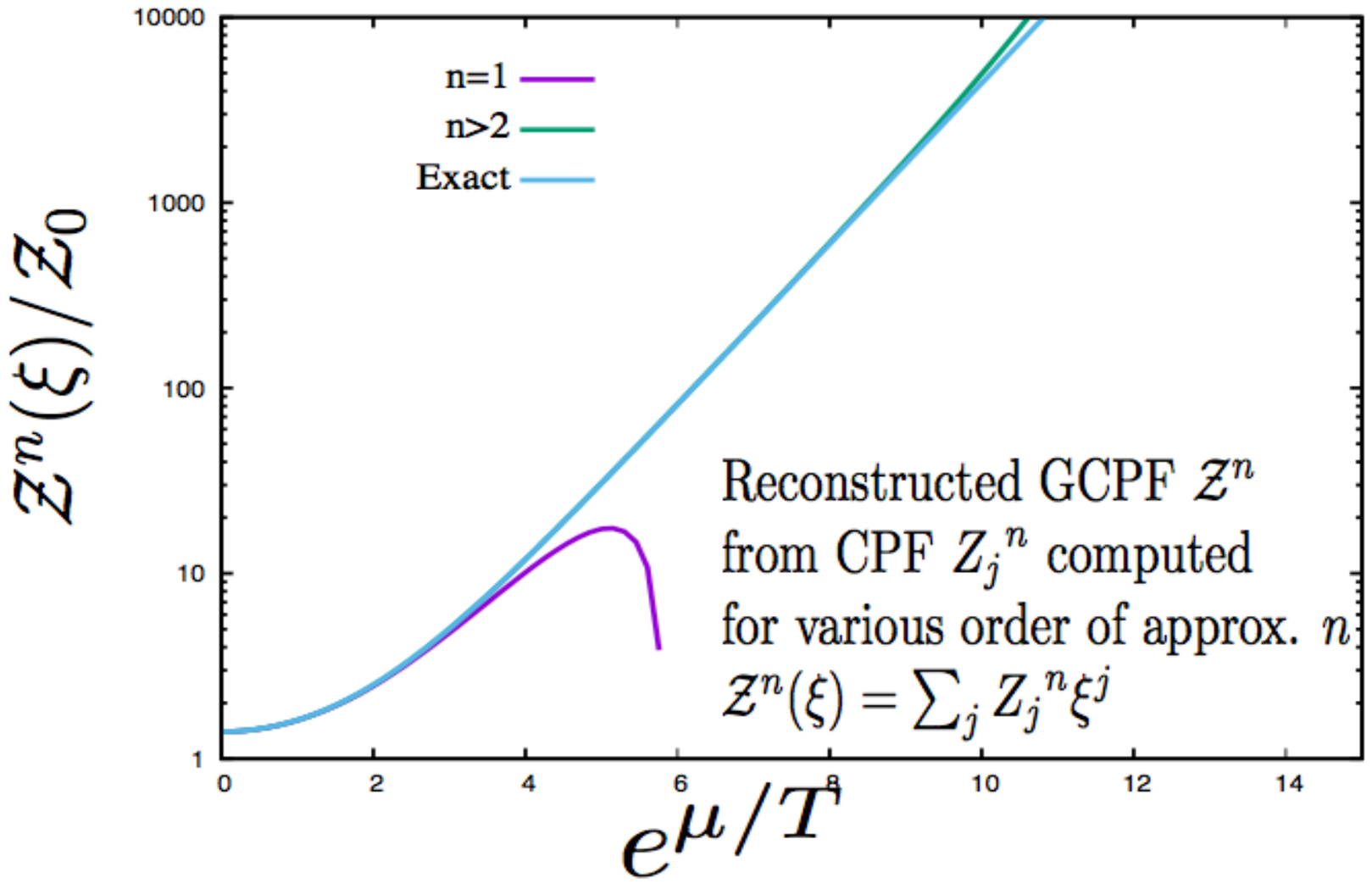
# The GCPF in QCD 1d beyond the radius of convergence




The convergence of the CPF to their exact result:
an insufficient approximation produces a *negative* Z2

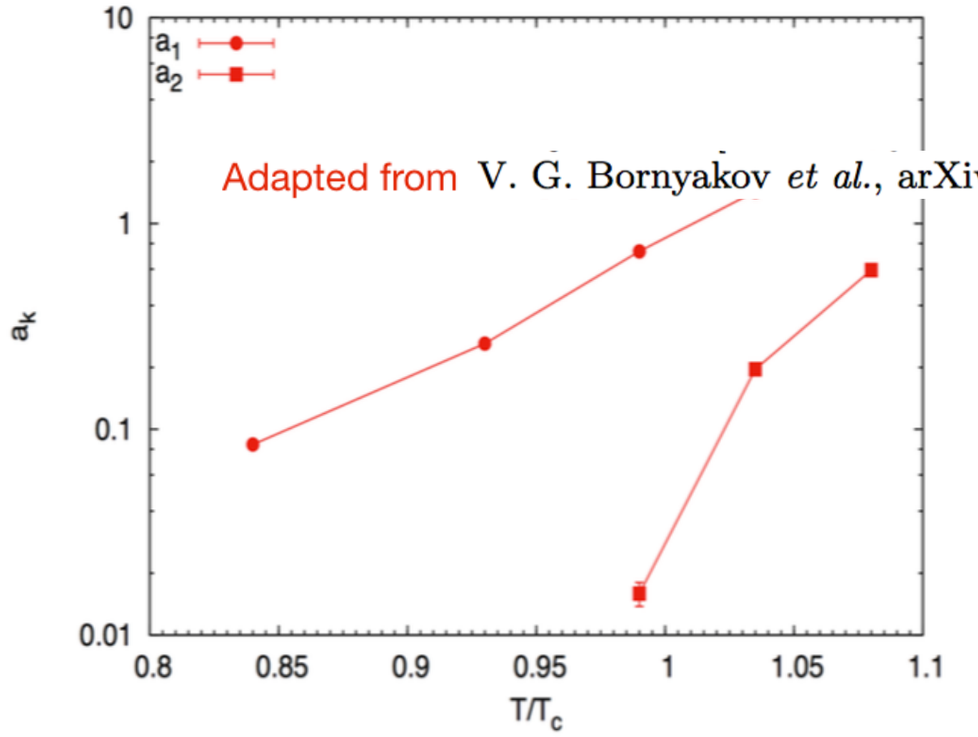
# CPF, Virial Expansion, radius of convergence and GCPF QCD 1d



# QCD - preliminary results .
# Virial expansion, radius of convergence, CEM

We use gauge field configurations generated in $N_f = 2$ lattice QCD with clover improved Wilson quarks and Iwasaki improved gauge field action. More details about the configurations can be found in Ref.[5]

The simulations were made on $16^3 \times 4$ lattices at temperatures $T/T_c = 1.35, 1.20, 1.08$, and $1.035$ in the deconfinement phase and $0.99, 0.93, 0.84$ in the confinement phase along the line of constant physics with $m_\pi/m_\rho = 0.8$. The parameters of the action, including $c_{SW}$ value were borrowed from the WHOT-QCD collaboration paper[6]. We compute the number density on samples of 1800 or 3800 configurations using every 10-th trajectory produced with Hybrid Monte Carlo algorithm.




Adapted from V. G. Bornyakov *et al.*, arXiv:1712.02830

We measure the quark number density $n_q$ at imaginary quark chemical potential $\mu_q = i\mu_{qI}$. It is defined as

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{GC} = \frac{N_f N_t^3}{N_s^3 Z_{GC}}\int \mathcal{D}U e^{-S_G}(\det\Delta(\mu_q))^{N_f}\mathrm{tr}\left[\Delta^{-1}\frac{\partial\Delta}{\partial\mu_q/\mathrm{T}}\right]$$

We then fitted the number density to functions

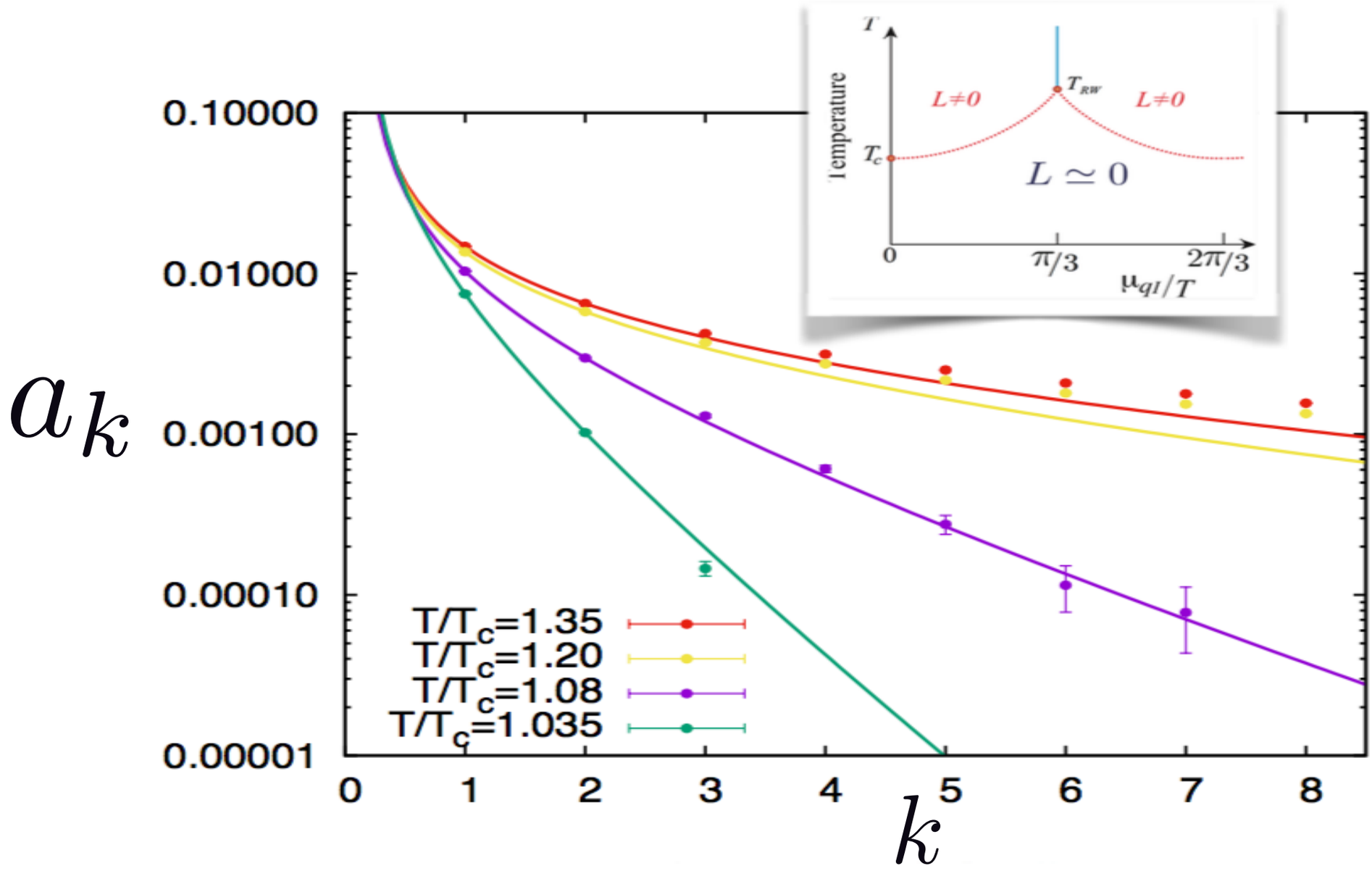
$$n_{qI}(\theta)/T^3 = \sum_{k=1}^{k_{max}} a_k \sin(3k\theta)$$

for $T < T_{RW}$ and

$$n_{qI}(\theta)/T^3 = \sum_{n=1}^{2} c_{2n-1}(\theta^{2n-1})$$

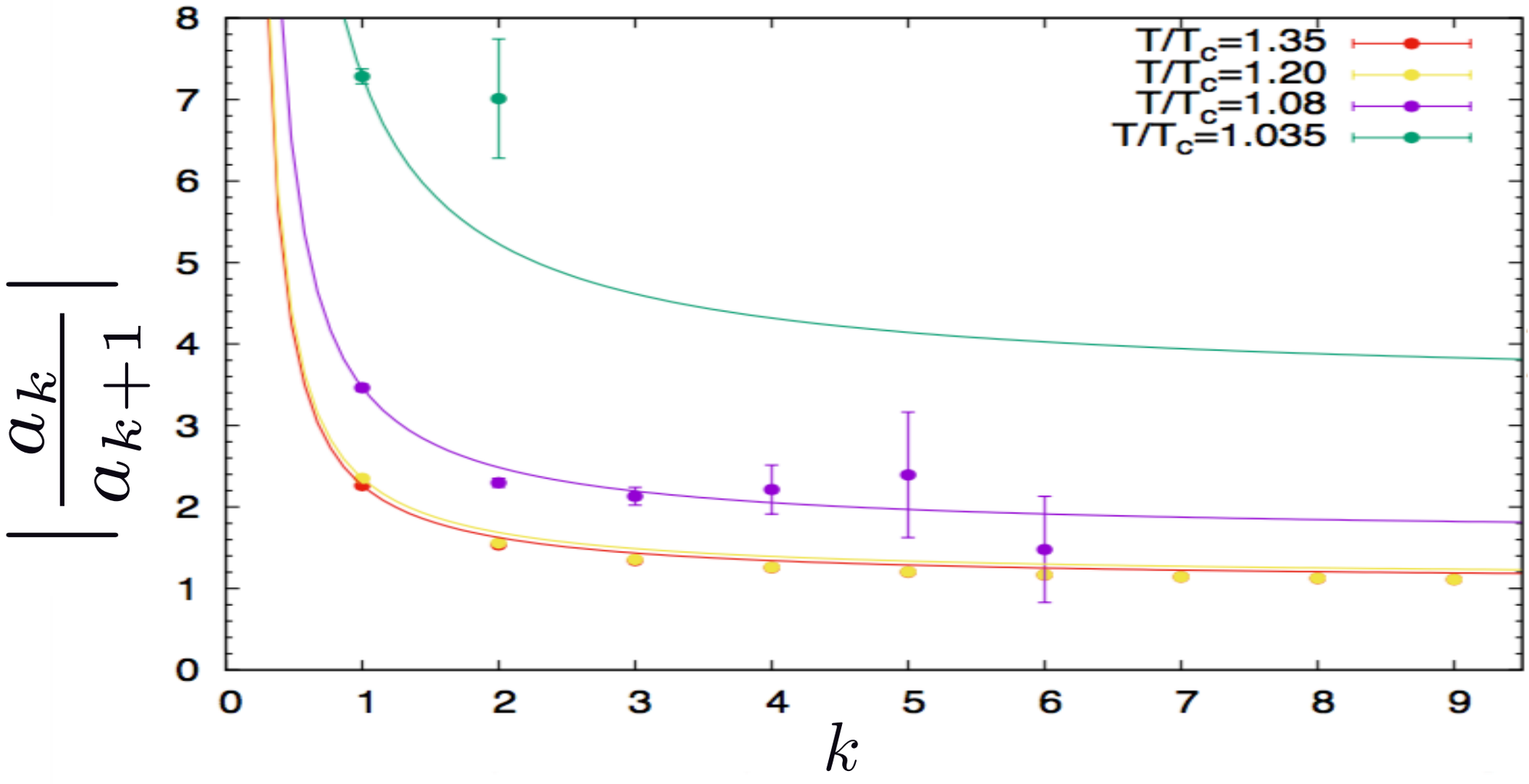
for $T > T_{RW}$ $(\theta = \mu_{qI}/T)$. In this case we used Fourier transform to compute the virial coefficients $a_k$.

# QCD - preliminary results .
# Virial expansion, radius of convergence, CEM



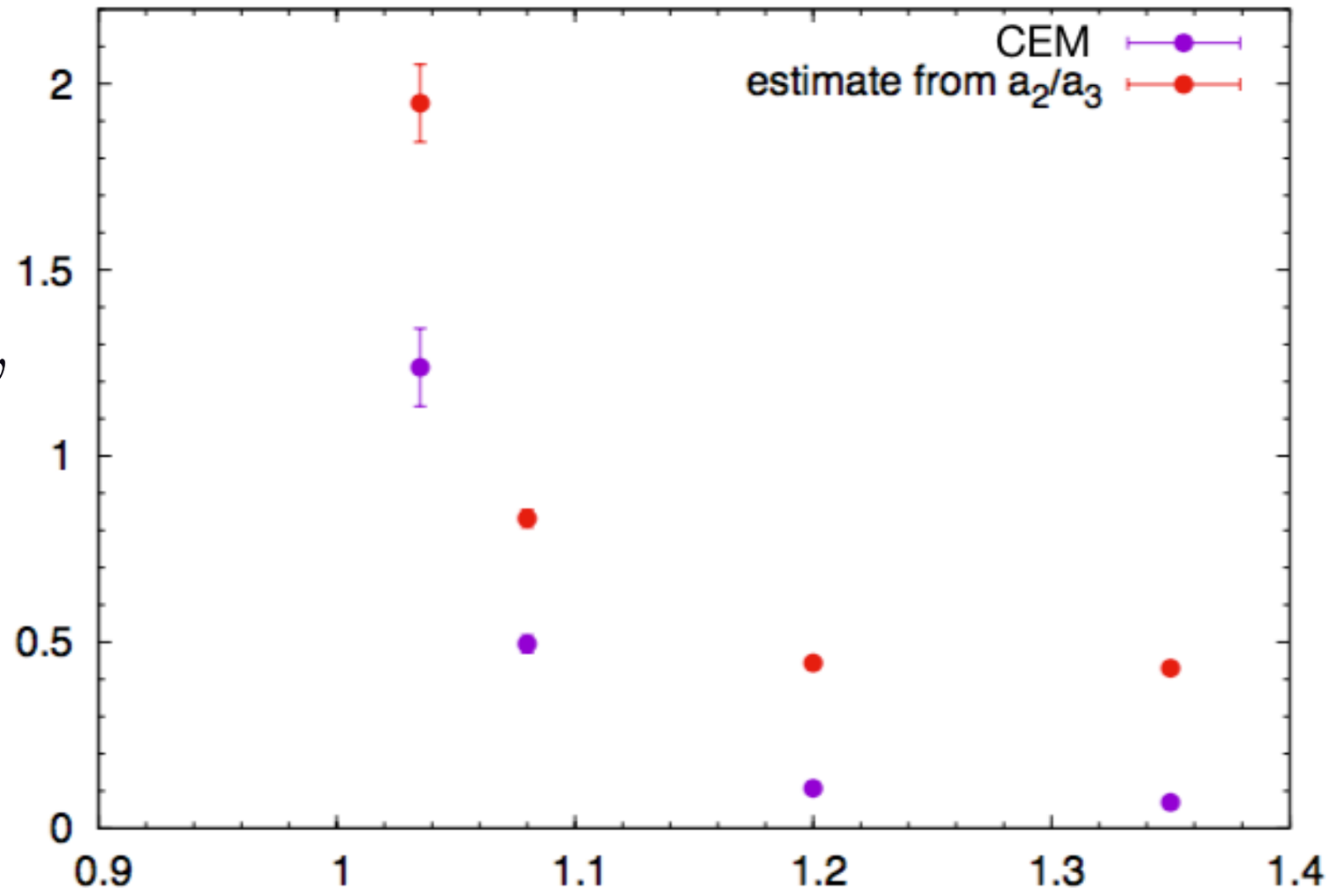

Comparison of $a_k$ (circles) with predictions of CEM (solid curves). There is a good agreement for $T < T_{RW}$. There is no agreement for $T > T_{RW}$.

# QCD - preliminary results .
# Virial expansion, radius of convergence, CEM




Ratio $|a_k/a_{k+1}|$ for four temperature values. Curves show predictions of CEM. There is good agreement for $T/T_c = 1.08$. Note, that for this temperature already $a_2/a_3$ can be taken as a good estimate of the $k \to \infty$ limit.

# QCD - preliminary results .
# Virial expansion, radius of convergence, CEM




Radius of convergence of the virial expansion $(\mu_B/T)_{conv}$ computed using CEM and estimated from the ratio $a_2/a_3$.

# Summary

Results at imaginary chemical potential  offer a simple way to compute the virial expansion.

The series may converge at imaginary chemical potential, but  not at real chemical potential .

However, the canonical partition functions computed at imaginary chempot allow the reconstruction of the GCPF  everywhere.

In the case of 1d QCD, an insufficient approximation may produce negative canonical partition functions.

Negative partition functions have been observed in real QCD and it remains to be seen whether the origin is the same as in QCD1d.

Virial coefficients can be computed within different phenomenological frameworks. The CEM parametrization puts limit on CEP..

Preliminary numerical results suggest discrepancies with CEM in QCD