Latest developments in EPOS

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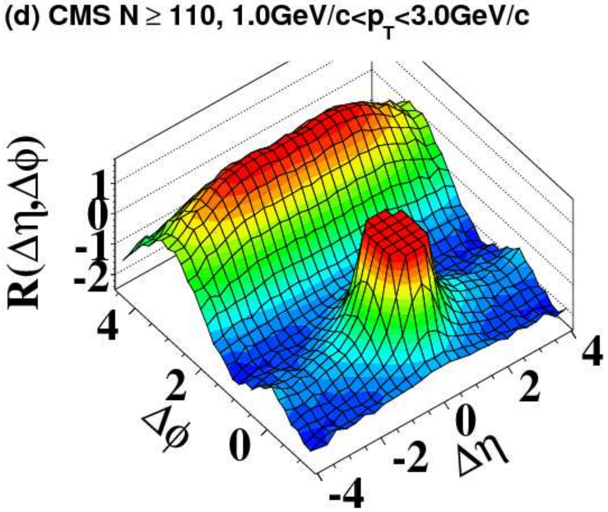
Many HI features observed in pp, so do we observe a QGP (or at least some hydro expansion)?



This talk

- **New trends on the foundations of hydrodynamics**
- **Microcanonical hadronization**

- **A systematic way get the equations of relativistic hydrodynamics is via a formal gradient expansion** (of $\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\nu\lambda} T^{\nu\lambda} + \Gamma^{\nu}_{\nu\lambda} T^{\mu\lambda} = 0$)

- **The hydrodynamic gradient expansion has (probably) a vanishing radius of convergence**

- **Good news: There are tools to deal with that. Need to go beyond perturbative expansions.**

New trends on the foundations of hydrodynamics

- **Resurgence theory => go beyond the case of "small gradients" (close to equilibrium).**
- **Systematic treatment of divergent power series, methods to include exponential corrections ("instantons"). Jean Ecalle (1981)**
- **Applied to hydrodynamics by several authors** (Michal P. Heller, Michal Spalinski, Phys. Rev. Lett. 115, 072501 (2015); Paul Romatschke and Ulrike Romatschke, arXiv:1712.05815; Buchel, Michal P. Heller, Jorge Noronha Phys. Rev. D 94, 106011 (2016))

Truncated conformal Bjorken hydrodyn.

Mueller-Israel-Steward (MIS) approach

(second order + shear stress tensor π and bulk pressure Π dynamical quantities, governed by relaxation equations)

+ imposing scale and boost invariance,

Michal P. Heller, M. Spalinski, Phys. Rev. Lett. 115, 072501 (2015)

$$\tau\dot{\epsilon} = -\frac{4}{3}\epsilon + \phi, \quad \tau_\pi\dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1\phi^2}{2\eta^2} - \frac{4\tau_\pi\phi}{3\tau} - \phi,$$

with $\phi = -\pi_y^y$ shear stress.

Equation considered (per def.) complete (not expansion), but one is investigating perturbative solutions.

With $\epsilon = T^4$, $\tau_\pi = C_{\tau\pi}/T$, $\lambda_1 = C_{\lambda_1}\eta/T$, $\eta = C_\eta s$, defining w and f as

$$w(\tau) = \tau T, \quad f(w) = \tau\frac{\dot{w}}{w},$$

=> diff. equation (DE) for $f(w)$

$$\boxed{\begin{gathered} C_{\tau\pi} w f f' + 4C_{\tau\pi} f^2 + \left(w - \frac{16C_{\tau\pi}}{3}\right) f \\ -\frac{4C_\eta}{9} + \frac{16C_{\tau\pi}}{9} - \frac{2w}{3} = 0. \end{gathered}}$$

$w = \tau^{2/3}$ for ideal hydro.

Perturbative solution: series in powers of w^{-1}

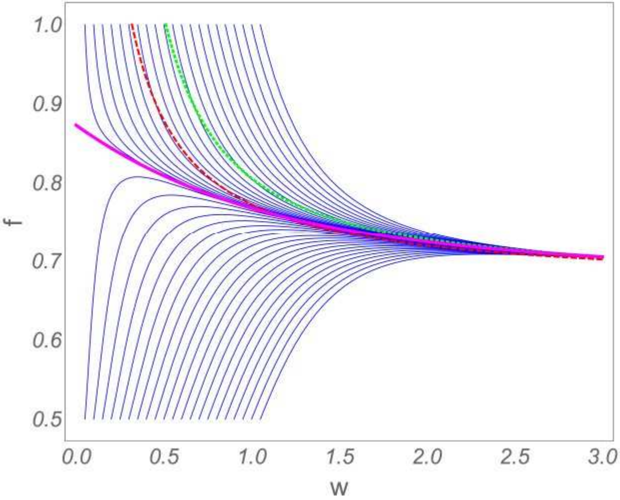
$$f = \sum_{n=0}^{\infty} a_n w^{-n},$$

called **hydrodynamical expansion** for large w (large times), coefficients obtained from DE:

$$a_n \sim n!$$

so the series is divergent.

Solving the equation numerically => **attractor**



Picture from Heller, M. Spalinski.

well defined solutions even at small w (small times),

contrary to the perturbative expansion.

=> well defined solutions "far off equilibrium"

Resummation

(a very systematic approch for divergent series)

$$f = \sum_{n=0}^{\infty} a_n w^{-n},$$

(computed up to $n = N = 200$) is **Borel transformed**

$$f_B(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} B_n x^n,$$

has a finite radius of convergence.

The **inverse Borel transform** is

$$f_{iB}(w) = w \int_0^\infty f_B(x) e^{-wx} dx.$$

Analytic continuation of f_B via **Padé approximants** having a sequence of singularities

$$f_{PB}(x) = h_0(x) + (a-x)^\gamma h_1(x) + (2a-x)^{2\gamma} h_1(x) + \ldots$$

These branch-cut singularities

=> **ambiguities** (for large w) of the form

$$\boldsymbol{w^{-m\gamma} e^{-maw}}$$

This ambiguity = feature of the hydrodynamic series

indication of physics outside the grad expansion.

The solution should have the form of a trans-series

$$f(w) = \sum_{m=0}^{\infty} c^m \, \boldsymbol{w^{-m\gamma} e^{-maw}} \, f_m(w)$$

with perturbative series f_m,
get coeficients by substituting the trans-series into the DE, then same procedure

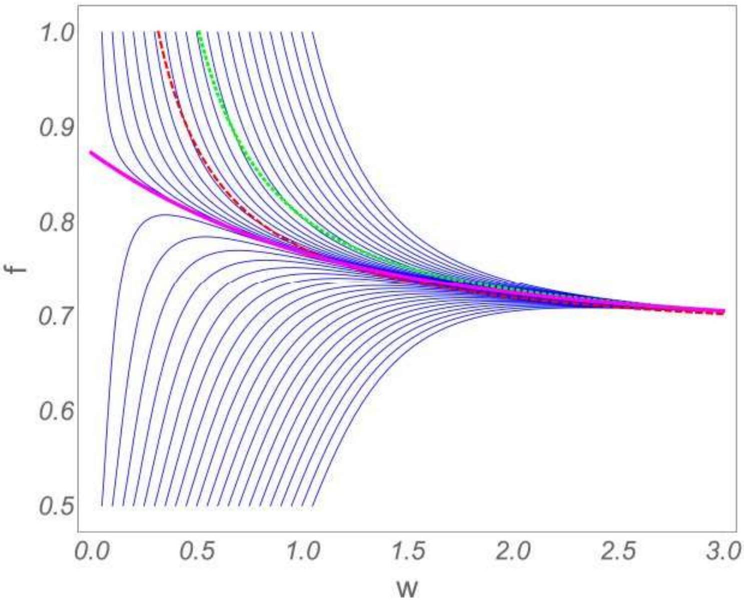
=> **unique result called "resummation result"**

One finds (on percent level):

Resummed result

= Hydrodynamical attractor

both being in general quite different compared to the perturbative expansions


1.0
0.9
0.8
0.7
0.6
0.5
f
0.0
0.5
1.0
1.5
2.0
2.5
3.0
w

Conclusion (part 1)

- **Hydro applicable even far off equilibrium (in particular relevant for small systems)**

- **=> True solution : Hydrodynamic attractor**
  Accessible (in principle) via resummation

- **Frequently asked question:**
  "Why do small systems thermalize so quickly"
  can be anwered: They don´t

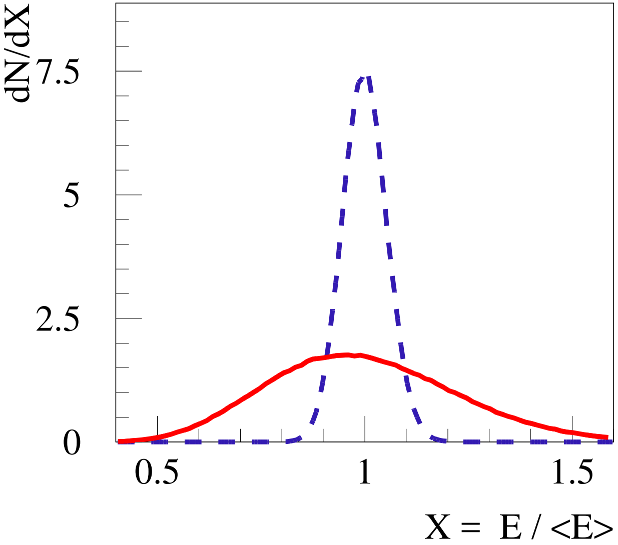
Microcanonical hadronization

- **No need to match dynamical part**

- **Energy and flavor conservation for small systems**

Grand canonical decay, T = 130 MeV

V=50 fm^3; V=1000 fm^3



Microcanonical hadronization in EPOS

(very preliminary)

Hadronization hyper-surface $x^\mu(\tau, \varphi, \eta)$:

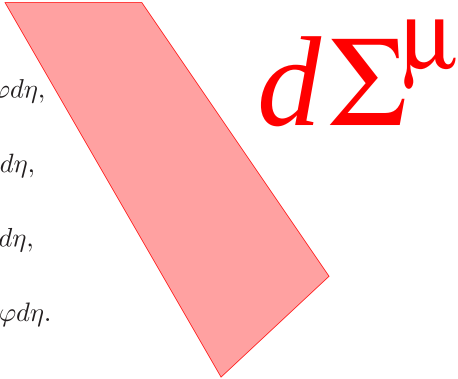
$$\boxed{\begin{aligned} x^0 &= \tau \cosh \eta, \\ x^1 &= r \cos \varphi, \\ x^2 &= r \sin \varphi, \\ x^3 &= \tau \sinh \eta \end{aligned}}$$

with $r = r(\tau, \varphi, \eta)$, representing the **FO condition**.

Hypersurface element:

$$\boldsymbol{d\Sigma_\mu = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\kappa}{\partial\varphi}\frac{\partial x^\lambda}{\partial\eta}\,d\tau\,d\varphi\,d\eta}.$$

$$\begin{aligned}
d\Sigma_0 &= \left\{-r\frac{\partial r}{\partial\tau}\tau\cosh\eta + r\frac{\partial r}{\partial\eta}\sinh\eta\right\}d\tau d\varphi d\eta,\\
d\Sigma_1 &= \left\{\frac{\partial r}{\partial\varphi}\tau\sin\varphi + r\,\tau\cos\varphi\right\}d\tau d\varphi d\eta,\\
d\Sigma_2 &= \left\{-\frac{\partial r}{\partial\varphi}\tau\cos\varphi + r\,\tau\sin\varphi\right\}d\tau d\varphi d\eta,\\
d\Sigma_3 &= \left\{r\frac{\partial r}{\partial\tau}\tau\sinh\eta - r\frac{\partial r}{\partial\eta}\cosh\eta\right\}d\tau d\varphi d\eta.
\end{aligned}$$

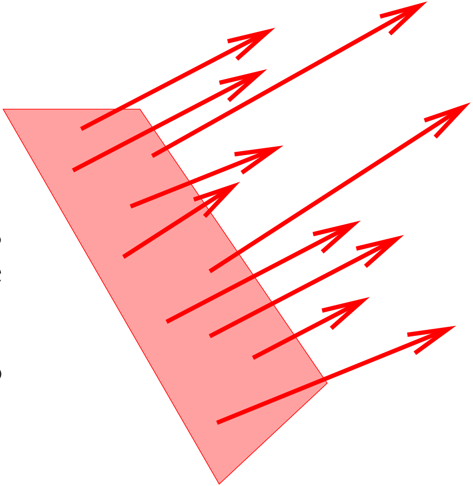


GC particle production via Cooper-Frye

$$E\frac{dn}{d^3p} = \int d\Sigma_\mu p^\mu f(up),$$

assuming that “matter” is a thermalized resonance gas

(adding δf for viscous hydro, close to equilibrium)



More general:

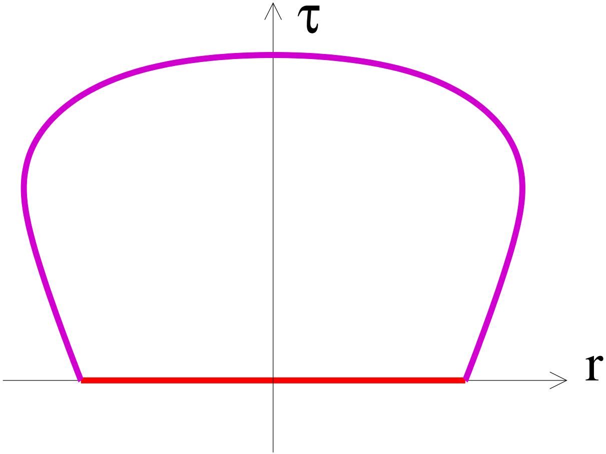
Flow of momentum vector dP^{μ} and conserved charges dQ_A through the surface element:

$$
\begin{aligned}
dP^{\mu} &= T^{\mu\nu} d\Sigma_{\nu}, \\
dQ_A &= J^{\nu}_{A} d\Sigma_{\nu}.
\end{aligned}
$$

Momentum and charges are conserved :

$$\int_{\Sigma_{\mathrm{FO}}} dP^{\mu} = P^{\mu}_{\mathrm{ini}},$$

$$\int_{\Sigma_{\mathrm{FO}}} dQ_{A} = Q_{A\,\mathrm{ini}}$$



Construct an **effective mass** by summing surface elements:

$$M = \int_{\text{surface area}} dM,$$

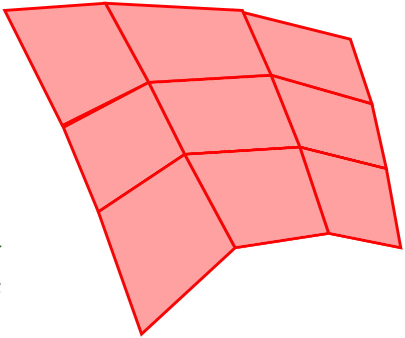
with

$$dM = \sqrt{dP^\mu dP_\mu},$$

knowing for each element four-velocity and volume element

$$U^\mu = dP^\mu / dM,$$

$$dV = u^\mu d\Sigma_\mu.$$



The four-velocity U^μ is NOT equal to the fluid velocity u^μ! (Only in case of zero pressure)

These effective masses we **decay microcanonically**:

$$dP = C_{\mathrm{vol}}\, C_{\mathrm{deg}}\, C_{\mathrm{ident}} \times \delta(E - \Sigma E_i)\, \delta(\Sigma \vec{p}_i) \prod_A \delta_{Q_A, \Sigma q_{A\,i}} \prod_{i=1}^{n} d^3 p_i,$$

$$C_{\mathrm{vol}} = \frac{V^n}{(2\pi\hbar)^{3n}}\,, \quad C_{\mathrm{deg}} = \prod_{i=1}^{n} g_i\,, \quad C_{\mathrm{ident}} = \prod_{\alpha\in\mathcal{S}} \frac{1}{n_\alpha!}\,,$$

(n_α is the number of particles of species α, $\mathcal{S}$ is the set of particle species)

then **boost the particles** according to velocities U^μ.

Microcanonical decay

$$dP \propto d\Phi_{\mathrm{NRPS}} = \delta(M - \Sigma E_i)\,\delta(\Sigma \vec{p}_i)\,\prod_{i=1}^{n} d^3p_i$$

- □ Hagedorn 1958 methods to compute Φ_{NRPS}
- □ Lorentz invariant phase space (LIPS) (James 1968)
- □ Hagedorn methods used for decaying QGP droplets (Werner, Aichelin, 1994, Becattini 2003)
- □ 2012 (Bignamini,Becattini,Piccinini) compute Φ_{NRPS} via the Lorentz invariant phase space (LIPS)

- **Hagedorn integral method can be made very efficient at large n,** but becomes VERY time consuming at small n

- **LIPS method very fast for small n**, gets time consuming at large n

- **around** $n \approx 30 - 40$ **both methods work (=> checks)**

Hagedorn integral method, optimized

The phase-space integral:

$$\phi_{\mathrm{NRPS}}(M, m_1, \ldots, m_n)$$
$$= (4\pi)^n \int \prod_{i=1}^{n} p_i^2 \, \delta(E - \sum_{i=1}^{n} E_i) \, W(p_1, \ldots, p_n) \prod_{i=1}^{n} dp_i,$$

with the "random walk function" W given as

$$W(p_1, \ldots, p_n) := \frac{1}{(4\pi)^n} \int \delta(\sum_{i=1}^{n} p_i \times \frac{\vec{p_i}}{p_i}) \prod_{i=1}^{n} d\Omega$$

We obtain (Werner, Aichelin 94)

$$\boldsymbol{\phi(M, m_1, \ldots, m_n) = \int_0^1 dr_1 \ldots \int_0^1 dr_{n-1} \psi(r_1, ..., r_{n-1})}$$

$$\boldsymbol{\psi = \frac{(4\pi)^n \, T^{n-1}}{(n-1)!} \prod_{i=1}^{n} p_i \, E_i \, W(p_1, \ldots, p_n),}$$

with $z_i = r_i^{1/i}$, $x_i = z_i x_{i+1}$, $s_i = x_i T$, $t_i = s_i - s_{i-1}$,
$E_i = t_i + m_i$, $T = M - \sum_{i=1}^n m_i$

Suitable for MC

The random walk function may be written as

$$W(p_1, \ldots, p_n) = \frac{1}{(4\pi)^n} \frac{1}{(2\pi)^3} \int \int e^{-i\vec{\lambda}\Sigma p_j \hat{p}_j} \prod_{j=1}^{n} d\Omega_j \, d^3\lambda,$$

which gives $W = \int_0^\infty F(\lambda)\, d\lambda$ with

$$F(\lambda) = \frac{1}{2\pi^2} \lambda^2 \prod_{j=1}^{n} \frac{\sin p_j \lambda}{p_j \lambda}.$$

For small λ :

$$\prod_{j=1}^{n} \frac{\sin p_j \lambda}{p_j \lambda} \approx \exp\left(-P^2 \lambda^2\right), \qquad P = \sqrt{\frac{1}{6} \sum_{j=1}^{n} p_j^2}$$

Approximation is stricly true for small λ, but for large n it provides a good approximation over the whole range of λ

=> estimate $W \approx \left(4\pi P^2\right)^{-3/2}$

In order to get more precise results, we define

$$F_0(\lambda) = F(\lambda) \times \exp\left(P^2\lambda^2\right),$$

with F_0/λ^2 being a slowly varying function of λ.

This allows to use the Gauss-Hermite formula

$$W = \frac{1}{P}\int_0^\infty F_0\left(\frac{x}{P}\right) \times \exp\left(-x^2\right)\, dx$$

$$\approx \frac{1}{P}\sum_{k=1}^{K} w_j^{GH} F_0\left(\frac{x_j^{GH}}{P}\right),$$

with Gauss-Hermite nodes and weights x_j^{GH} **and** w_j^{GH} **found in text books.**

With only six nodes we get excellent results.

Sampling via Markov chains

To generate $K = \{h_1, \ldots, h_n; r_1, \ldots r_m\}$ ($m = 3n - 1$ or $m = 3n - 4$) according to $\Omega(K)$, consider random configurations

$$K_0\,,\; K_1\,,\; K_2, \ldots$$

with Ω_t being the law for K_t. Per def

$$\Omega_{t+1}(B) = \sum_A \Omega_t(A)\, p(A \to B)$$

Convergence in case of detailled balance:

$$\Omega(A)\, p(A \to B) = \Omega(B)\, p(B \to A)$$

Use

$$\boldsymbol{p(A \to B) = w_{AB} \;\times\; u_{AB}\;,}$$

with a so-called proposal matrix $\boldsymbol{w}$ and an acceptance matrix $\boldsymbol{u}$. Detailed balance now reads

$$\frac{u_{AB}}{u_{BA}} = \frac{\Omega_B}{\Omega_A}\frac{w_{BA}}{w_{AB}}\;,$$

which is fulfilled for

$$\boldsymbol{u_{AB} = \min\left(\frac{\Omega_B}{\Omega_A}\frac{w_{BA}}{w_{AB}}, 1\right)}$$

(more generally using some function F fulfilling $F(z)\,/\,F(z^{-1}) = z$)

Grand canonical limit

For very large M we should recover the "grand canonical limit" for single particle spectra:

$$f_k = \frac{g_k V}{(2\pi\hbar)^3} \exp\left(-\frac{E_k}{T}\right),$$

The average energy is

$$\bar{E} = \frac{g_k V}{(2\pi\hbar)^3} \sum_k \int_0^\infty E_k \exp\left(-\frac{E_k}{T}\right) 4\pi p^2 dp$$

Changing variables via $E_k dE_k = p dp$, and using $K_1(z) = z \int_1^\infty \exp(-zx)\sqrt{x^2-1}dx$, and $3\,K_2(z) = z^2 \int_1^\infty \exp(-zx)\sqrt{x^2-1}^3 dx$,

=>

$$\bar{E} = \frac{4\pi g_k V}{(2\pi\hbar)^3} m^2 T \left(3TK_2(\frac{m}{T}) + mK_1(\frac{m}{T}) \right).$$

The microcanonical decay of an object of mass M and volume V should converge (for $M \to \infty$) to the GC single particle spectra

with T obtained from $M = \bar{E}$.

First:

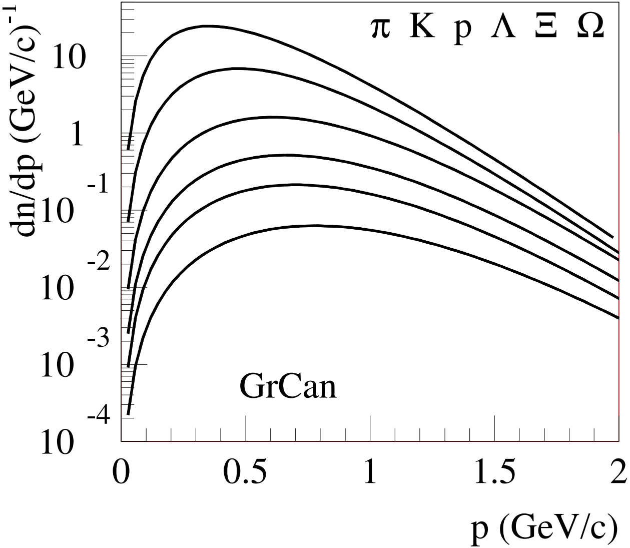
Pseudoparticles

- **Normal hadron masses**

- **no flavor**

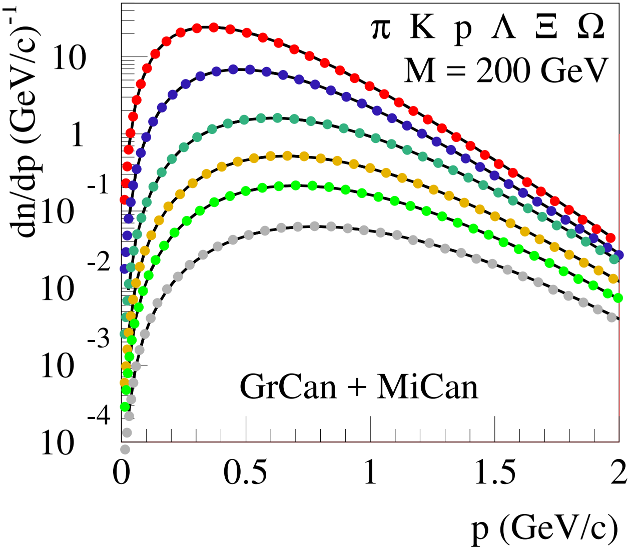
Check effect of energy conservation

GC decay, E/V= 0.333 GeV/fm^3 T=164 MeV



V=150 fm^3

GC+MiC decay, E/V= 0.333 GeV/fm^3 M=200 GeV

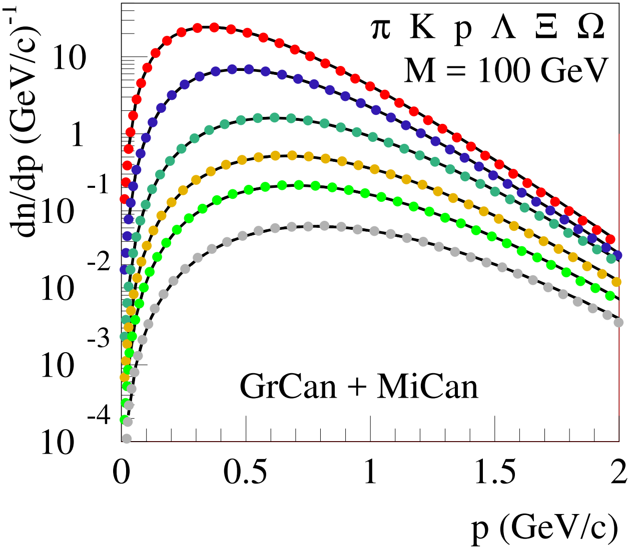


V=600 fm^3

$\times \frac{1}{4}$

good test for
Metropolis proposal

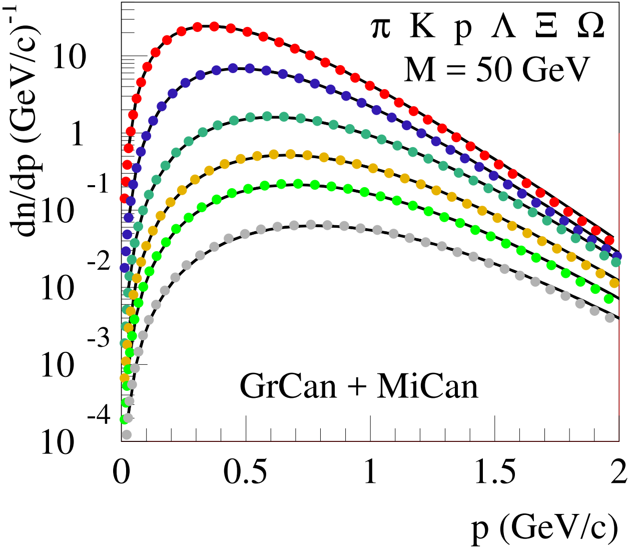
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=100 GeV



V=300 fm^3

$\times \frac{1}{2}$

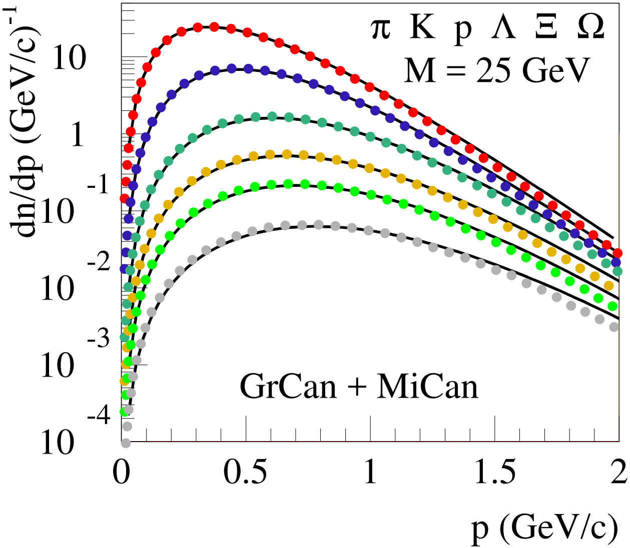
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=50 GeV



V=150 fm^3

×1

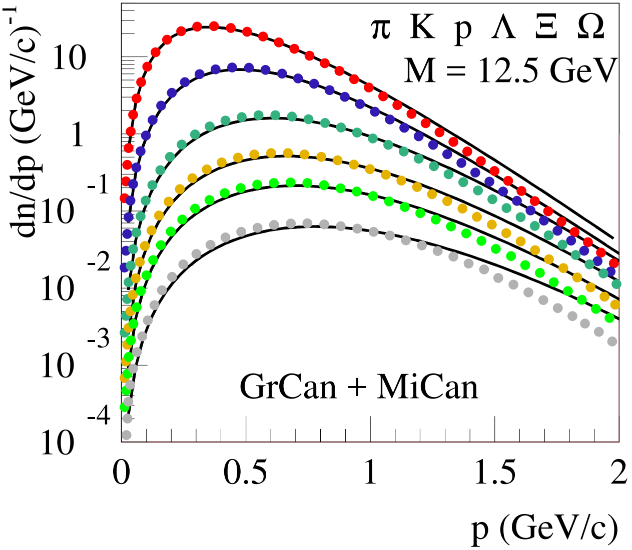
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=25 GeV



V=75 fm^3

$\times 2$

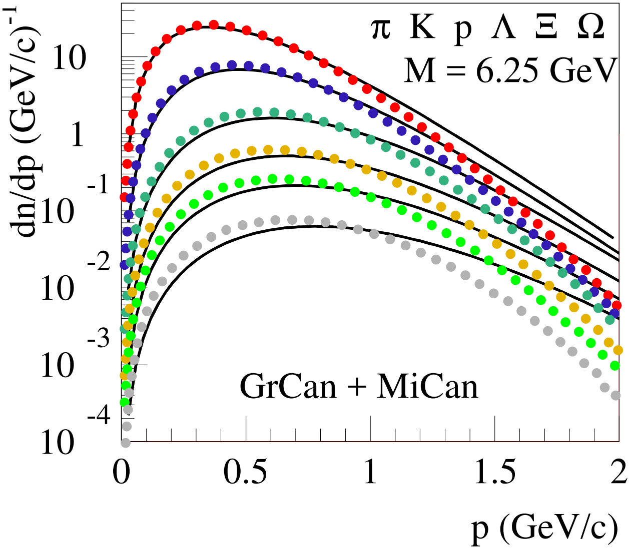
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=12.5 GeV



V=37.5 fm^3

×4

GC+MiC decay, E/V= 0.333 GeV/fm^3 M=6.25 GeV



V=18.75 fm^3

×8

Now:

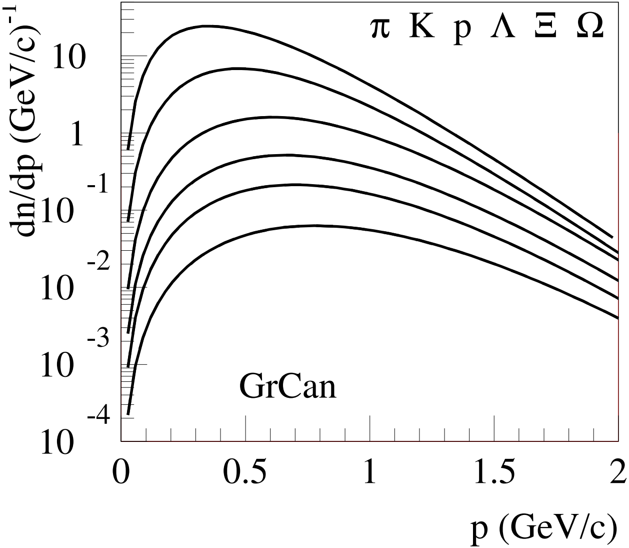
Normal particles

- **Normal hadron masses**

- **Normal flavor content**

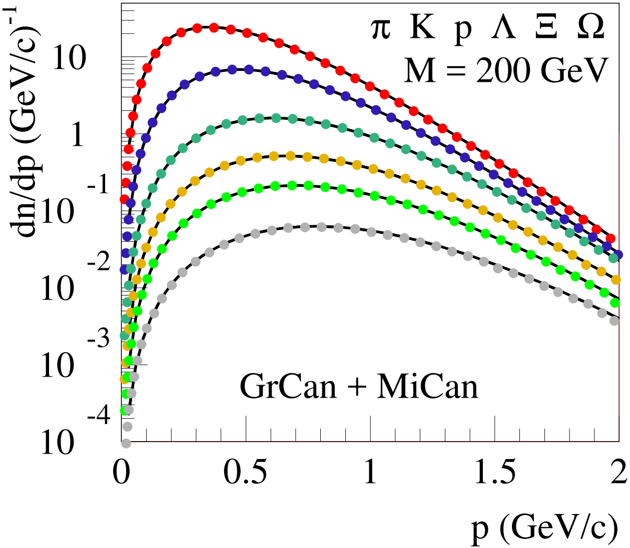
Check effect of energy + flavor conservation

GC decay, E/V= 0.333 GeV/fm^3 T=164 MeV



V=150 fm^3

GC+MiC decay, E/V= 0.333 GeV/fm^3 M=200 GeV

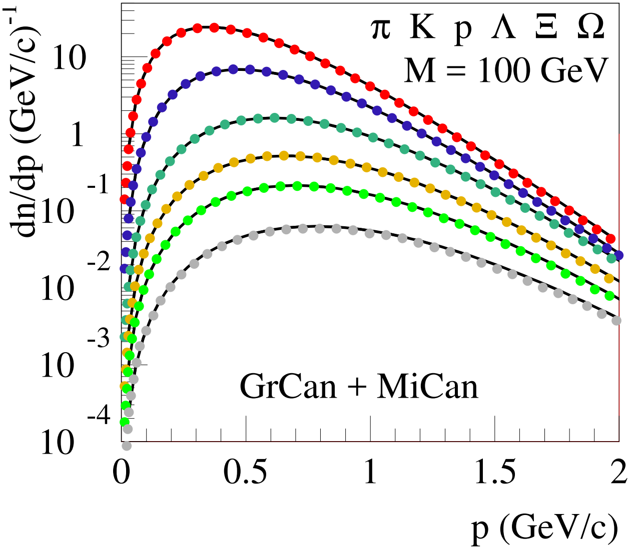


V=600 fm^3

$\times\frac{1}{4}$

good test for
Metropolis proposal

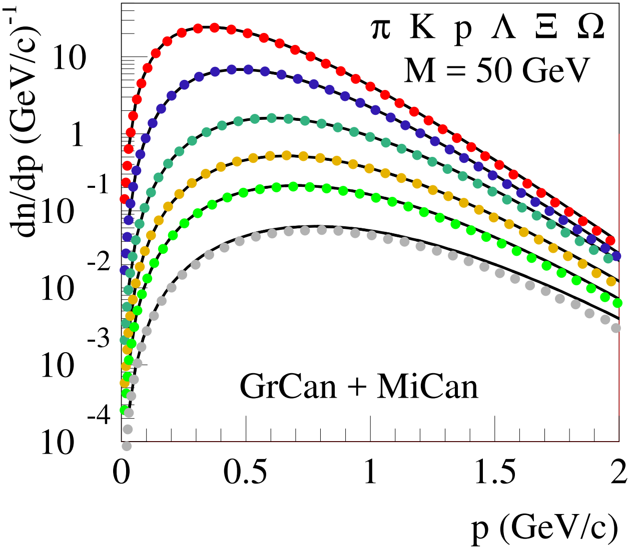
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=100 GeV



V=300 fm^3

$\times \frac{1}{2}$

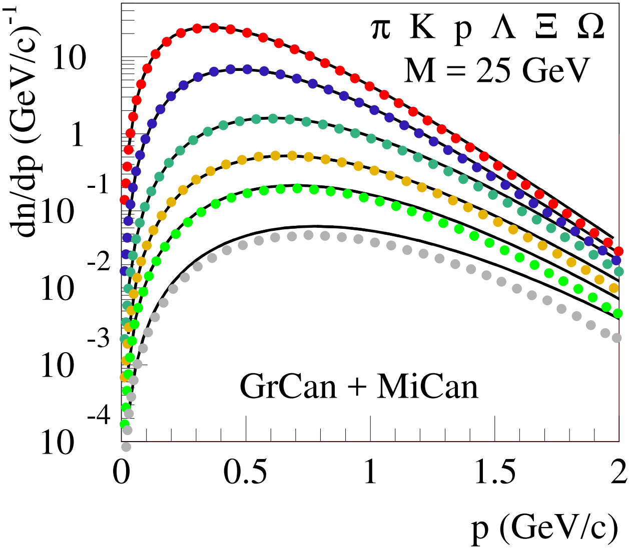
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=50 GeV



V=150 fm^3

×1

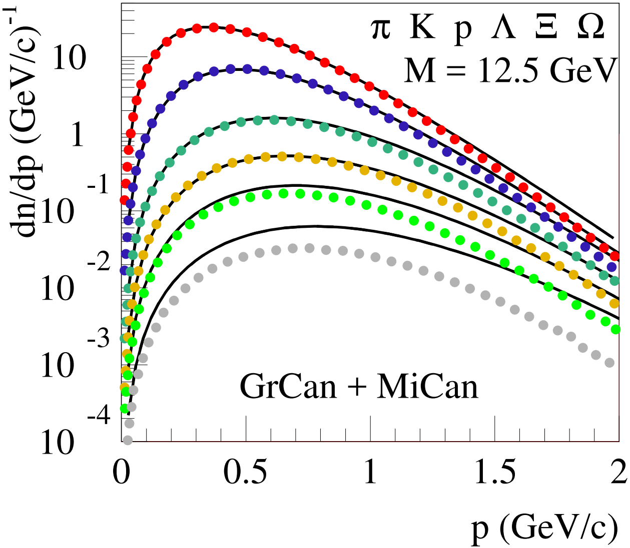
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=25 GeV



V=75 fm^3

$\times 2$

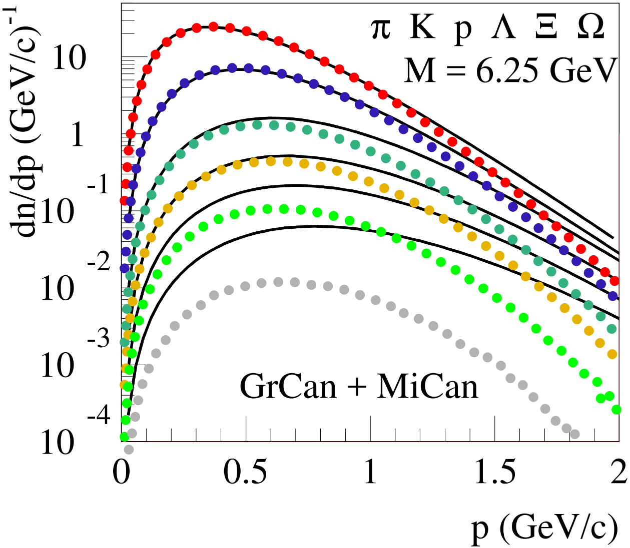
GC+MiC decay, E/V= 0.333 GeV/fm^3 M=12.5 GeV



V=37.5 fm^3

×4

GC+MiC decay, E/V= 0.333 GeV/fm^3 M=6.25 GeV



V=18.75 fm^3

×8

Status on microcanonical hadronization:

- **Reliable and fast methods, even for large systems**
- **Todo:**
  - **larger hadron set (54 hadrons presently)**
  - **Implementation to do plasma hadronization**

Thank you

Mueller-Israel-Steward (MIS) approach

(second order + π and Π dynamical quantities, governed by relaxation equations)

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\nu\lambda} T^{\nu\lambda} + \Gamma^\nu_{\nu\lambda} T^{\mu\lambda} = 0 \tag{1}$$

with the Christoffel symbols defined as $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$. The energy-momentum tensor may be expressed via a systematic expansion in terms of gradients (of $\ln\varepsilon$ and u):

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + ..., \tag{2}$$

with the “equilibrium term” $T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu}$, where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projector orthogonal to u^μ. One usually writes

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + -\Pi\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{3}$$

(shear stress tensor, bulk pressure). Mueller-Israel-Steward (MIS) theory: Promote π and Π to dynamical quantities, governed by relaxation equations. Details concerning second order expressions see Paul Romatschke and Ulrike Romatschke, arXiv:1712.05815.

MIS approach (Yuri Karpenko)

$\eta - \tau$ coordinates, $\eta/S = 0.08$, $\zeta/S = 0$

$$\partial_{;\nu} T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\nu\lambda} T^{\nu\lambda} + \Gamma^{\nu}_{\nu\lambda} T^{\mu\lambda} = 0$$

$$\gamma\left(\partial_t + v_i \partial_i\right) \pi^{\mu\nu} = -\frac{\pi^{\mu\nu} - \pi^{\mu\nu}_{\mathrm{NS}}}{\tau_\pi} + I^{\mu\nu}_\pi \qquad \gamma\left(\partial_t + v_i \partial_i\right) \Pi = -\frac{\Pi - \Pi_{\mathrm{NS}}}{\tau_\Pi} + I_\Pi$$

- $T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$,
- $\partial_{;\nu}$ denotes a covariant derivative,
- $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projector orthogonal to u^μ,
- $\pi^{\mu\nu}$, Π shear stress tensor, bulk pressure
- $\pi^{\mu\nu}_{\mathrm{NS}} = \eta(\Delta^{\mu\lambda}\partial_{;\lambda} u^\nu + \Delta\nu\lambda\partial_{;\lambda} u^\mu) - \frac{2}{3}\eta\Delta^{\mu\nu}\partial_{;\lambda} u^\lambda$
- $\Pi_{\mathrm{NS}} = -\zeta \partial_{;\lambda} u^\lambda$
- $I^{\mu\nu}_\pi = -\frac{4}{3}\pi^{\mu\nu}\partial_{;\gamma} u^\gamma - [u^\nu \pi^{\mu\beta} + u^\mu \pi^{\nu\beta}] u^\lambda \partial_{;\lambda} u_\beta$
- $I_\Pi = -\frac{4}{3}\Pi\partial_{;\gamma} u^\gamma$