Towards the continuum limit in thermal charmonium physics[1,2]

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[2] Based on Y. Burnier, H.-T. Ding, O. Kaczmarek, A.-L. Kruse, ML, H. Ohno, H. Sandmeyer, in preparation.

Introduction

Recall challenges for thermal charmonium ($J/\psi, \eta_c$)

Non-relativistic effective theories (NRQCD, pNRQCD) are probably not accurate $\Rightarrow$ use the **relativistic formulation**.

In the relativistic formulation, imaginary-time measurement is generically hampered by a contribution from a **transport peak**, which originates from completely different physics (diffusion).[3]

There is however a particular case, the **pseudoscalar correlator** (η_c), in which there is no transport peak at leading order.[4] We have verified its absence at NLO as well, and focus on it now.

[3] T. Umeda, *Constant contribution in meson correlators at finite temperature*, hep-lat/0701005.

[4] G. Aarts and J.M. Martínez Resco, *Continuum and lattice meson spectral functions at nonzero momentum and high temperature*, hep-lat/0507004.

There are conflicting results in the literature

Traditional potential models:[5] J/ψ, η_c dissolve below $2T_\mathrm{c}$.

Modern potential model:[6] $J/\psi, \eta_c$ dissolve at $T \simeq 1.4T_\mathrm{c}$.

Quenched lattice QCD:[7] "*Our analysis suggests that both S wave states (J/ψ and η_c) ... disappear already at about $1.5T_\mathrm{c}$.*"

Unquenched lattice QCD:[8] "*The highest temperature studied is approximately $1.4T_\mathrm{c}$. Up to this temperature no significant variation is seen in the pseudoscalar channel.*"

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5 e.g. Á. Mócsy *et al*, *Quarkonia in the Quark Gluon Plasma*, 1302.2180.

6 Y. Burnier, O. Kaczmarek and A. Rothkopf, *Quarkonium at finite temperature: Towards realistic phenomenology from first principles*, 1509.07366.

7 H.-T. Ding *et al*, *Charmonium properties in hot quenched lattice QCD*, 1204.4945.

8 S. Borsányi *et al*, *Charmonium spectral functions from 2+1 flavour lattice QCD*, 1401.5940.

Perturbative side

Basic issues

$$G_{\rm P}(\tau) \equiv M_{\rm B}^2 \int_{\bf x} \langle (\bar\psi i\gamma_5\psi)(\tau,{\bf x})\ (\bar\psi i\gamma_5\psi)(0,{\bf 0})\rangle_{\rm c}\ .$$

There are many headaches already in vacuum ($T = 0$):

- The pseudoscalar density has an **anomalous dimension** $\Rightarrow$ use $M_{\rm B}\bar\psi i\gamma_5\psi$ which is renormalizable / RG-invariant.
- The **definition of** γ_5 in dimensional regularization requires typically the use of an additional finite renormalization factor.
- Many **quark masses** play a role: pole mass ($\equiv M$) for threshold physics, $\overline{\rm MS}$ mass ($\equiv m(\bar\mu)$) for UV asymptotics.

$T = 0$ result for spectral function[9] often given with M

$$\frac{\rho_{\mathrm{P}}(\omega)}{\omega^2 M^2}\bigg|^{\mathsf{vac}} \equiv \frac{N_{\mathrm{c}}}{8\pi} R^p_{\mathsf{c}}(\omega) \; ,$$

$$\begin{aligned} R^p_{\mathsf{c}}(\omega) &= R^{p(0)}(\omega) + \frac{\alpha_{\mathrm{s}}(\bar\mu)}{\pi} C_{\mathrm{F}} R^{p(1)}(\omega) \\ &+\Big(\frac{\alpha_{\mathrm{s}}(\bar\mu)}{\pi}\Big)^2 \Big[C_{\mathrm{F}}^2 R^{p(2)}_{A}(\omega) + C_{\mathrm{F}} N_{\mathrm{c}} R^{p(2)}_{NA}(\omega) + C_{\mathrm{F}} T_{\mathsf{f}} N_{\mathrm{f}} R^{p(2)}_{l}(\omega)\Big] \\ &+ \mathcal{O}(\alpha_{\mathrm{s}}^3) \; . \end{aligned}$$

But the coefficients contain $\ln(\frac{\bar\mu}{\omega})$, $\ln(\frac{\bar\mu}{M})$, $\ln(\frac{\omega}{M})$, whereas $\alpha_{\mathrm{s}} \sim 1/\ln(\frac{\bar\mu}{\Lambda})$; corrections cannot be eliminated simultaneously.

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[9] S. Caron-Huot, *Asymptotics of thermal spectral functions*, 0903.3958.

The result[10] can be re-expressed in the $\overline{\mathrm{MS}}$ scheme

$$\frac{\rho_{\mathrm{P}}(\omega)}{\omega^2 m^2(\bar\mu)}\bigg|^{\mathsf{vac}} \equiv \frac{N_{\mathrm{c}}}{8\pi}\,\tilde{R}^p_{\mathsf{c}}(\omega,\bar\mu)\ ,$$

$$\tilde{R}^{p(0)} \overset{\omega\gg m(\bar\mu)}{\approx} 1\ ,$$

$$\tilde{R}^{p(1)} \overset{\omega\gg m(\bar\mu)}{\approx} \frac{3}{2}\ln\Big(\frac{\bar\mu^2}{\omega^2}\Big) + \frac{17}{4}\ ,$$

$$\tilde{R}^{p(2)}_{A} \overset{\omega\gg m(\bar\mu)}{\approx} \frac{9}{8}\ln^2\Big(\frac{\bar\mu^2}{\omega^2}\Big) + \frac{105}{16}\ln\Big(\frac{\bar\mu^2}{\omega^2}\Big) + \frac{691}{64} - \frac{9\zeta_2}{4} - \frac{9\zeta_3}{4}\ ,$$

$$\tilde{R}^{p(2)}_{NA} \overset{\omega\gg m(\bar\mu)}{\approx} \frac{11}{16}\ln^2\Big(\frac{\bar\mu^2}{\omega^2}\Big) + \frac{71}{12}\ln\Big(\frac{\bar\mu^2}{\omega^2}\Big) + \frac{893}{64} - \frac{11\zeta_2}{8} - \frac{31\zeta_3}{8}\ .$$

Now get good convergence by choosing $\bar\mu \propto \omega$.

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[10] R. Harlander and M. Steinhauser, *Higgs decay to top quarks at* $\mathcal{O}(\alpha_{\mathrm{s}}^2)$, hep-ph/9704436; A. Maier and P. Marquard, *Low- and High-Energy Expansion of Heavy-Quark Correlators at Next-To-Next-To-Leading Order*, 1110.5581.

But for threshold physics we also need the pole mass.

Strict $m(\bar{\mu}) \leftrightarrow M$ relation shows no convergence.[11]

We came up with our own recipe, suitable for $\rho_{\mathrm{P}}(\omega)$:

$$M_x^2 \equiv m^2(\bar{\mu}) \; \frac{\tilde{R}^p_{\mathsf{c}}(\omega,\bar{\mu})}{R^p_{\mathsf{c}}(\omega)}\Bigg|_{\bar{\mu}=\omega,\ \omega=xM_x} , \quad x = 4\ldots8 \, .$$

On the $\overline{\mathrm{MS}}$ side, running is available up to 5 loops.[12]

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[11] P. Marquard *et al*, *$\overline{\mathrm{MS}}$-on-shell quark mass relation up to four loops in QCD and a general $SU(N)$ gauge group*, 1606.06754.

[12] P.A. Baikov *et al*, *Quark Mass and Field Anomalous Dimensions to $\mathcal{O}(\alpha_s^5)$*, 1402.6611; T. Luthe *et al*, *Five-loop quark mass and field anomalous dimensions for a general gauge group*, 1612.05512; P.A. Baikov *et al*, *Five-Loop Running of the QCD coupling constant*, 1606.08659; F. Herzog *et al*, *The five-loop beta function of Yang-Mills theory with fermions*, 1701.01404.

Some examples

	$m(\bar{\mu}_{\mathrm{ref}} \equiv 2$ GeV$)$	M_x	α_{s}
[$\sim$ charm]	1.0 GeV	1.19(2) GeV	0.280
	2.0 GeV	2.11(2) GeV	0.208
	3.0 GeV	2.99(2) GeV	0.181
	4.0 GeV	3.84(2) GeV	0.166
[$\sim$ bottom]	5.0 GeV	4.67(3) GeV	0.155

Close to threshold, thermal effects need to be included.

In this regime we use NRQCD/pNRQCD type techniques.[13]

$$\rho_{\rm P}^{\rm NRQCD} = \frac{M^2}{3}\rho_{\rm V}^{\rm NRQCD} \;, \quad \omega \approx 2M \;,$$

$$\rho_{\rm V}^{\rm NRQCD}(\omega) = \frac{1}{2}(1 - e^{-\frac{\omega}{T}}) \int_{-\infty}^{\infty} {\rm d}t \, e^{i\omega t} \; C_{>}(t; \mathbf{0}, \mathbf{0}) \;,$$

$$\Big\{ i\partial_t - \Big[2M + V_T(r) - \frac{\nabla_{\mathbf{r}}^2}{M} \Big] \Big\} C_{>}^{V}(t; \mathbf{r}, \mathbf{r}') \quad = \quad 0 \;, \quad t \neq 0 \;,$$

$$C_{>}^{V}(0; \mathbf{r}, \mathbf{r}') \quad = \quad 6 N_{\rm c} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}') \;.$$

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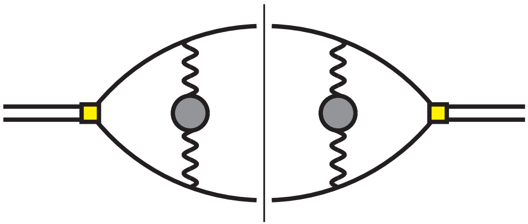
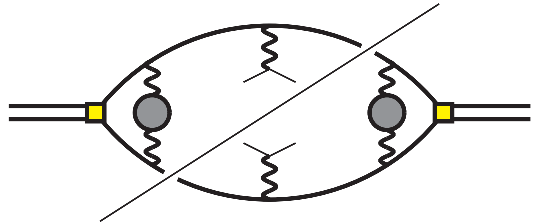
[13] Y. Burnier, ML and M. Vepsäläinen, *Heavy quarkonium in any channel in resummed hot QCD*, 0711.1743.

The thermal potential includes virtual and real processes.





At large r use the leading-order real-time static potential:[14]

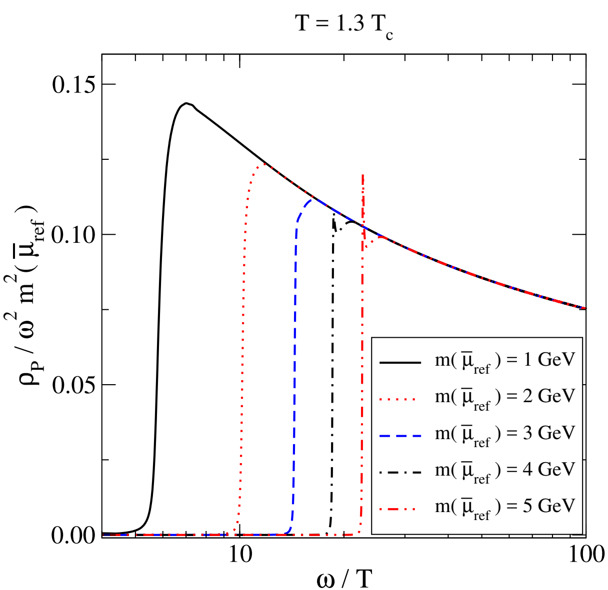
$$V_T(r) = -\alpha_s C_F \Big[m_D + \frac{\exp(-m_D r)}{r} \Big] - i \alpha_s C_F T\, \phi(m_D r) \; .$$

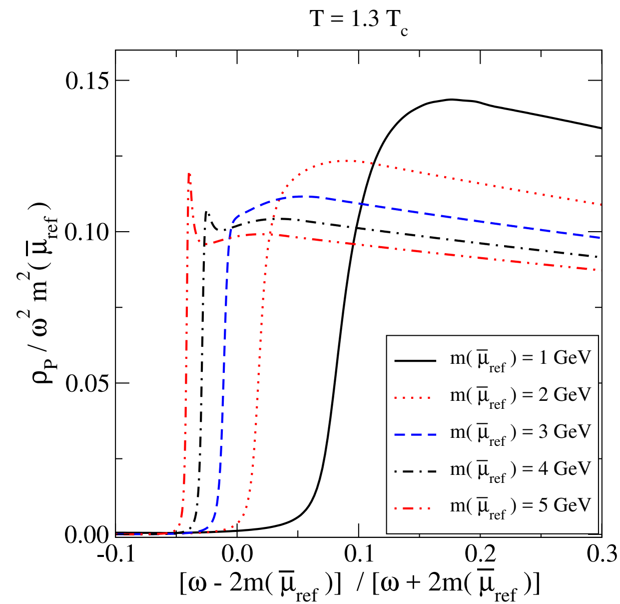
At small r, replace $\mathrm{Re}\, V_T$ by the 2-loop vacuum potential.[15]

[14] ML *et al*, *Real-time static potential in hot QCD*, hep-ph/0611300; A. Beraudo *et al*, *Real and imaginary-time $Q\overline{Q}$ correlators in a thermal medium*, 0712.4394; N. Brambilla *et al*, *Static quark-antiquark pairs at finite temperature*, 0804.0993.

[15] Y. Schröder, *The Static potential in QCD to two loops*, hep-ph/9812205; R.N. Lee *et al*, *Analytic three-loop static potential*, 1608.02603.

Using $M_{x=6}$ and fitting an overall normalization of ρ_P^{NRQCD}, a full spectral function can be parsed together.





Lattice side

Ensembles for quenched SU(3)

a	lattice size	T/T_c	statistics
0.0177 fm	$96^3 \times 48$	0.74	237
($\beta = 7.192$)	$\vdots$	$\vdots$	$\vdots$
	$96^3 \times 16$	2.23	237
0.0139 fm	$120^3 \times 60$	0.76	171
($\beta = 7.394$)	$\vdots$	$\vdots$	$\vdots$
	$120^3 \times 20$	2.27	226
0.0116 fm	$144^3 \times 72$	0.75	221
($\beta = 7.544$)	$\vdots$	$\vdots$	$\vdots$
	$144^3 \times 24$	2.26	237
0.0087 fm	$192^3 \times 96$	0.76	224
($\beta = 7.793$)	$\vdots$	$\vdots$	$\vdots$
	$192^3 \times 32$	2.27	235

Technical details

Wilson fermions with non-perturbative $O(a)$ improvement.[16] Mass corrections of $O(aM_{\rm B})$ are only known perturbatively.[17]

At each point, quark propagators are determined for 6 κ-values (i.e. $M_{\rm B}$). Many "sources" are employed for improved statistics.

[16] B. Sheikholeslami and R. Wohlert, *Improved Continuum Limit Lattice Action for QCD with Wilson Fermions*, NPB 259 (1985) 572; M. Lüscher *et al*, *Nonperturbative* $O(a)$ *improvement of lattice QCD*, hep-lat/9609035.

[17] M. Lüscher, S. Sint, R. Sommer and P. Weisz, *Chiral symmetry and* $O(a)$ *improvement in lattice QCD*, hep-lat/9605038; S. Sint and P. Weisz, *Further results on* $O(a)$ *improved lattice QCD to one loop order of perturbation theory*, hep-lat/9704001.

Recall topological freezing close to the continuum limit.

Our measurements were separated by 500 sweeps, each consisting of 1 heatbath and 4 overrelaxation updates.

We do observe some evolution between the topological sectors up to $\beta \simeq 6.8$, however at the values $\beta \gtrsim 7.2$ that play a role in our analysis, we are **frozen to the trivial sector**.[18]

Errors from here should be suppressed by inverse volume (?).[19]

Scale setting through our own estimates of r_0 at $\beta > 7.0$.[20]

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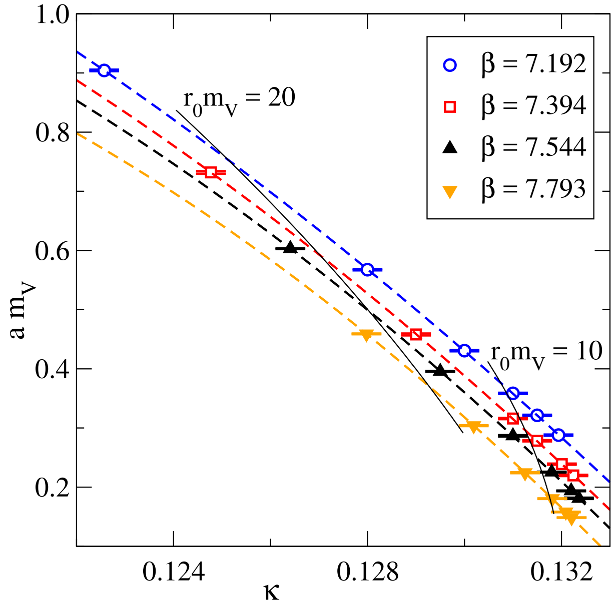
[18] S. Schaefer *et al.* [ALPHA Collaboration], *Critical slowing down and error analysis in lattice QCD simulations*, 1009.5228.

[19] e.g. M. Lüscher, *Stochastic locality and master-field simulations of very large lattices*, 1707.09758.

[20] R. Sommer, *A New way to set the energy scale in lattice gauge theories and its applications to the static force and* $\alpha_{\rm S}$ *in SU(2) Yang-Mills theory*, hep-lat/9310022; *Scale setting in lattice QCD*, 1401.3270.

Idea of mass interpolation ($T \approx 0$)



$r_0 m_\mathrm{V} = 20$

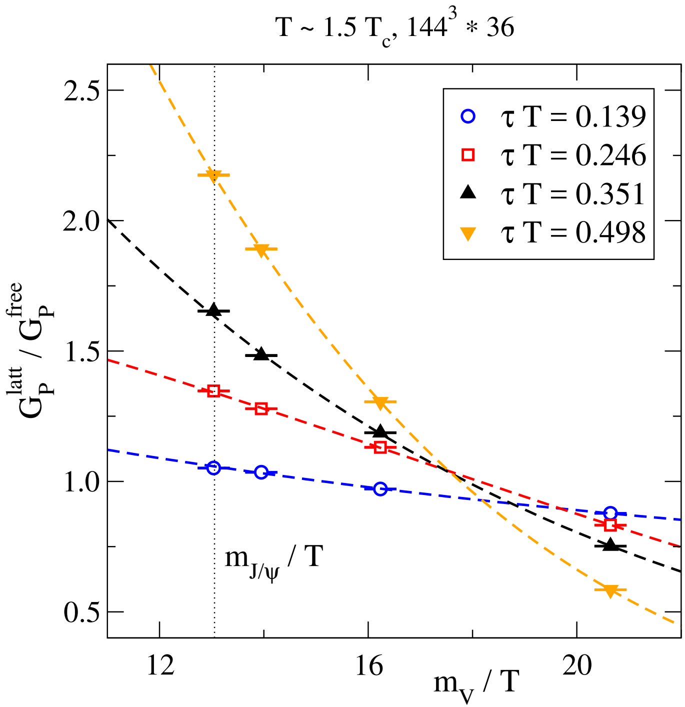
$\Leftrightarrow m_\mathrm{V} \simeq 8$ GeV

$r_0 m_\mathrm{V} = 10$

$\Leftrightarrow m_\mathrm{V} \simeq 4$ GeV

Dependence on κ is rather smooth $\Rightarrow$ correlation functions can be quadratically interpolated/extrapolated to the physical point.

Mass interpolation applied to a thermal correlation function



Renormalization factors

Only 1-loop and 2-loop factors Z_{P} are known.[21]

Great care is needed in the inclusion of an additional finite factor Z_5, related to how γ_5 is treated on the pQCD side.[22]

$$Z_5 = 1 - \frac{g^2 C_{\mathrm{F}}}{2\pi^2} + \frac{g^4 C_{\mathrm{F}}}{128\pi^4}\frac{N_{\mathrm{c}} + 2N_{\mathrm{f}}}{9} + \mathcal{O}(g^6) \approx 0.8 \; .$$

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[21] S. Capitani *et al*, *Renormalization and off-shell improvement in lattice perturbation theory*, hep-lat/0007004; A. Skouroupathis and H. Panagopoulos, *Two-loop renormalization of scalar and pseudoscalar fermion bilinears on the lattice*, 0707.2906.

[22] S.A. Larin, *The Renormalization of the axial anomaly in dimensional regularization*, hep-ph/9302240.

Additional renormalization due to anomalous dimension

$Z_{\mathrm{P}} Z_5$ brings us to the $\overline{\mathrm{MS}}$ scheme at the scale $\bar{\mu} = a^{-1}$.

For scale-independent results they should be multiplied by a quark mass in the same scheme and scale, i.e. $m^2(a^{-1})$.

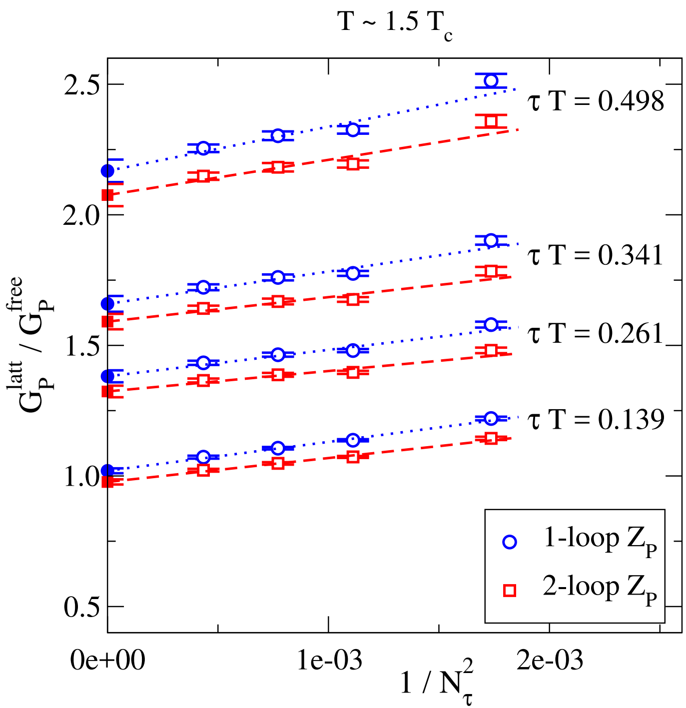
Subsequently the masses can be evolved to the scale $\bar{\mu}_{\mathsf{ref}}$:

$$m^2(a^{-1}) = m^2(\bar{\mu}_{\mathsf{ref}}) \underbrace{\left[m(a^{-1})/m(\bar{\mu}_{\mathsf{ref}})\right]^2}_{\approx 0.6} .$$

Normalization

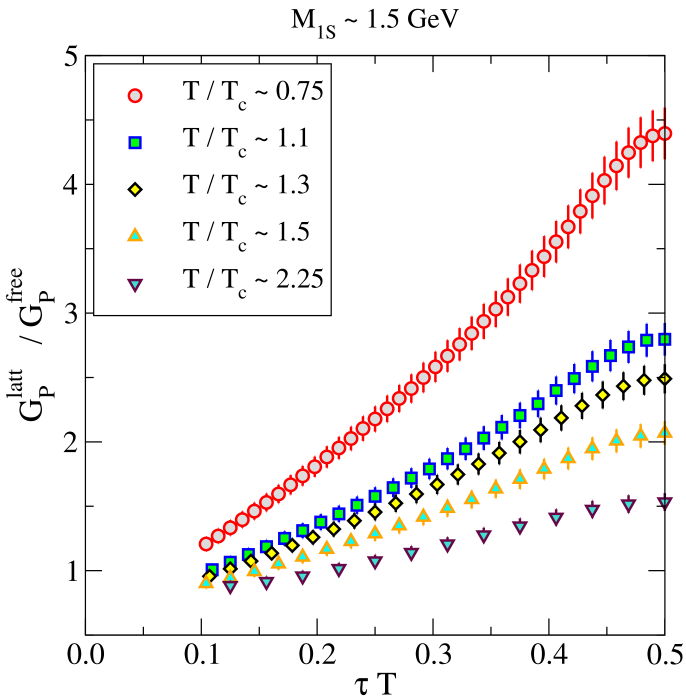
$$\frac{G_{\mathrm{P}}^{\mathsf{free}}(\tau)}{m^2(\bar{\mu}_{\mathsf{ref}})} \equiv \int_{2M_{1\mathrm{S}}}^{\infty} \frac{\mathrm{d}\omega}{\pi} \left\{ \frac{N_{\mathrm{c}} \omega^2}{8\pi} \tanh\left(\frac{\omega}{4T}\right) \sqrt{1 - \frac{4M_{1\mathrm{S}}^2}{\omega^2}} \right\} \frac{\cosh\left(\frac{1}{2T} - \tau\right)\omega}{\sinh\frac{\omega}{2T}} .$$

All that done, the extrapolation looks reasonably good



Treat difference of 1-loop and 2-loop factors as an error estimate.

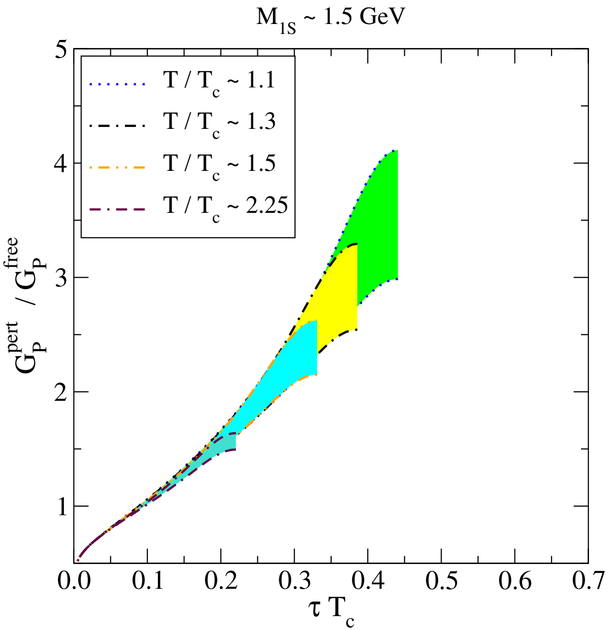
Final results at different temperatures

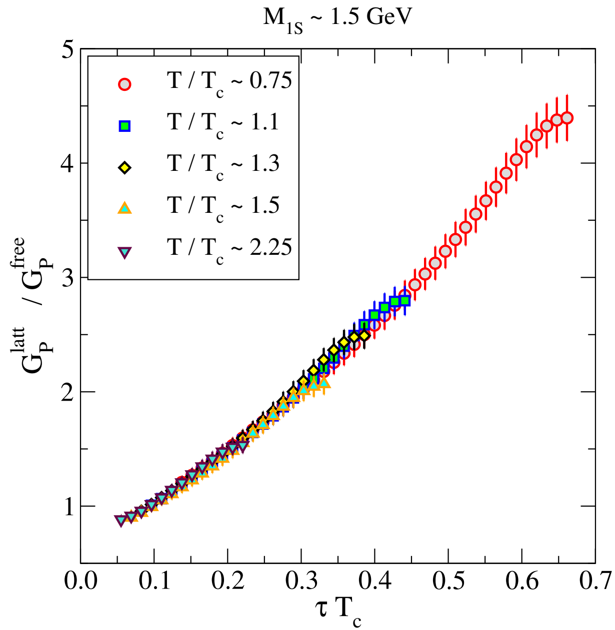


Errors include statistics and $\sim$ systematics.

Comparison of pQCD and lattice

A first comparison shows minor differences (note x-axis)





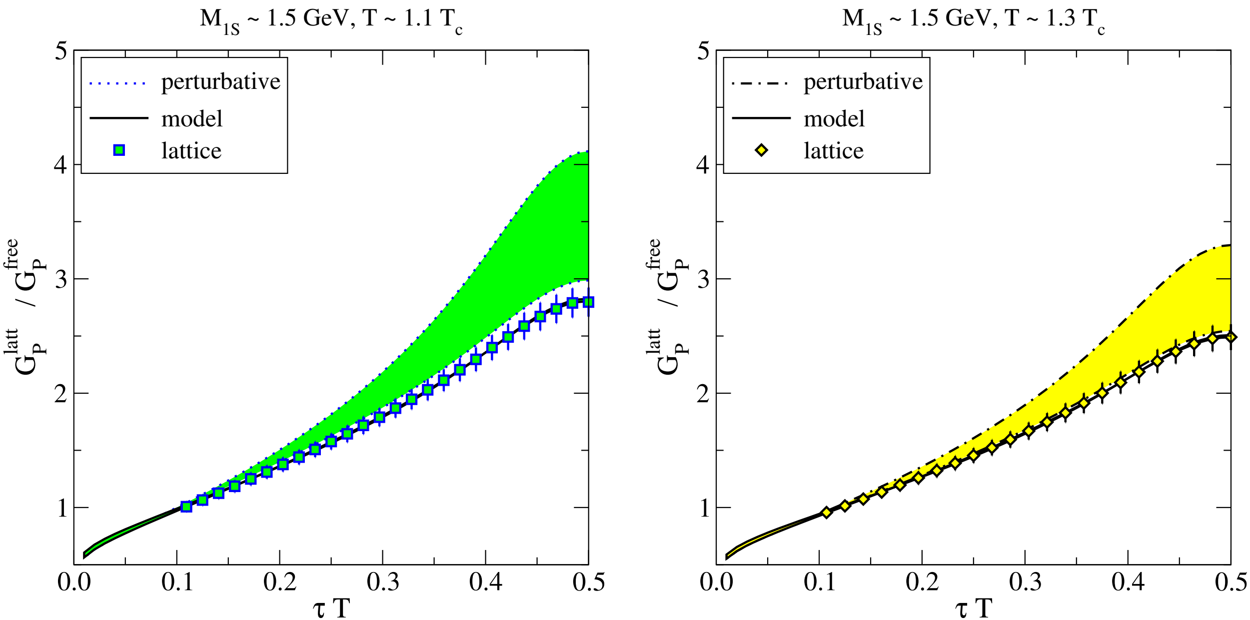
... but there is no reason for worry

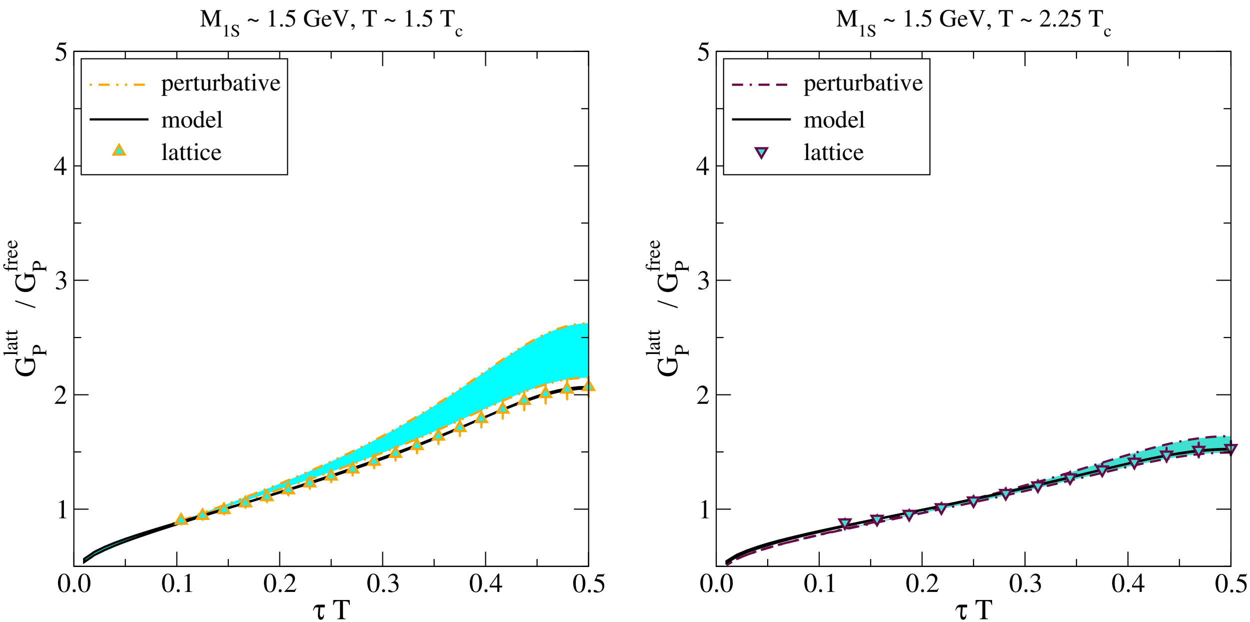
- The overall renormalization factors (1-loop/2-loop) contained uncertainties.
- The relation between $m(\bar{\mu}_{\text{ref}})$ and M and therefore the threshold location on the pQCD side is poorly controlled.

To account for these we allow for two free parameters and carry out a simple χ^2 minimization (no BR or MEM attempted):

$$\rho_{\mathrm{P}}^{\text{model}}(\omega) \;\equiv\; A\,\rho_{\mathrm{P}}^{\text{pert}}(\omega + B)\ .$$

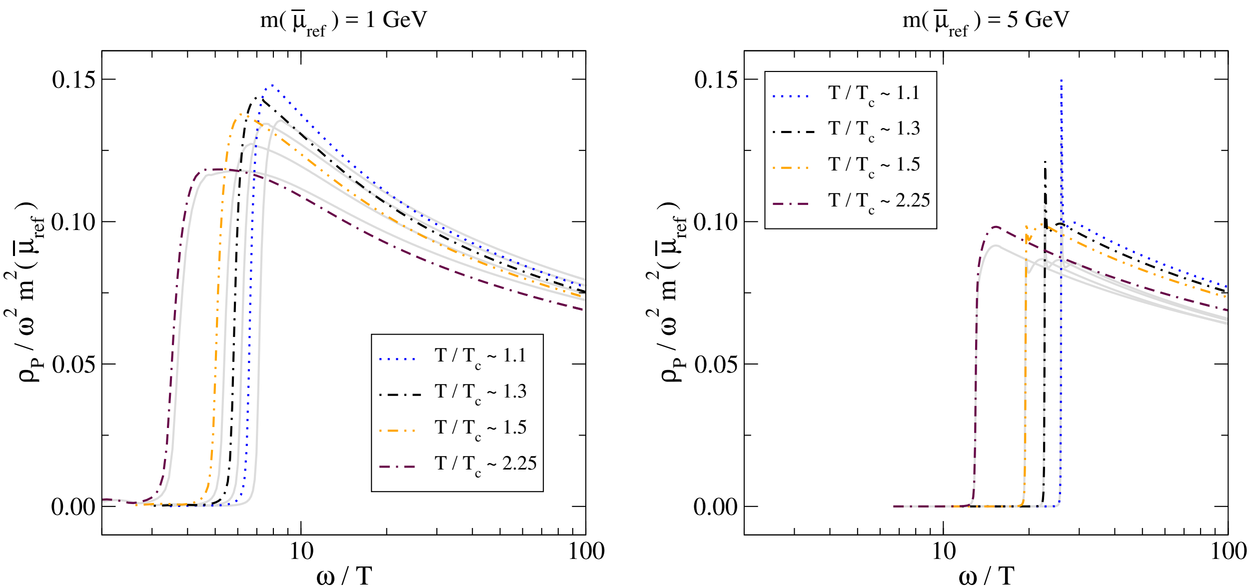
The fits are pretty much perfect!





M1S ~ 1.5 GeV, T ~ 1.5 Tc
perturbative
model
lattice
G_P^latt / G_P^free
τ T
0.0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5
M1S ~ 1.5 GeV, T ~ 2.25 Tc
perturbative
model
lattice

The corresponding spectral functions (charm and bottom)



Charm: threshold shifts to larger energy.

Bottom: normalization uncertainty is visible ($aM_{\rm B} \sim 1!$).

Conclusions

Whenever using a relativistic formulation, the pseudoscalar density is theoretically ideal for thermal quarkonium physics.

But it comes with subtleties, both on pQCD and lattice sides.

Yet, once the dust settles, amazing agreement is found.

No resonance peaks are needed for η_c at $T > T_\mathrm{c}$ in the **quenched case** (though we cannot strictly exclude their existance either). For η_b a resonance peak can exist up to $T \simeq 1.5 T_\mathrm{c}$.

It is crucial to extend the analysis to unquenched QCD!

Epilogue

Conjecture

Quenched QCD **is** weakly coupled down to $T \approx T_c$, if we look at observables with an external scale (mass, momentum) $\gtrsim \pi T$.

Recall numbers (scale setting through $r_0 \simeq 0.47$ fm)

$\Lambda_{\overline{\mathrm{MS}}}|_{N_\mathrm{f}=0} \approx 255$ MeV.[23]

$\Lambda_{\overline{\mathrm{MS}}}|_{N_\mathrm{f}=3} \approx 340$ MeV.[24]

$T_\mathrm{c}|_{N_\mathrm{f}=0} \approx 1.24\ \Lambda_{\overline{\mathrm{MS}}}|_{N_\mathrm{f}=0}$.[25]

$T_\mathrm{c}|_{N_\mathrm{f}=3} \approx 0.45\ \Lambda_{\overline{\mathrm{MS}}}|_{N_\mathrm{f}=3}$.[26]

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[23] K.I. Ishikawa *et al, Non-perturbative determination of the Λ-parameter in the pure SU(3) gauge theory from the twisted gradient flow coupling*, 1702.06289.

[24] M. Bruno *et al., The strong coupling from a nonperturbative determination of the Λ parameter in three-flavor QCD*, 1706.03821.

[25] A. Francis *et al, Critical point and scale setting in SU(3) plasma: An update*, 1503.05652.

[26] review: H.B. Meyer, *QCD at non-zero temperature from the lattice*, 1512.06634.

Physics argument for the difference in $T_\mathrm{c}/\Lambda_{\overline{\mathrm{MS}}}$

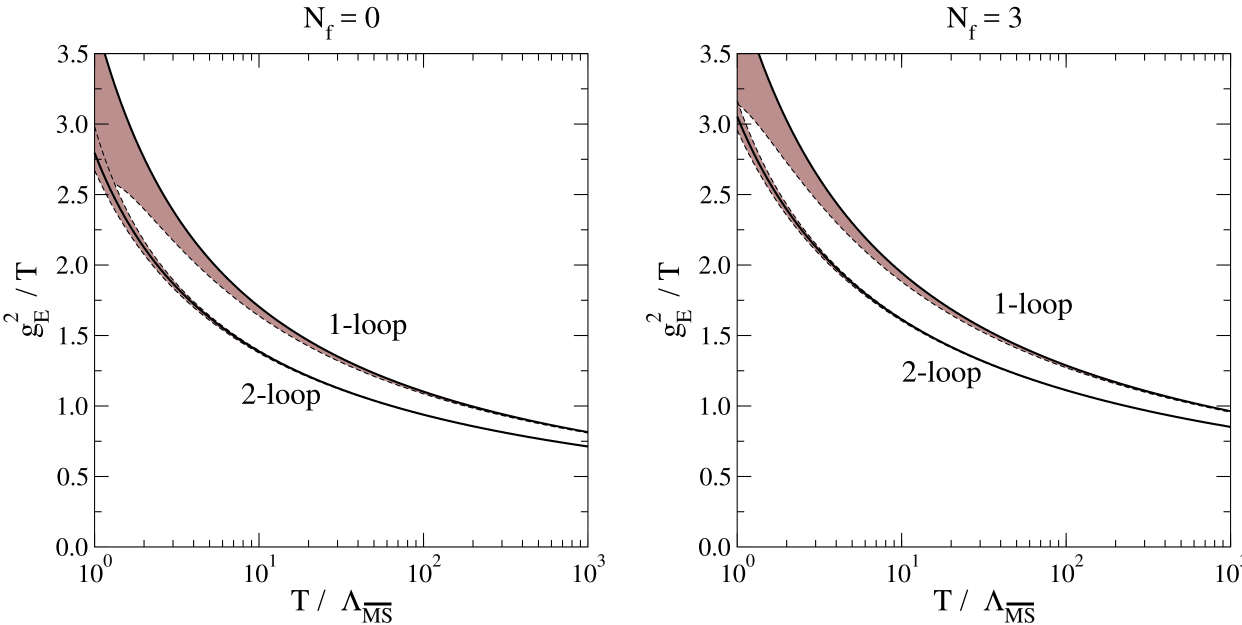
$N_\mathrm{f} = 0$:

$m_{0_{++}} \gg 1$ GeV $\Rightarrow$ need to heat the system "a lot" in order for something to happen $\Rightarrow T_\mathrm{c}$ is large.

$N_\mathrm{f} = 3$:

$m_\pi \ll 1$ GeV $\Rightarrow$ already a mild heating excites (exponentially) many hadrons $\Rightarrow T_\mathrm{c}$ is low.

Quantitately, via an effective coupling [27]

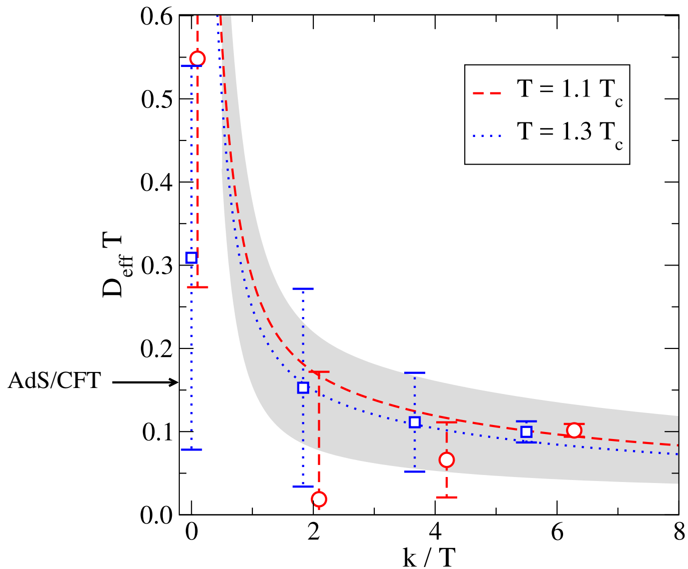


$$\Longrightarrow \alpha_s|_{T\simeq T_c, N_f=0} \simeq 0.2, \; \alpha_s|_{T\simeq T_c, N_f=3} > 0.3.$$

[27] ML and Y. Schröder, *Two-loop QCD gauge coupling at high temperatures*, hep-ph/0503061.

There is already evidence for good convergence at $N_\mathrm{f} = 0$

The thermal photon production rate appears to agree with pQCD as soon as momentum is large, $k \gtrsim 3T$.[28]



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28 J. Ghiglieri, O. Kaczmarek, ML and F. Meyer, *Lattice constraints on the thermal photon rate*, 1604.07544.