The Six Gluon One-Loop Amplitude

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This article brings together in a single place the different components of the six gluon one-loop amplitude

1. Introduction

The one-loop scattering amplitude for six-gluons in pure QCD has been an interesting test case for the development of analytic techniques in perturbative gauge theories. The amplitude can be decomposed into various subamplitudes which were calculated between 1993 and 2006 [1–13] in thirteen different publications by 23 authors.

The results for the amplitude are spread across a large number of papers. The purpose of this contribution is to bring as many of these as practical together in a common source with a reasonably consistent notation. There is no original material in this process although some of the original results have been expressed in alternate forms.

The different pieces have been used in the development of various techniques: this article will focus upon the results not the processes by which they were created. The amplitudes discussed here are available at *http : //pyweb.swan.ac.uk/ ∼ dunbar/sixgluon.html* in `Mathematica` format. Many of the helicity amplitudes are included in expressions which are valid for n legs. We will try to show how these specialise to the six gluon case and give the six-gluon case explicitly.

2. Organisation

The organisation of loop amplitudes into physical sub-amplitudes is an important step toward computing these amplitudes: although eventually all the pieces must be reassembled.

For one-loop amplitudes of adjoint representation particles in the loop, one may perform a colour decomposition similar to the tree-level decomposition [14]. The one-loop decomposition is [15],

$$\mathcal{A}_n^{\text{1-loop}} = g^n \sum_{c=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma)\; A_{n;c}(\sigma)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. The leading colour-structure factor,

$$\mathrm{Gr}_{n;1}(1) \;=\; N_c \;\; \mathrm{Tr}\left(T^{a_1}\cdots T^{a_n}\right) \,,$$

is just N_c times the tree colour factor, and the subleading colour structures ($c>1$) are given by,

$$\mathrm{Gr}_{n;c}(1) \;=\; \mathrm{Tr}\left(T^{a_1}\cdots T^{a_{c-1}}\right)\; \mathrm{Tr}\left(T^{a_c}\cdots T^{a_n}\right) \,.$$

S_n is the set of all permutations of n objects and $S_{n;c}$ is the subset leaving $\mathrm{Gr}_{n;c}$ invariant [15]. The contributions with fundamental representation quarks can be obtained from the same partial amplitudes, except that sum runs only over the $A_{n;1}$ and the overall factor of N_c in $\mathrm{Gr}_{n;1}$ is dropped. For one-loop amplitudes of gluons the $A_{n;c}$, $c>1$ can be obtained from the $A_{n;1}$ by summing over permutations [15,3]. Hence it is suffice to compute $A_{n;1}$ in what follows. The partial amplitudes $A_{n;1}$ have cyclic symmetry rather than full crossing symmetry

The amplitudes are also organised according to the helicity of the outgoing gluon which may be $\pm$. We use polarisation tensors formed from Weyl spinors [16]

$$\epsilon^+_\mu(k;q) = \frac{\langle q^- |\, \gamma_\mu \,| k^-\rangle}{\sqrt{2}\,\langle q\, k\rangle} \,,\; \epsilon^-_\mu(k;q) = \frac{\langle q^+ |\, \gamma_\mu \,| k^+\rangle}{\sqrt{2}\,[k\, q]}$$

where k is the gluon momentum and q is an arbitrary null 'reference momentum' which drops out of final gauge-invariant amplitudes. The plus and minus labels on the polarization vectors refer to the gluon helicities and we use the notation $\langle ij\rangle \equiv \langle k_i^-|k_j^+\rangle\,, [ij] \equiv \langle k_i^+|k_j^-\rangle$. In the twistor-inspired studies of gauge theory amplitudes the two component Weyl spinors are often expressed as

$$\lambda_a = |k^+\rangle \quad \bar\lambda_{\dot a} = |k^-\rangle$$

Helicity amplitudes are related to those with all legs of opposite helicity by conjugation $\langle a\, b\rangle \leftrightarrow [b\, a]$. Consequently, up to conjugation and relabeling, there are eight independent helicity color ordered amplitudes for the six-gluon amplitude as given in table 1. Of these amplitudes the $A(1^+2^+3^+4^+5^+6^+)$ and $A(1^-2^+3^+4^+5^+6^+)$ have the simplest one-loop structure: a consequence that the tree partial amplitudes vanish. In fact, these amplitudes vanish to all orders in perturbation theory within any supersymmetry theory. Amplitudes with exactly two negative helicities are referred to as MHV ("maximally helicity violating") amplitudes and those with three negative helicities as NMHV ("next to MHV") amplitudes.

Using spinor helicity leads to amplitudes which are functions of the spinor variables $\langle a\, b\rangle$ and $[a\, b]$. It is also useful to define combinations of spinor products

$$[a|K_{b\cdots f}|m\rangle \equiv [a\, b]\,\langle b\, m\rangle + \cdots + [a\, f]\,\langle f\, m\rangle$$

etc. In terms of Dirac traces

$$\mathrm{tr}_+(\not k_a \not k_b \not k_c \not k_d) = [a\, b]\,\langle b\, c\rangle\,[c\, d]\,\langle d\, a\rangle$$

A general one-loop amplitude for massless particles can be expressed, after an appropriate Passarino-Veltman reduction [17], in terms of scalar integral functions with rational coefficients,

$$A_n^{1-\mathrm{loop}} = \sum_{i\in\mathcal{C}} a_i\, I_4^i + \sum_{j\in\mathcal{D}} b_j\, I_3^j + \sum_{k\in\mathcal{E}} c_k\, I_2^k + R_n\,, \tag{2.1}$$

where a_i, b_i, c_i and R_n are rational functions of the $|k_i^\pm\rangle$ (or equivalently of λ_a and $\bar\lambda_{\dot a}$). The I_4, I_3, and I_2 are scalar box, triangle and bubble functions respectively and these contain the logarithms and dilogarithms of the amplitude. The

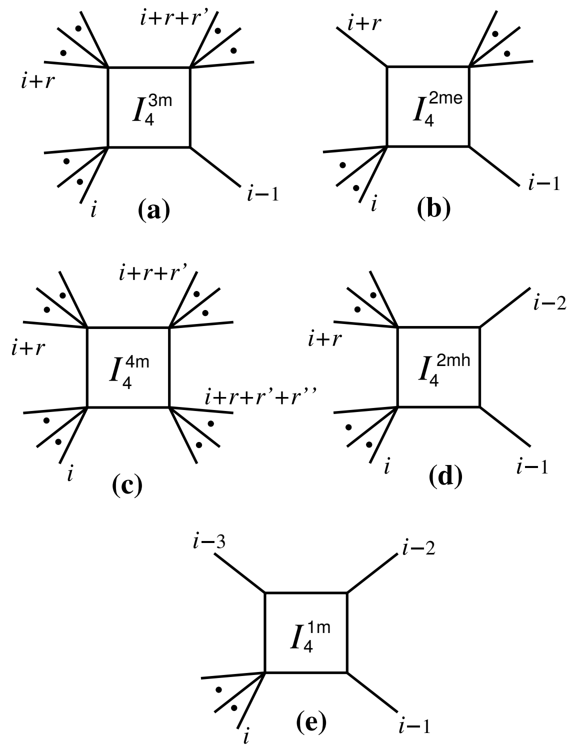

Figure 1. The various box integral functions

functional form of the scalar integrals depends upon the number of legs with non-null momenta inflowing. These are frequently referred to as massive legs although strictly we are dealing with massless states throughout.

We also choose to use a *supersymmetric decomposition.* Instead of calculating the one-loop contributions from massless gluons, $A_n^{[1]}$ or quarks $A_n^{1/2]}$ circulating in the loop it is considerable more convenient to calculate the contributions from a full $N=4$ multiplet, a $N=1$ chiral multiplet and a complex scalar. In terms of these

$$\begin{aligned} A_n^{[1]} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1\ \mathrm{chiral}} + A_n^{[0]}\,, \\ A_n^{[1/2]} &= A_n^{\mathcal{N}=1\ \mathrm{chiral}} - A_n^{[0]}\,. \end{aligned} \tag{2.2}$$

Finally, we find it useful to split an amplitude into its *cut-constructible part* and *rational part.*

The I_n span the cut-constructible part and

$$A_n = C_n + R_n$$

This is a convenient rather than a physical split. The split into R_n and C_n is, to some extent arbitrary. Some of the integral functions eg. the bubble function I_2 contain rational pieces however it is much more logical to keep these within C_n. When doing so, for the supersymmetric contributions $R_n = 0$. In fact, it is useful to redefine different integral functions which include as much of of the rational terms within C_n as possible. When this happens we will use the notation $\hat{R}_n$.

Finally, we present out results in the "four-dimensional-helicity" (FDH) scheme of dimensional regularisation which since it respects supersymmetry merges well with the supersymmetric decomposition. The translation to 't Hooft-Veltman scheme is immediate with

$$A_n^{N=4\ 'tHV} = A_n^{N=4\ FDH} - \frac{c_\Gamma}{3} A_n^{\rm tree}$$

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$[0]^C$	$[0]^R$
$(++++++)$	0	0	0	93 [2]
$(-+++++)$	0	0	0	93 [1]
$(--++++)$	94 [3]	94 [4]	94 [4]	05 [8]
$(-+-+++)$	94 [3]	94 [4]	04 [6]	06 [12]
$(-++-++)$	94 [3]	94 [4]	04 [6]	06 [12]
$(---+++)$	94 [4]	04 [5]	05 [9]	06 [11]
$(--+-++)$	94 [4]	05 [7]	06 [10]	06 [13]
$(-+-+-+)$	94 [4]	05 [7]	06 [10]	06 [13]

Table 1
The components of the six-gluon amplitude with the year of computation and original references

3. $\mathcal{N} = 4$ amplitudes

The one-loop amplitudes for gluon scattering within $\mathcal{N} = 4$ super-Yang-Mills are particularly simple being heavily constrained by the large symmetry. In terms of the integral basis they can be expressed entirely in terms of scalar boxes [3]

$$A_n^{\mathcal{N}=4} = \sum_{i\in C} a_i\, I_4^i$$

The scalar box functions are illustrated in figure 1. It is convenient to define rescaled box functions F_4^i where

$$\begin{aligned}
I_{4:i}^{\rm 1m} &= -2r_\Gamma \frac{F_{4:i}^{\rm 1m}}{t_{i-3}^{[2]} t_{i-2}^{[2]}} \\
I_{4:r;i}^{\rm 2me} &= -2r_\Gamma \frac{F_{4:r;i}^{\rm 2m\,\it e}}{t_{i-1}^{[r+1]} t_i^{[r+1]} - t_i^{[r]} t_{i+r+1}^{[n-r-2]}}\ , \\
I_{4:r;i}^{\rm 2m\it h} &= -2r_\Gamma \frac{F_{4:r;i}^{\rm 2m\,\it h}}{t_{i-2}^{[2]} t_{i-1}^{[r+1]}}\ ,
\end{aligned}$$

where $r_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$. The F-functions are the box functions with the appropriate momentum prefactor removed. For example, the scalar box function with one-massive leg is

$$\begin{aligned}
&F_4^{1m}(s_{12}, s_{23}, m_4^2) = \\
&\quad -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-m_4^2}\right)^\epsilon \\
&\quad +{\rm Li}_2[1-\frac{-m_4^2}{-s_{12}}] + {\rm Li}_2[1-\frac{-m_4^2}{-s_{23}}] + \frac{1}{2}\ln^2(\frac{-s_{12}}{-s_{23}})
\end{aligned}$$

These integrals are expressed in the Euclidean region where all momentum invariants are negative i.e. $-s > 0$ etc. m_4^2 is the momentum invariant inflowing to the fourth leg not a physical mass.

The expressions are given in the physical region through the usual analytic continuation. We use notations,

$$t_j^{[m]} = K_{j\cdots j+m}^2 \equiv (k_j + k_{j+1} + \cdots + k_{j+m})^2$$

but also the shorthand $s_{ij} = (k_i + k_j)^2$ and $t_{ijk} = (k_i + k_j + k_k)^2$.

Closed analytic expressions are known for the $\mathcal{N} = 4$ MHV and NMHV n-point one-loop amplitudes. The MHV amplitudes are given by

$$A_n^{\mathcal{N}=4,MHV} = c_\Gamma A_n^{\rm tree} \times \left(\sum_i F_4^{1m} + \sum_i F_4^{2me}\right)$$

where the sum over F-functions is over all possible inequivalent functions with the appropriate cyclic ordering of legs. For the six-point amplitude this consists of the six one-mass box functions and the three independent F_4^{2me} box functions. The three different MHV amplitudes only differ by the overall $A_6^{\rm tree}$ factor. The amplitude has an overall factor of

$$c_\Gamma = \frac{(4\pi)^\epsilon}{16\pi^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{r_\Gamma}{(2\pi)^{2-\epsilon}}$$

In subsequent amplitudes we will usually suppress this factor. In terms of elementary functions we can express the six-point amplitude as

$$\begin{aligned}
&\left(\sum_i F_4^{1m} + \sum_i F_4^{2me}\right) = \\
&\sum_{i=1}^{6} -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-t_i^{[2]}}\right)^\epsilon - \sum_{i=1}^{6} \ln\left(\frac{-t_i^{[2]}}{-t_i^{[3]}}\right)\ln\left(\frac{-t_{i+1}^{[2]}}{-t_i^{[3]}}\right) \\
&-\mathrm{Li}_2\left[1-\frac{t_1^{[2]}t_6^{[4]}}{t_1^{[3]}t_6^{[3]}}\right] - \mathrm{Li}_2\left[1-\frac{t_2^{[2]}t_1^{[4]}}{t_2^{[3]}t_1^{[3]}}\right] \\
&-\mathrm{Li}_2\left[1-\frac{t_3^{[2]}t_2^{[4]}}{t_3^{[3]}t_2^{[3]}}\right] + \\
&-\frac{1}{4}\sum_{i=1}^{6}\ln\left(\frac{-t_i^{[3]}}{-t_{i+4}^{[3]}}\right)\ln\left(\frac{-t_{i+1}^{[3]}}{-t_{i+3}^{[3]}}\right) + \pi^2
\end{aligned}$$

The three independent NMHV amplitudes also contain the one-mass box functions but contain F_4^{2mh} boxes. The boxes which appear in the six-point NMHV amplitudes appear in the very special combination

$$\begin{aligned}
W_6^{(i)} &\equiv F^{\rm 1m}_{4:i} + F^{\rm 1m}_{4:i+3} + F^{\rm 2m\,\mathit{h}}_{4:2;i+1} + F^{\rm 2m\,\mathit{h}}_{4:2;i+4} \\
&= -\frac{1}{2\epsilon^2}\sum_{j=1}^{6}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^\epsilon \\
&- \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i,i+1}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+1,i+2}}\right) \\
&\quad - \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+4,i+5}}\right) \\
&+ \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+2,i+3}}\right)\ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+5,i}}\right) \\
&\quad + \frac{1}{2}\ln\left(\frac{-s_{i,i+1}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-s_{i+1,i+2}}{-s_{i+4,i+5}}\right) \\
&+ \frac{1}{2}\ln\left(\frac{-s_{i-1,i}}{-s_{i,i+1}}\right)\ln\left(\frac{-s_{i+1,i+2}}{-s_{i+2,i+3}}\right) \\
&\quad + \frac{1}{2}\ln\left(\frac{-s_{i+2,i+3}}{-s_{i+3,i+4}}\right)\ln\left(\frac{-s_{i+4,i+5}}{-s_{i+5,i}}\right) + \frac{\pi^2}{3}\,.
\end{aligned} \tag{3.1}$$

As we can see the dilogarithms drop out of this expression. It is an example where the expansion in terms of scalar boxes is probably not optimal in some sense. This feature of the $\mathcal{N}=4$ NMHV six-point amplitude persists for amplitudes involving external states other than gluons [18] but not beyond six-points.

The first NMHV $\mathcal{N}=4$ amplitude is given by

$$\begin{aligned}
&A_{6;1}^{N=4}(1^-2^-3^-4^+5^+6^+) \\
&\qquad = ic_\Gamma\ \left[B_1\,W_6^{(1)} + B_2\,W_6^{(2)} + B_3\,W_6^{(3)}\right]
\end{aligned} \tag{3.2}$$

The coefficients B_i are given in terms of the B_0 function

$$B_0 \;=\; \frac{t_{123}^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\ \ [1|K_{23}|4\rangle[3|K_{12}|6\rangle}\,.$$

by

$$\begin{aligned}
B_1 &= B_0\,, \\
B_2 &= \left(\frac{[4|K_{123}|1\rangle}{t_{234}}\right)^4 B_0^+ + \left(\frac{\langle 2\,3\rangle\,[5\,6]}{t_{234}}\right)^4 \bar{B}_0^+\,, \\
B_3 &= \left(\frac{[6|K_{345}|3\rangle}{t_{345}}\right)^4 B_0^- + \left(\frac{\langle 1\,2\rangle\,[4\,5]}{t_{345}}\right)^4 \bar{B}_0^-\,.
\end{aligned}$$

where

$$B_0^+ \equiv B_0|_{i\longrightarrow i+1}\quad B_0^- \equiv B_0|_{i\longrightarrow i-1}$$

and

$$\bar{B}_0 \equiv B_0|_{\langle a\, b\rangle \leftrightarrow [a\, b]}$$

i.e. conjugation.

The other amplitudes are

$$A_{6;1}^{N=4}(1^-2^-3^+4^-5^+6^+) = ic_\Gamma \left[D_1\, W_6^{(1)} + D_2\, W_6^{(2)} + D_3\, W_6^{(3)} \right],$$

where

$$\begin{aligned} D_1 &= \left(\frac{[3|K_{123}|4\rangle}{t_{123}}\right)^4 B_0 + \left(\frac{\langle 1\,2\rangle\,\langle 5\,6\rangle}{t_{123}}\right)^4 \bar{B}_0\ , \\ D_2 &= \left(\frac{[3|K_{234}|1\rangle}{t_{234}}\right)^4 B_0^+ + \left(\frac{\langle 2\,4\rangle\,[5\,6]}{t_{234}}\right)^4 \bar{B}_0^+\ , \\ D_3 &= \left(\frac{[6|K_{345}|4\rangle}{t_{345}}\right)^4 B_0^- + \left(\frac{\langle 1\,2\rangle\,[3\,5]}{t_{345}}\right)^4 \bar{B}_0^-, \end{aligned}$$

and

$$A_{6;1}^{N=4}(1^-2^+3^-4^+5^-6^+) = ic_\Gamma \left[G_1\, W_6^{(1)} + G_2\, W_6^{(2)} + G_3\, W_6^{(3)} \right],$$

where

$$\begin{aligned} G_1 &= \left(\frac{[2|K_{456}|5\rangle}{t_{123}}\right)^4 B_0 + \left(\frac{\langle 1\,3\rangle\,[4\,6]}{t_{123}}\right)^4 \bar{B}_0\ , \\ G_2 &= \left(\frac{[6|K_{234}|3\rangle}{t_{234}}\right)^4 \bar{B}_0^+ + \left(\frac{\langle 5\,1\rangle\,[2\,4]}{t_{234}}\right)^4 B_0^+\ , \\ G_3 &= \left(\frac{[4|K_{612}|1\rangle}{t_{345}}\right)^4 \bar{B}_0^- + \left(\frac{\langle 3\,5\rangle\,[6\,2]}{t_{345}}\right)^4 B_0^-\ . \end{aligned}$$

4. $\mathcal{N} = 1$ amplitudes $(- - + + + +)$,$(- - - + + +)$

The simplest of the non-zero $\mathcal{N} = 1$ amplitudes is the MHV with the two-minus helicities arranged adjacent to each other [4]. For this configuration the amplitude contains no boxes (or triangles) and is given by

$$A^{N=1}(1^-2^-3^+\dots n^+) = \frac{c_\Gamma A^{\rm tree}}{2}\Bigg\{ \left({\rm K}_0(-t_2^{[2]}/\mu^2) + {\rm K}_0(-t_n^{[2]}\mu^2) \right) - \frac{1}{t_1^{[2]}} \sum_{m=4}^{n-1} c_{12}^m \frac{{\rm L}_0\left(-t_2^{[m-2]}/(-t_2^{[m-1]})\right)}{t_2^{[m-1]}} \Bigg\}$$

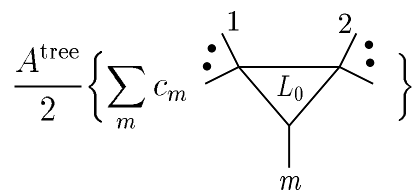

Figure 2. The N=1 one-loop contribution to the simplest of the MHV amplitudes $A(1^-2^-3^+4^+\cdots n^+)$. The ${\rm L}_0$ function can be thought of as either a combination of bubble functions or as arising from a Feynman parameter integral of the two mass triangle indicated.

where

$$c_{12}^m = \left({\rm tr}_+[\not{k}_1 \not{k}_2 \not{k}_m \not{q}_{m,1}] - {\rm tr}_+[\not{k}_1 \not{k}_2 \not{q}_{m,1} \not{k}_m] \right)$$

using

$$q_{m,l} = \begin{cases} \displaystyle\sum_{i=m}^{l} k_i, & m \le l, \\ \displaystyle\sum_{i=m}^{n} k_i + \sum_{i=1}^{l} k_i, & m > l, \end{cases}$$

and

$${\rm L}_0(r) = \frac{\ln(r)}{1-r} \quad {\rm K}_0(r) = \frac{1}{\epsilon} - \ln(r) + 2$$

The ${\rm K}_0$ function is just the bubble integral function $I_2(s) = {\rm K}_0(-s/\mu^2)$. We have expressed the amplitude using ${\rm L}_0(s/s')/s'$ expressions to avoid spurious singularities. In the limit $s/s' \longrightarrow 1$,

$$\frac{{\rm L}_0(-s/-s')}{s'} = \frac{\ln(-s) - \ln(-s')}{s'-s} \longrightarrow -1$$

which is obviously non-singular however if we express the amplitude in terms of the scalar bubble functions I_2 using

$$\frac{{\rm L}_0(-s/-s')}{s'} = \frac{I_2(-s') - I_2(-s)}{s'-s}$$

then the coefficients of $I_2(s)$ and $I_2(s')$ both contain the spurious singularity. For the six-point case there are just two L_0 functions in the summations and

$$A^{N=1}(1^-2^-3^+4^+5^+6^+) = \frac{c_\Gamma A^{\rm tree}}{2}\Bigg\{\Big(\mathrm{K}_0(-s_{23}/\mu^2)+\mathrm{K}_0(-s_{61}/\mu^2)\Big) - \frac{c_{12}^4}{s_{12}}\frac{\mathrm{L}_0\left(-s_{23}/(-t_{234})\right)}{t_{234}} - \frac{c_{12}^5}{s_{12}}\frac{\mathrm{L}_0\left(-s_{61}/(-t_{234})\right)}{t_{234}}\Bigg\}$$

The next simplest amplitude we present is not one of the remaining MHV amplitudes but the amplitude $A(---+++)$. This example of a "split helicity" configurations contains many simplifications and can be expressed using L_0 and K_0 functions. The amplitude is symmetrical under the operations

$$A \longrightarrow A|_{123456\longrightarrow 321654}$$
$$A \longrightarrow \bar{A}|_{123456\longrightarrow 456123}$$

There is an all-n expression for the configuration with three adjacent negative helicities [19],

$$A_n^{\mathcal{N}=1\ \rm chiral}(1^-2^-3^-4^+5^+\cdots n^+) = \frac{c_\Gamma A^{\rm tree}}{2}\left(\mathrm{K}_0(-s_{n1}/\mu^2)+\mathrm{K}_0(-s_{34}/\mu^2)\right) - \frac{ic_\Gamma}{2}\left(\sum_{r=4}^{n-1}\hat{d}_{n,r}\,\frac{\mathrm{L}_0[q_{3,r}^2/q_{2,r}^2]}{q_{2,r}^2} + \sum_{r=4}^{n-2}\hat{g}_{n,r}\,\frac{\mathrm{L}_0[q_{2,r}^2/q_{2,r+1}^2]}{q_{2,r+1}^2} + \sum_{r=4}^{n-2}\hat{h}_{n,r}\,\frac{\mathrm{L}_0[q_{3,r}^2/q_{3,r+1}^2]}{q_{3,r+1}^2}\right)$$

where

$$\hat{d}_{n,r} = \frac{\langle 3|q_{3,r}q_{2,r}|1\rangle^2\,\langle 3|q_{3,r}\left[k_2,q_{2,r}\right]q_{2,r}|1\rangle\,\langle r\,r+1\rangle}{[2|q_{2,r}|r\rangle[2|q_{2,r}|r+1\rangle\prod_{k=3}^n\langle k\,k+1\rangle\;q_{2,r}^2\,q_{3,r}^2},$$

$$\hat{g}_{n,r} = \sum_{j=4}^{r}\frac{\langle 3|q_{3,j}q_{2,j}|1\rangle^2\langle 3|q_{3,j}q_{2,j}\left[k_{r+1},q_{2,r}\right]|1\rangle\,\langle j\,j+1\rangle}{[2|q_{2,j}|j\rangle[2|q_{2,j}|j+1\rangle\prod_{k=3}^n\langle k\,k+1\rangle\;q_{3,j}^2\,q_{2,j}^2}$$

$$\hat{h}_{n,r} = (-1)^n\,\hat{g}_{n,n-r+2}\Big|_{(123..n)\to(321n..4)}.$$

This general expression reduces to the explicit form for the six-point case,

$$A_6^{N=1}(1^-2^-3^-4^+5^+6^+) = a_1\,\mathrm{K}_0[-s_{61}/\mu^2]+a_2\,\mathrm{K}_0[-s_{34}/\mu^2] - \frac{ic_\Gamma}{2}\left[c_1\frac{\mathrm{L}_0[t_{345}/s_{61}]}{s_{61}}+c_2\frac{\mathrm{L}_0[t_{234}/s_{34}]}{s_{34}} + c_3\frac{\mathrm{L}_0[t_{234}/s_{61}]}{s_{61}}+c_4\frac{\mathrm{L}_0[t_{345}/s_{34}]}{s_{34}}\right]$$

where the coefficients are

$$a_1 = a_2 = \frac{c_\Gamma}{2}A_6^{\rm tree}(1^-2^-3^-4^+5^+6^+),$$

and

$$c_1 = \frac{[6|K|3\rangle^2[6|(k_2K-Kk_2)K|3\rangle}{[2|K|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,K^2},\quad K=K_{345}$$

$$c_2 = c_1|_{123456\longrightarrow 321654}\quad c_4 = c_3|_{123456\longrightarrow 321654}$$
$$c_3 = \bar{c}_1|_{123456\longrightarrow 654321}$$

5. Basis of Box Functions

At this point we must discuss a suitable basis for expressing the amplitudes. We could use the basis (2.1) however this is not the most efficient option. By choosing a suitable basis of box functions we can considerably simplify the structure of the triangle coefficients.

Triangle integral functions may have one, two or three massless legs: $I_3^{\rm 2m}$, $I_3^{\rm 1m}$, $I_3^{\rm 3m}$. The one-mass triangle depends only on the momentum invariant of the massive leg K_1 and is

$$I_3^{\rm 1m} = \frac{r_\Gamma}{\epsilon^2}(-K_1^2)^{-1-\epsilon}\ .$$

whilst the two-mass triangle integral with non-null momenta K_1 and K_2 is,

$$I_3^{\rm 2m} = \frac{r_\Gamma}{\epsilon^2}\frac{(-K_1^2)^{-\epsilon}-(-K_2^2)^{-\epsilon}}{(-K_1^2)-(-K_2^2)}\ .$$

Both these integral functions contain $\ln(K^2)/\epsilon$ IR singularities. The key point is that the IR singularity of an amplitude must be [20]

$$A_{IR}^{N=1\ \rm chiral} = \frac{c_\Gamma}{\epsilon}A^{\rm tree}\qquad A_{IR}^{[0]} = \frac{c_\Gamma}{3\epsilon}A^{\rm tree}$$

so that the $\ln(K^2)/\epsilon$ singularities must cancel. This constraint effectively determines the coefficients of $I_3^{\rm 1m}$ and $I_3^{\rm 2m}$ in terms of the box coefficients.

Specifically, the one and two mass triangles are linear combinations of the set of functions,

$$G(-K^2) = r_\Gamma \frac{(-K^2)^{-\epsilon}}{\epsilon^2} \ ,$$

with

$$I_3^{1m} = G(-K_1^2) \ ,$$
$$I_3^{2m} = \frac{1}{(-K_1^2)-(-K_2^2)} \left(G(-K_1^2) - G(-K_2^2) \right) .$$

The $G(-K^2)$ are labeled by the independent momentum invariants K^2 and in fact form an independent basis of functions, unlike the one and two-mass triangles which are not all independent.

In practice we need never calculate the coefficients of the G functions once we know the box coefficients. The only functions containing $\ln(s)/\epsilon$ terms are the box functions and I_4^{1m} and I_4^{2m} so

$$\sum a_i I_4|_{\ln(K^2)/\epsilon} + b_G \frac{\ln(K^2)}{\epsilon} = 0$$

This equation fixes the single b_G in terms of the a_i.

The simplest approach to implement this simplification is to express the amplitude in terms of truncated finite F-functions. If we define the function

$$\mathcal{F}_4^{1m}(s_{12}, s_{23}, K_4^2) = F_4^{1m}(s_{12}, s_{23}, K_4^2)$$
$$+ \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon$$

and use these, together with the other truncated functions, as a basis then the $\mathcal{N}=1$ and scalar amplitudes can be expressed as

$$A^{\rm 1-loop} = \sum a_i \mathcal{F}_4^i + \sum b_j^{3m} I_3^{3m,j} + \sum c_k I_2^k + R$$

with no I_3^{1m} and I_3^{2m} present. The truncated two-mass-easy box functions is

$$\mathcal{F}_4^{2me}(S,T,K_2^2,K_4^2) = F_4^{2me}(S,T,K_2^2,K_4^2)$$
$$+ \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon$$
$$- \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_2^2}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon$$

where $S=(k_1+K_2)^2$ and $T=(K_2+k_3)^2$. This function was labeled $M_0(S,T,K_2^2,K_4^2)$ in ref. [4] and fig. 2. It has the feature that its soft limit is smooth

$$\mathcal{F}_4^{2me}(S,T,0,K_4^2) = \mathcal{F}_4^{1m}(S,T,K_4^2)$$

which means the F^{2me} and F^{1m} can be combined in a single summation. The truncated two-mass-hard box functions is

$$\mathcal{F}^{2mh}(S,T,K_3^2,K_4^2) = F^{2mh}(S,T,K_3^2,K_4^2)$$
$$+ \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon$$
$$- \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_3^2}\right)^\epsilon - \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon$$

6. $\mathcal{N}=1$ amplitudes $(-+-+++)$,$(-++-++)$

These are the two remaining MHV amplitudes.

$$A(1^-2^+3^-4^+5^+6^+) \quad A(1^-2^+3^+4^-5^+6^+)$$

These two amplitudes can be expressed in a general way. Suppose we have a MHV amplitude with the negative helicity legs being i and j then the amplitude is a combination of F_4^{2me}, F^{1m}, L_0 and K_0 functions

$$A^{\rm tree} \times \left(\sum a_{ij}^{n_1 n_2} \mathcal{F}_4^{2me} + \sum c_{m,a}^{ij}\, \mathrm{L}_0 \right)$$

In the above the functions $\mathcal{F}^{1m}$ and K_0 are implicit as special cases. When one of the Kinematic invariants is zero we *replace* $\mathrm{L}_0(s/s')/s'$ by $\mathrm{K}_0(-s/\mu^2)/s$. This is *not* a smooth limit. The boxes which are present are a restricted set of the $I_4^{2m\ e}$ and I_4^{1m}. To be included in the sum the two negative helicities must lie in the two massive legs of the integral function. For such a $I_4^{2m\ e}$ if we label the two massless legs by n_1 and n_2 then the coefficient of this box is

$$a_{ij}^{n_1 n_2} = 2 \frac{\langle n_1\, i\rangle \langle n_1\, j\rangle \langle n_2\, i\rangle \langle n_2\, j\rangle}{\langle i\, j\rangle^2 \langle n_1\, n_2\rangle^2}$$

The formula also holds when one of the two masses is zero. For the L_0 functions there is a

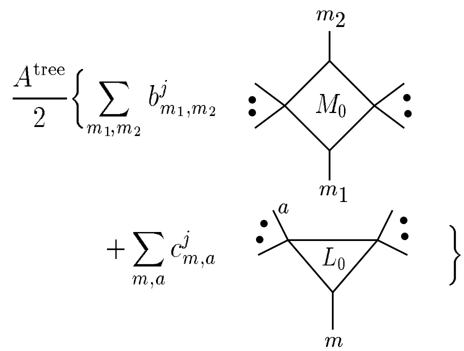

Figure 3. The N=1 one-loop contributions to a generic MHV configuration $A(1^-,2^+,\cdots j^-,\cdots n^+)$. The summation of M_0 terms runs over all (scaled and truncated) box functions where the negative helicity legs 1 and j lie in the massive legs. Explicitly $1<m_1<j<m_2$. The sum include the case where one massive leg is in fact null.

visual realisation of the summation if we recognize that L_0 is also the result of carrying out a Feynman parameter integral of a two-mass triangle $\mathrm{L}_0(s/s')/s' = I_3^{2m}(s,s')[a_2]$. In this case the summation runs over L_0 where one of the negative helicities lies in each of the massive legs. The general summation formula is given in ref. [4]. The coefficient is given by

$$c^{ij}_{m,a} = \frac{(\mathrm{tr}_+[\not k_i \not k_j \not k_m \not q_{m,a}] - \mathrm{tr}_+[\not k_i \not k_j \not q_{m,a} \not k_m])}{[(k_i+k_j)^2]^2} \times \langle m\, i\rangle\,[i\, j]\,\langle j\, m\rangle\;\frac{\langle a\, a+1\rangle}{\langle a\, m\rangle\,\langle m\, a+1\rangle}$$

The six-point case is rather a degenerate form of the general case so it is useful to present here the explicit form

$$\begin{aligned}
A(1^-2^+3^-4^+5^+6^+) &= \frac{c_\Gamma A^{\rm tree}}{2}\Big(\\
a^{13}_{62}\,\mathcal{F}^{1m}&(s_{61},s_{12},t_{345}) + a^{13}_{24}\,\mathcal{F}^{1m}(s_{23},s_{34},t_{561}) \\
&+ a^{13}_{25}\,\mathcal{F}^{2me}(t_{234},t_{345},s_{34},s_{61}) \\
&+ c^{13}_{2,4}\frac{\mathrm{L}_0(s_{34}/t_{234})}{t_{234}} + c^{13}_{2,5}\frac{\mathrm{L}_0(s_{61}/t_{612})}{t_{612}} \\
&+ c^{13}_{4,1}\frac{\mathrm{L}_0(s_{23}/t_{234})}{t_{234}} + c^{13}_{5,1}\frac{\mathrm{L}_0(s_{61}/t_{561})}{t_{561}} \\
&+ c^{13}_{5,2}\frac{\mathrm{L}_0(s_{34}/t_{345})}{t_{345}} + c^{13}_{6,2}\frac{\mathrm{L}_0(s_{12}/t_{612})}{t_{612}}) \\
&+ c^{13}_{2,6}\frac{\mathrm{K}_0(-s_{12}/\mu^2)}{s_{12}} + c^{13}_{6,1}\frac{\mathrm{K}_0(-s_{23}/\mu^2)}{s_{23}} \\
&+ c^{13}_{4,2}\frac{\mathrm{K}_0(-s_{34}/\mu^2)}{s_{34}} + c^{13}_{2,3}\frac{\mathrm{K}_0(-s_{23}/\mu^2)}{s_{23}}\Big)
\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
A(1^-2^+3^+4^-5^+6^+) &= \frac{c_\Gamma A^{\rm tree}}{2}\Big(\\
b^{14}_{62}\,\mathcal{F}^{1m}&(s_{61},s_{12},t_{345}) + b^{14}_{35}\,\mathcal{F}^{1m}(s_{34},s_{45},t_{612}) \\
&+ b^{14}_{63}\,\mathcal{F}^{2me}(t_{612},t_{123},s_{12},s_{45}) \\
&+ b^{14}_{52}\,\mathcal{F}^{2me}(t_{561},t_{612},s_{61},s_{34}) \\
&+ c^{14}_{2,5}\frac{\mathrm{L}_0(s_{61}/t_{612})}{t_{612}} + c^{14}_{3,6}\frac{\mathrm{L}_0(s_{12}/t_{123})}{t_{123}} \\
&+ c^{14}_{6,3}\frac{\mathrm{L}_0(s_{45}/t_{456})}{t_{456}} + c^{14}_{5,2}\frac{\mathrm{L}_0(s_{34}/t_{345})}{t_{345}} \\
&+ c^{14}_{6,2}\frac{\mathrm{L}_0(s_{12}/t_{612})}{t_{612}} + c^{14}_{5,1}\frac{\mathrm{L}_0(s_{61}/t_{561})}{t_{612}}) \\
&+ c^{14}_{2,4}\frac{\mathrm{L}_0(s_{34}/t_{234})}{t_{234}} + c^{14}_{3,5}\frac{\mathrm{L}_0(s_{45}/t_{345})}{t_{345}}) \\
&+ c^{14}_{2,6}\frac{\mathrm{K}_0(-s_{12}/\mu^2)}{s_{12}} + c^{14}_{6,1}\frac{\mathrm{K}_0(-s_{23}/\mu^2)}{s_{23}} \\
&+ c^{14}_{4,3}\frac{\mathrm{K}_0(-s_{45}/\mu^2)}{s_{45}} + c^{14}_{2,4}\frac{\mathrm{K}_0(-s_{34}/\mu^2)}{s_{34}}\Big)
\end{aligned} \tag{6.2}$$

7. $N=1$ amplitudes $A_6(--+-++)$, $A_6(-+-+-+)$

These amplitudes are not part of any known all-n series. Let us consider the case $(-+-+-+)$ first because it has the most symmetry being symmetrical under the operations

$$A \longrightarrow A|_{i\longrightarrow i+2}$$
$$A \longrightarrow \bar{A}|_{i\longrightarrow i+1}$$

The amplitude contains all six one-mass and all six two-mass-hard boxes.

$$A(1^-2^+3^-4^+5^-6^+)_{\rm box} = \sum_i a_i^{1m}\mathcal{F}_i^{1m} + \sum_i a_i^{2m}\mathcal{F}_i^{2mh}$$

We need only quote a_1^{1m} and a_2^{2m} since

$$a_{i+2} = a_i|_{j\longrightarrow j+2} \quad a_{i+1} = \bar{a}_i|_{j\longrightarrow j+1}$$

with

$$a_4^{1m} = \frac{[2|K|5\rangle^2[1|K|5\rangle[3|K|5\rangle}{[1\,3]^2\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|K|6\rangle[1|K|4\rangle K^2} \; K = K_{123}$$
$$a_5^{2m} = \frac{[2|K|5\rangle^2[3|K|5\rangle[2|K|4\rangle}{[1\,2]\,\langle 5\,6\rangle\,[3|K|6\rangle[1|K|4\rangle[3|K|4\rangle^2} \; K = K_{123}$$

The amplitude also has bubble integral functions, and, two, three-mass triangle functions

$$\sum_{i=1}^{6} c_{i,2} I_2(s_{ii+1}) + \sum_{i=1}^{3} c_{i,3} I_2(t_{ii+1i+2}) + b_1^{3m} I_3^{3m} + b_2^{3m} I_3^{3m}$$

The bubble functions are of two types depending upon whether they are bubbles in s_{ii+1} or $t_{ii+1i+2}$ Again we need only specify one of each c_i since

$$c_{i+2,a} = c_{i,a}|_{j\longrightarrow j+2} \quad c_{i+1,a} = \bar{c}_{i,a}|_{j\longrightarrow j+1}$$
$$b_2^{3m} = \bar{b}_1^{3m}|_{j\longrightarrow j+1}$$

This amplitude (and $A_6(--+-++)$) were originally calculated in ref [7] using the fermionic integration unitarity method. We choose to express these amplitude using functions which are explicitly rational in the (components) of the Weyl spinor [21].

To describe the bubble coefficients, it is useful to introduce

$$H_1(a;b;K) \equiv \frac{\langle b|K|a]}{\langle a|K|a]} = \frac{\langle b|K|a]}{a\cdot K}$$

$$G_1(a;b,c;B;Q,K) \equiv - \frac{[B|K(KQ-QK)|a\rangle\langle b|(KQ-QK)|c\rangle}{2\langle a|KQ|a\rangle\Delta_3(K,Q)} + \frac{[B|K|a\rangle(\langle b\,a\rangle\,[a|K|c\rangle + \langle c\,a\rangle\,[a|K|d\rangle)}{2\langle a|KQ|a\rangle[a|K|a\rangle}$$

where

$$\Delta_3(K,Q) \equiv 4(K\cdot Q)^2 - 4K^2Q^2$$

is the Gram determinant of the three mass triangle defined by having two legs with massive momenta K and Q. Its appearance is a clear indication of the links between the bubble and triangle functions implied by the absence of spurious singularities [22,23].

We also define extended versions H_n,G_n

$$H_n(a_1\cdots a_n;b_1\cdots b_n;K) = \sum_{i=1}^{n} \frac{\prod_{j=2}^{n}\langle b_j\,a_i\rangle}{\prod_{j\neq i}\langle a_j\,a_i\rangle} H_1(a_i,b_1;K)$$

$$G_n(a_1\cdots a_n;B,b_1\cdots b_{n+1};K,Q) = \sum_{i=1}^{n} \frac{\prod_{j=2}^{n}\langle b_j\,a_i\rangle}{\prod_{j\neq i}\langle a_j\,a_i\rangle} G_1(a_i;B;b_1,b_{n+1};K,Q)$$

Using these functions we can express the bubble coefficients as

$$c_{1,3} = -\frac{[2|K|5\rangle^2}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,[1\,2]\,[2\,3]\,t_{123}} \times H_4(4,6,K|3],K|1];5,5,K|2],K|2],K)$$

where $K = K_{123}$, and

$$c_{1,2} = \frac{[2|K|5\rangle^2}{[1\,2]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[3|K|6\rangle t_{123}} \times H_3(2,K|3],K'K|4\rangle;1,K'K|5\rangle,K'K|5\rangle;K') + \frac{1}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,[5\,6]} \times G_3(2,Q|5],K'K|4\rangle;6;1,3,X,X;Q';K') + \frac{[4|K_{345}|1\rangle^2}{\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,[3|K_{345}|6\rangle t_{345}} \times H_3[2,6,K_{345}|5];1,K_{345}|4],K_{345}|4];K'];$$

where $K = K_{123}, K' = K_{12}, Q = K_{34}, Q' = K_{56}$ and

$$|X\rangle = -|1\rangle[6|K_{51}|3\rangle - |2\rangle\,[6\,2]\,\langle 1\,3\rangle$$

This coefficient contains many spurious singularities. Singularities $[a|K|a\rangle$ generally cancel between bubble functions

$$cI_2[K^2] + c'I_2[(K+a)^2] \longrightarrow \text{finite}$$

The Δ^{-1} singularities vanish between the bubble coefficients and the three-mass triangle functions [22]

The three-mass triangle coefficient is the most complicated function so far being, in the form given in ref. [23],

$$
\begin{aligned}
b_1^{3m} =& \\
&-\frac{[4|K_{345}|1\rangle[5|K_{345}|1\rangle[4|K_{345}|6\rangle}{[5|K_{345}|2\rangle[3|K_{345}|6\rangle[5|K_{345}|6\rangle t_{345}}\times\Bigg(\langle 3\,5\rangle[2\,6] \\
&+\frac{[4|K_{345}|1\rangle\,(2s_{12}s_{34}+(s_{56}-s_{12}-s_{34})t_{345})}{2\,\langle 1\,2\rangle\,[3\,4]\,[5|K_{345}|6\rangle}\Bigg) \\
&-\frac{[6|K_{561}|2\rangle[1|K_{561}|3\rangle[6|K_{561}|3\rangle}{[1|K_{561}|2\rangle[5|K_{561}|2\rangle[1|K_{561}|4\rangle t_{561}}\times\Bigg(\langle 5\,1\rangle[4\,2] \\
&+\frac{[6|K_{561}|3\rangle\,(2s_{34}s_{56}+(s_{12}-s_{34}-s_{56})t_{561})}{2\,\langle 3\,4\rangle\,[5\,6]\,[1|K_{561}|2\rangle}\Bigg) \\
&-\frac{[2|K_{123}|4\rangle[2|K_{123}|5\rangle[3|K_{123}|5\rangle}{[1|K_{123}|4\rangle[3|K_{123}|4\rangle[3|K_{123}|6\rangle t_{123}}\times\Bigg(\langle 1\,3\rangle[6\,4] \\
&+\frac{[2|K_{123}|5\rangle\,(2s_{12}s_{56}-(s_{12}+s_{56}-s_{34})t_{123})}{2\,\langle 5\,6\rangle\,[1\,2]\,[3|K_{123}|4\rangle}\Bigg) \\
&-\frac{1}{\Delta_3(K_{12},K_{34})}\Bigg(\langle 1\,3\rangle\,\langle 3\,5\rangle\,[2\,6]\,[3\,4] \\
&+\langle 1\,3\rangle\,\langle 1\,5\rangle\,[1\,2]\,[4\,6]+\langle 1\,5\rangle\,\langle 3\,5\rangle\,[2\,4]\,[5\,6]\Bigg)\times \\
&\Bigg(\frac{[5|K_{34}|1\rangle[4|K_{35}|6\rangle(t_{345}-t_{346})}{[5|K_{345}|2\rangle[3|K_{345}|6\rangle[5|K_{345}|6\rangle} \\
&+\frac{[6|K_{561}|2\rangle[1|K_{561}|3\rangle(t_{561}-t_{562})}{[1|K_{561}|2\rangle[5|K_{561}|2\rangle[1|K_{561}|4\rangle} \\
&+\frac{[2|K_{123}|4\rangle[3|K_{123}|5\rangle(t_{123}-t_{124})}{[1|K_{123}|4\rangle[3|K_{123}|4\rangle[3|K_{123}|6\rangle} \\
&-2\frac{[6|K_{34}|2\rangle[2|K_{56}|4\rangle[4|K_{12}|6\rangle}{[5|K_{561}|2\rangle\,[1|K_{456}|4\rangle\,[3|K_{456}|6\rangle} \\
&+2\frac{[5|K_{34}|1\rangle[1|K_{56}|3\rangle[3|K_{12}|5\rangle}{[5|K_{561}|2\rangle\,[1|K_{456}|4\rangle\,[3|K_{456}|6\rangle}\Bigg)
\end{aligned}
$$

The amplitude $A(--+-++)$ is a little more complicated since there is less symmetry amongst the coefficients. The amplitude is symmetrical under the operation

$$A \longrightarrow \bar{A}|_{123456\longrightarrow 654321}$$

We can split up the amplitudes

$$
\begin{aligned}
&A^{N=1}(1^-2^-3^+4^-5^+6^+) \\
&\qquad = A|_{boxes} + A|_{three-mass-triangle} + A|_{bubbles}
\end{aligned}
$$

There are three two-mass hard boxes,and two one-mass boxes and the box part of the amplitude is

$$A|_{boxes} = a_1\mathcal{F}^{2\mathrm{m}\,h}_{4:4}+a_2\mathcal{F}^{2\mathrm{m}\,h}_{4:6}+a_3\mathcal{F}^{2\mathrm{m}\,h}_{4:2}+a_4\mathcal{F}^{1\mathrm{m}}_{4:2}+a_5\mathcal{F}^{1\mathrm{m}}_{4:3}$$

where

$$
\begin{aligned}
a_1 &= \frac{[3|K_{24}|1\rangle^2[3|K_{234}|5\rangle\,\langle 5\,1\rangle}{[4|K_{234}|5\rangle^2[2|K_{234}|5\rangle\,[2\,3]\,\langle 5\,6\rangle\,\langle 6\,1\rangle} \\
a_2 &= \frac{[3|K_{123}|4\rangle^2\,[3\,1]\,\langle 6\,4\rangle}{[1|K_{123}|6\rangle^2\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle} \\
a_4 &= \frac{[3|K_{234}|1\rangle^2[2|K_{234}|1\rangle}{[2\,4]^2\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,[2|K_{234}|5\rangle K^2_{234}} \\
a_5 &= \bar{a}_4|_{123456\longrightarrow 654321} \quad a_3 = \bar{a}_1|_{123456\longrightarrow 654321}
\end{aligned}
$$

Note

$$a_2 = \bar{a}_2|_{123456\longrightarrow 654321}$$

The amplitude contains a single three mass triangle, that with massive legs 61, 23 and 45. The coefficient of this is [23]

$$
\begin{aligned}
&b_{3m}^{[\{2^-3^+\},\{4^-5^+\},\{6^+1^-\}]} = \\
&\frac{[2\,6]\,[2|K_{35}|4\rangle[6|K_{35}|4\rangle}{[1\,2]\,[2|K_{45}|3\rangle[2|K_{34}|5\rangle t_{345}}\times\Bigg(\langle 1\,2\rangle\,[3\,5]+ \\
&\frac{[6|K_{345}|4\rangle(2s_{61}s_{45}+(s_{23}-s_{61}-s_{45})t_{345})}{2\,[6\,1]\,\langle 4\,5\rangle\,[2|K_{345}|3\rangle}\Bigg) \\
&+\frac{\langle 5\,1\rangle\,[3|K_{234}|5\rangle[3|K_{234}|1\rangle}{\langle 5\,6\rangle\,[4|K_{234}|5\rangle[2|K_{234}|5\rangle t_{234}}\times\Bigg(\langle 2\,4\rangle\,[6\,5]+ \\
&\frac{[3|K_{234}|1\rangle(2s_{23}s_{61}+(s_{45}-s_{23}-s_{61})t_{234})}{2\,[2\,3]\,\langle 1\,6\rangle\,[4|K_{234}|5\rangle}\Bigg) \\
&+\frac{[1\,3]\,\langle 6\,4\rangle\,[3|K_{123}|4\rangle}{[1\,2]\,\langle 5\,6\rangle\,[1|K_{123}|6\rangle t_{123}}\times\Bigg(\langle 1\,2\rangle\,[6\,5]+ \\
&\frac{[3|K_{123}|4\rangle(2s_{45}s_{23}+(s_{61}-s_{45}-s_{23})t_{123})}{2\,[2\,3]\,\langle 4\,5\rangle\,[1|K_{123}|6\rangle}\Bigg)
\end{aligned}
$$

$$-\frac{1}{\Delta_3}\Big(\langle 4\,2\rangle\langle 2\,1\rangle[3\,2][6\,5]$$
$$+\langle 4\,1\rangle\langle 2\,1\rangle[6\,1][3\,5]+\langle 4\,2\rangle\langle 4\,1\rangle[4\,5][3\,6]\Big)$$
$$\times\Big(2\frac{[2\,6]\langle 6\,5\rangle[3\,6]\langle 6\,4\rangle-\langle 5\,1\rangle[1\,2]\langle 4\,1\rangle[1\,3]}{[1\,2]\langle 5\,6\rangle[2|K_{61}|5\rangle}$$
$$+\frac{[1\,3]\langle 4\,6\rangle(t_{123}-t_{623})}{\langle 5\,6\rangle[1\,2][1|K_{23}|6\rangle}+\frac{[2\,6][2|K_{345}|4\rangle(t_{345}-t_{245})}{[1\,2][2|K_{34}|5\rangle[2|K_{45}|3\rangle}$$
$$+\frac{\langle 5\,1\rangle[3|K_{234}|5\rangle(t_{234}-t_{235})}{\langle 5\,6\rangle[2|K_{34}|5\rangle[4|K_{23}|5\rangle}\Big),$$

The bubble part of this amplitude is

$$c_{2,2}\,\mathrm{K}_0(-s_{23}/\mu^2)+c_{2,3}\,\mathrm{K}_0(-s_{34}/\mu^2)$$
$$+c_{2,4}\,\mathrm{K}_0(-s_{45}/\mu^2)+c_{2,6}\,\mathrm{K}_0(-s_{61}/\mu^2)$$
$$+c_{3,1}\,\mathrm{K}_0(-t_{123}/\mu^2)+c_{3,2}\,\mathrm{K}_0(-t_{234}/\mu^2)$$
$$+c_{3,3}\,\mathrm{K}_0(-t_{345}/\mu^2)$$

From the symmetry of the amplitude we have

$$c_{3,3}=\bar{c}_{3,2}|_{123456\longrightarrow 654321}\quad c_{2,4}=\bar{c}_{2,2}|_{123456\longrightarrow 654321}$$

so have five bubble functions which we must define. Firstly we have

$$c_{3,1}=\frac{-[3|K_{123}|4\rangle^2}{\langle 4\,5\rangle\langle 5\,6\rangle[1\,2][2\,3]\,t_{123}}\times$$
$$H_2(6,K_{123}|1];4,K_{123}|3],K_{123})$$
$$c_{3,2}=\frac{-[3|K_{234}|1\rangle^2}{\langle 5\,6\rangle\langle 6\,1\rangle[2\,3][3\,4]\,t_{234}}\times$$
$$H_3(5,K_{234}|2],K_{234}|4];1,K_{234}|3],K_{234}|3],K_{234})$$

and

$$c_{2,3}=\frac{[3|K_{234}|1\rangle^2}{\langle 5\,6\rangle\langle 6\,1\rangle[3\,4][2|K_{234}|5\rangle t_{234}}$$
$$\times H_2(3,K_{34}|2];4,K_{34}K_{56}|1];K_{34})+$$
$$\frac{[6|K_{345}|4\rangle^2}{\langle 3\,4\rangle[6\,1][1\,2][2|K345|5\rangle t_{345}}$$
$$\times H_2[3,5;4,K_{12}|6];K_{34}]$$

$$c_{2,2}=-\frac{\langle 2\,4\rangle^2[5\,6]^3}{\langle 2\,3\rangle[6\,1][1|K_{56}|4\rangle t_{561}}\times$$
$$H_2[3,K_{61}|5];2,4;K_{23}]$$
$$+\frac{1}{\langle 1\,6\rangle\langle 2\,3\rangle[4\,5]}\times$$
$$G_4[3,K_{23}|4],K_{61}|5],K_{23}K_{45}|6\rangle;$$
$$5;K_{23}|5],2,1,Y,Y;K_{45};K_{23}]$$
$$+\frac{[3|K_{123}|4\rangle^2}{\langle 4\,5\rangle\langle 5\,6\rangle[2\,3][1|K_{56}|4\rangle t_{123}}\times$$
$$\bar{H}_3[2,1,K_{45}|6\rangle;3,K_{56}|4\rangle,K_{56}|4\rangle;K_{23}]$$

where

$$|Y\rangle=|2\rangle[5|K_{24}|1\rangle+|3\rangle\langle 1\,2\rangle[3\,5]$$

and

$$c_{2,6}=$$
$$\frac{[6|K_{612}|4\rangle^2}{\langle 3\,4\rangle\langle 4\,5\rangle[6\,1][2|K_{612}|5\rangle t_{612}}\times$$
$$H_3[6,K_{612}|2],K_{61}K_{612}|3\rangle;$$
$$1,K_{61}K_{612}|4\rangle,K_{61}K_{612}|4\rangle;K_{61}]$$
$$+\frac{1}{\langle 1\,6\rangle\langle 2\,3\rangle[4\,5]}\times$$
$$G_3[6,K_{23}|4],K_{61}K_{612}|3\rangle;5;1,2,Z,Z;K_{45},K_{61}]$$
$$-\frac{[3|K_{612}|1\rangle^2}{\langle 6\,1\rangle[2\,3][3\,4][2|K_{612}|5\rangle t_{612}}\times$$
$$H_3[6,5,K_{234}|4];1,K_{234}|3],K_{234}|3];K_{61}]$$

where

$$|Z\rangle=-|1\rangle[5|K_{41}|2\rangle-|6\rangle[5\,6]\langle 1\,2\rangle$$

8. $A_6^{[0]}(++++++)$

This first of the scalar amplitudes is relatively simple: it vanishes at tree level and consequently is purely rational at one-loop. It was originally deduced by examining collinear limits in ref. [2] and consequently proven to be correct in [1] using off-shell recursion. The general n-point form is

$$A_n(1^+\cdots n^+)=\frac{ic_\Gamma}{12}\frac{E_n+O_n}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

where

$$E_n = -\sum_{1\le i_1<i_2<i_3<i_4\le n} \mathrm{tr}(\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4})$$
$$O_n = -\sum_{1\le i_1<i_2<i_3<i_4\le n} \mathrm{tr}(\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}\gamma_5)$$

The two trace terms can obviously be combined as $2\,\mathrm{tr}_+(\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4})$.

For the six-point amplitude the above sum yields twelve terms.

9. $A_6^{[0]}(-+++++)$

This amplitude also vanishes at tree level and consequently is purely rational at one-loop. It was first calculated in [1] using off-shell recursion. We present the form [24] which was obtained using on-shell recursion

$$A_6^{[0]}(1^-2^+3^+4^+5^+6^+) = \frac{ic_\Gamma}{6}\Bigg[\frac{[6|K_{23}|1\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 4\,5\rangle^2[6|K_{12}|3\rangle t_{123}} + \frac{[2|K_{34}|1\rangle^3}{\langle 3\,4\rangle^2\langle 5\,6\rangle\langle 6\,1\rangle[2|K_{34}|5\rangle t_{234}} + \frac{[2\,6]^3}{[1\,2][6\,1]t_{345}}\left(\frac{[2\,3][3\,4]}{\langle 4\,5\rangle[2|K_{34}|5\rangle} - \frac{[4\,5][5\,6]}{\langle 3\,4\rangle[6|K_{12}|3\rangle} + \frac{[3\,5]}{\langle 3\,4\rangle\langle 4\,5\rangle}\right) - \frac{\langle 1\,3\rangle^3[2\,3]\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle} + \frac{\langle 1\,5\rangle^3\langle 4\,6\rangle[5\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle^2\langle 5\,6\rangle^2} + \frac{\langle 1\,4\rangle^3\langle 3\,5\rangle[4|K_{23}|1\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^2\langle 4\,5\rangle^2\langle 5\,6\rangle\langle 6\,1\rangle}\Bigg].$$

10. $A_6^{[0]}(---+++)$

Surprisingly this NMHV amplitude is the simplest of the "non-trivial" scalar amplitudes. A general form is known and is

$$A_n^{[0]}(1^-2^-3^-4^+5^+\cdots n^+) = \frac{1}{3}A_n^{\mathcal{N}=1\text{ chiral}} - \frac{ic_\Gamma}{3}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{2,r}]}{t_{2,r}^3} - \frac{ic_\Gamma}{3}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_2[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}^3} - \frac{ic_\Gamma}{3}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}^3} + c_\Gamma\hat{R}_n\,,$$

The functions

$$\mathrm{L}_2(r) = \frac{\ln(r)-(r-r^{-1})/2}{(1-r)^3}$$
$$\mathrm{L}_1(r) = \frac{\ln(r)-(1-r)}{(1-r)^2}$$

are non-singular as $r \longrightarrow 1$. Using these functions includes a part of the rational terms into the integral functions. $\hat{R}_n$ is the remaining rational terms CITE.

For the six-point the amplitude reduces to

$$A_6^{[0]}(1^-2^-3^-4^+5^+6^+) = \frac{1}{3}A^{N=1} + -\frac{i}{2}\left[c_1\frac{\mathrm{L}_2[t_{345}/s_{61}]}{s_{61}^3} + c_2\frac{\mathrm{L}_2[t_{234}/s_{34}]}{s_{34}^2} + c_3\frac{\mathrm{L}_2[t_{234}/s_{61}]}{s_{61}^3} + c_4\frac{\mathrm{L}_2[t_{345}/s_{34}]}{s_{34}^3}\right] + c_\Gamma\hat{R}_6$$

where the coefficients are

$$c_1 = \frac{[6|Kk_2K|3\rangle[6|k_2|3\rangle[6|(k_2K-Kk_2)K|3\rangle}{[2|K|5\rangle[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle}$$

where $K = K_{345}$ and

$$c_2 = c_1|_{123456\longrightarrow 321654} \quad c_4 = c_3|_{123456\longrightarrow 321654}$$
$$c_3 = \bar{c}_1|_{123456\longrightarrow 456123}$$

and the rational term is

$$\hat{R}_6 = X_6 + X_6|_{123456\longrightarrow 321654}$$

where

$$X_6 = \frac{i}{6}\frac{1}{[2\,3]\,\langle 5\,6\rangle\;[2|K_{34}|5\rangle}\Bigg\{-\frac{[4\,6]^3\,[2\,5]\,\langle 5\,6\rangle}{[1\,2]\,[3\,4]\,[6\,1]}$$
$$-\frac{\langle 1\,3\rangle^3\,\langle 2\,5\rangle\,[2\,3]}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle}-\frac{\langle 1\,3\rangle^2(3[4|K_2|1\rangle+[4|K_3|1\rangle)}{\langle 3\,4\rangle\,\langle 6\,1\rangle}$$
$$+\frac{[4|K_{23}|1\rangle^2}{[3\,4]\,\langle 6\,1\rangle}\left(\frac{[4|K_2|1\rangle-[4|K_5|1\rangle}{t_{234}}+\frac{\langle 1\,3\rangle}{\langle 3\,4\rangle}-\frac{[4\,6]}{[6\,1]}\right)$$
$$+\frac{[4\,6]^2(3[4|K_5|1\rangle+[4|K_6|1\rangle)}{[3\,4]\,[6\,1]}\Bigg\}$$

11. $A^{[0]}(--++++)$

There exist general expressions for the MHV scalar amplitudes [25,12]. Within these the case of adjacent negative helicities simplifies enormously and we have

$$A^{[0]}(1^-2^-3^+4^+5^+6^+) = \frac{1}{3}A_6^{N=1}+\frac{2c_\Gamma}{9}A_6^{\rm tree}+C_6+c_\Gamma\hat{R}_6$$

where

$$C_6 = -\frac{c_\Gamma A^{\rm tree}}{3s_{12}^2}\Big(c_4\frac{{\rm L}_2(-s_{23}/-t_{234})}{t_{234}^3}$$
$$+c_5\frac{{\rm L}_2(-t_{561}/-s_{61})}{s_{61}^3}\Big)$$

with

$$c_m = {\rm tr}[k_1k_2k_mq_{m,2}]\,{\rm tr}[k_1k_2q_{m,2}k_m]$$
$$\times\,{\rm tr}[k_1k_2(q_{m,2}k_m-k_mq_{m,2})]$$

and where

$$\hat{R}_6 = \frac{1}{6}\Bigg\{-2\frac{\langle 3\,5\rangle\,[3\,5]\,[3|K_{12}|4\rangle[4|K_{12}|6\rangle[6|K_{12}|5\rangle}{[1\,2]\,\langle 3\,4\rangle^2\,\langle 4\,5\rangle^2\,[6\,1]\,[2|K_{34}|5\rangle[3|K_{12}|6\rangle}$$
$$-2\frac{\langle 3\,5\rangle\,[3\,6]\,[6|K_{12}|4\rangle^2}{[1\,2]\,\langle 3\,4\rangle^2\langle 4\,5\rangle^2\,[6\,1]\,[2|K_{34}|5\rangle}$$
$$+2\frac{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 3\,5\rangle\,[3\,5]^2\,[5\,6]\,[6|K_{(1+2)}|5\rangle}{\langle 3\,4\rangle^2\,\langle 4\,5\rangle\,[6\,1]\,[5|K_{16}|2\rangle[2|K_{34}|5\rangle[3|K_{12}|6\rangle}$$
$$+2\frac{\langle 1\,2\rangle^2\,[3\,5]^2\,\langle 5|K_{34}K_2|1\rangle+\langle 5|K_3K_5|1\rangle}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle\,[5|K_{16}|2\rangle[2|K_{34}|5\rangle[3|K_{12}|6\rangle}$$
$$-\frac{\langle 1\,2\rangle^3\,\langle 3\,5\rangle\,[4\,6]\,[5\,6]}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,[4|K_{23}|1\rangle[6|K_{12}|3\rangle}$$
$$+2\frac{[3\,6]^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle^2\,[6\,1]}$$
$$-\frac{[5\,6]\,[6|K_{12}|5\rangle^2(2\langle 4|K_{35}K_{12}|5\rangle+\langle 1\,2\rangle\,[1\,2]\,\langle 4\,5\rangle)}{[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle^2\,\langle 5\,6\rangle\,[6\,1]\,[6|K_{12}|3\rangle[2|K_{34}|5\rangle}$$
$$+2\frac{\langle 1\,5\rangle^2[3\,4]^2\,[5\,6]\,(\langle 1\,6\rangle\,[3\,4]\,\langle 4\,5\rangle-[3|K_{24}|1\rangle\,\langle 5\,6\rangle)}{[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle^2t_{234}[4|K_{23}|1\rangle[2|K_{34}|5\rangle}$$
$$-\frac{\langle 1\,2\rangle\,\langle 1\,5\rangle\,[3\,4]\,[5\,6]\,\langle 1|K_{56}K_{34}|5\rangle}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,t_{234}[4|K_{23}|1\rangle[2|K_{34}|5\rangle}$$
$$+2\frac{\langle 3\,5\rangle\,[3|K_{24}|1\rangle^3}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,t_{234}[2|K_{34}|5\rangle}$$
$$-\frac{\langle 1\,2\rangle\,[3|K_{24}|1\rangle(2[3|K_{24}|1\rangle+[3|K_4|1\rangle)}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,t_{234}}$$
$$+2\frac{\langle 1\,2\rangle^3[4\,6]^2[5|K_{46}|5\rangle}{\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,t_{123}[4|K_{23}|1\rangle[6|K_{12}|3\rangle}$$
$$+2\frac{\langle 1\,2\rangle^3[3\,5]^2[4|K_{35}|4\rangle}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 6\,1\rangle\,t_{612}[5|K_{16}|2\rangle[3|K_{12}|6\rangle}$$
$$-\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle}\left[\frac{[3|K_4|1\rangle}{[2\,3]}+\frac{[6|K_5|2\rangle}{[6\,1]}\right]\Bigg\}$$

$++++++$	$-+++++$	$--++++$	$---+++$	$-+-+++$
1088	1541	5210	1837	20733

Table 2
LeafCount of the rational terms of some of the amplitudes. This is the LeafCount of the terms expressed as polynomial in $\lambda_a, \bar{\lambda}_{\dot{a}}$ without any attempt at simplification. As such it is of comparative use only. The LeafCount of the MHV tree amplitude, expressed equivalently is about 100

12. $A^{[0]}(-+-+++)$ **and** $A^{[0]}(-++-++)$

The amplitude of course splits into cut-constructible and rational pieces. The cut constructible part of the amplitudes is just, for a MHV amplitude with negative helicities i and j

$$A^{\rm tree}\times\Bigg(\sum(b_{ij}^{n_1n_2})^2\mathcal{F}_4^{2me}+\sum\hat{e}_{m,a}^{ij}\frac{{\rm L}_0(s/s')}{s'}$$
$$+\sum\hat{f}_{m,a}^{ij}\frac{{\rm L}_1(s/s')}{(s')^2}+\sum g_{m,a}^{ij}\frac{{\rm L}_2(s/s')}{(s')^3}\Bigg)+\hat{R}_6$$

The box coefficients satisfy

$$b^{[0]} = \frac{(b^{N=1})^2}{b^{N=4}}$$

a feature which is shared by the $A(--+-++)$ and $A(-++-+-+)$ amplitudes but which needs extending for $n > 6$ point amplitudes [26]. The coefficients of the L_i take the form,

$$\begin{aligned}
\hat{e}^{ij}_{m,a} &= \frac{1}{3} c^{ij}_{m,a} + I^{ij}_{m,a} \\
\hat{f}^{ij}_{m,a} &= \frac{1}{3} S^{ij}_{m,m+a} \\
\hat{g}^{ij}_{m,a} &= \frac{1}{3} \frac{c^{ij}_{m,a}}{s^2_{ij}} \,\mathrm{tr}(k_i k_j k_m q_{m,a})\,\mathrm{tr}(k_i k_j q_{m,a} k_m)
\end{aligned}$$

where

$$\begin{aligned}
S^{ij}_{m,a} &= \frac{\mathrm{tr}_+(k_i k_j k_m k_{a+1})\,\mathrm{tr}_+(k_i k_j k_{a+1} k_m)}{s^2_{am+1}} \\
&- \frac{\mathrm{tr}_+(k_i k_j k_m k_a)\,\mathrm{tr}_+(k_i k_j k_a k_m)}{s^2_{am}} \\
I^{ij}_{m,a} &= \frac{\mathrm{tr}^2_+(k_i k_j k_m k_{a+1})\,\mathrm{tr}^2_+(k_i k_j k_{a+1} k_m)}{s^3_{am+1}} \\
&- \frac{\mathrm{tr}_+(k_i k_j k_m k_a)\,\mathrm{tr}_+(k_i k_j k_a k_m)}{s^3_{am}}
\end{aligned}$$

For the six-point MHV amplitudes the summations of the L_i functions run over the same variables as in eq. (6.1) and eq. (6.2).

The rational terms for these two amplitude we do not produce here. They are quite extensive. The `LeafCount` of these expression (as a naive rational function of $|k^{\pm}\rangle$) is given in table 2. For comparison, the six-point MHV tree amplitude has a `LeafCount` of about 100. The `LeafCount` of expressions is very sensitive to the way the functions are presented. For example the general expression for the rational terms of $A(1^-2^-3^+\cdots n^+)$ when specialised to $n = 6$ has a `LeafCount` of 34642 as opposed to 5210 for the specialised form. The `Mathematica` expressions of these will be available to download.

13. $A^{[0]}(--+-++)$ and $A^{[0]}(-+-+-+)$

The cut-constructible parts of these amplitudes were calculated in ref. [10] using fermionic integration unitary methods. The rational terms were calculated using Feynman diagrams in ref. [13] with a particular notation. We shall not produce these amplitudes here. The expressions are rather complicated and we have no innovative way to present these. The results have been checked by comparison to the results for the six-gluon obtained using numerical methods [27].

14. Summary

It is hoped that bringing together as much as practical, at this point, of the six-gluon amplitude is a useful exercise. The amplitudes presented here will be available from *http://pyweb.swan.ac.uk/~dunbar/sixgluon.html* in `Mathematica` format. The amplitudes available there been tested against the numerical results of ref. [27] and as such should be error free. It is intended to correct these as any typographic (or other) problems arise.

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