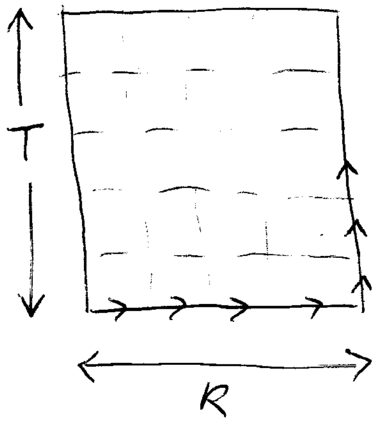
Strong Coupling Analysis

We can use the group integration rules to develop an alternative series expansion – one in powers of $\beta = \frac{2N}{g^2}$ – this is called a strong coupling series — in statistical mechanics the analogous expansion is known as the <u>high temperature series</u>, since the terms are ordered by powers of $(kT)^{-1}$

We will calculate the Wilson loop expectation $\langle W(R,T)\rangle$ to leading order in β, ie the trace of an ordered product of links around a rectangular $R \times T$ contour Γ



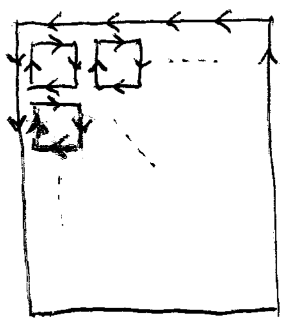
ie. $$W(R,T) = \frac{1}{N} \operatorname{tr} \mathcal{P} \prod_{\langle x\, x+\hat{\mu}\rangle \in \Gamma} U_\mu(x)$$

$$\langle W\rangle = \frac{1}{Z}\int DU \; W[U] \exp\left(\frac{\beta}{N}\sum_{x\mu\nu} \operatorname{Re} \operatorname{tr} U_{\mu\nu}(x)\right)$$

Now, to $O(\beta^0)$, the $\exp(\;) = 1$, so $\langle W\rangle$ factorises into

$$\operatorname{tr} \mathcal{P} \prod \int dU \; U = 0$$

we'll only get non-vanishing terms by expanding exponential to bring down products of U's from the action. The lowest non-trivial term corresponds to a "tiling" of Γ with plaquettes



$\Rightarrow$ brings down a factor $\left(\frac{\beta}{2N}\right)^{RT} \times \frac{1}{(RT)!}$ from action ... but there are $(RT)!$ possible ways of choosing the terms

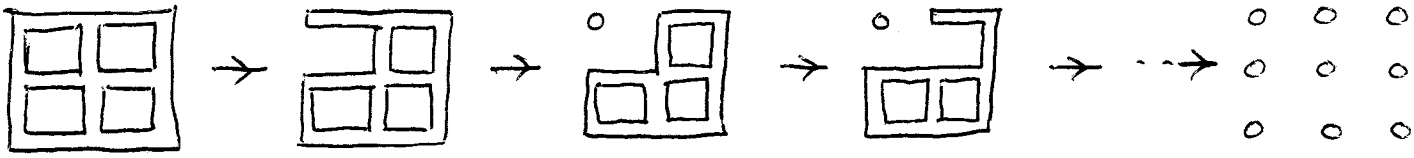
$$\Rightarrow \left(\frac{\beta}{2N}\right)^{RT}$$

Now do $\int dU$ on each $\uparrow\downarrow$ link: there are $T(R+1) + R(T+1)$ of them

$$\int dU \; U_{ij} U^+_{kl} = \frac{1}{N}\delta_{il}\delta_{jk} \qquad \uparrow\downarrow \;\Longrightarrow\; \begin{matrix}\smile \\ \frown\end{matrix}$$

$\Rightarrow$ $\int DU$ yields a numerical factor $\left(\frac{1}{N}\right)^{2RT+T+R}$

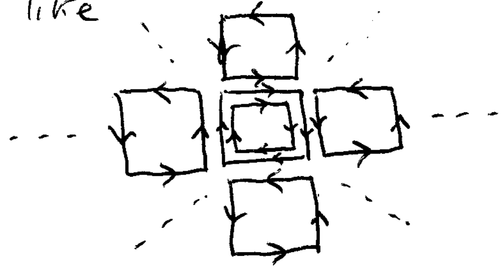
Diagramatically, successive $\int dU$ looks like...



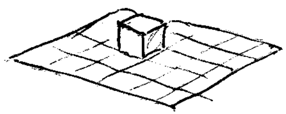
Now, each $\circ$ looks like $\delta_{ij}\delta_{ji} = \delta_{ii} = N$
There are $(T+1)(R+1)$ such points $\Rightarrow$ factor $(N)^{RT+R+T+1}$

$$\Rightarrow \quad \langle W(\Gamma)\rangle = \frac{1}{N}\left(\frac{\beta}{2N}\right)^{RT}\left(\frac{1}{N}\right)^{2RT+T+R} N^{RT+T+R+1} = \left(\frac{\beta}{2N^2}\right)^{RT}$$
$$= \left(\frac{1}{g^2 N}\right)^{RT}$$
where g is Yang-Mills coupling constant

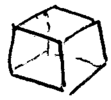
Exercise: for $N=3$ calculate the next-to-leading contribution; this comes from considering tilings like



Still higher orders: non-planar excitations



disconnected contributions



etc.

The main result is that all successive contributions depend on the product RT, ie. the <u>area</u> of the loop
$\Rightarrow$ can write $W(R,T) = \exp(-KRT)$

with the <u>string tension</u> $K = -\frac{1}{a^2}\left[\ln\left(\frac{\beta}{2N^2}\right) + O(\beta)\right]$

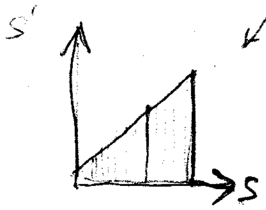
Path-Ordered exponential

$$U_P(z,y) = P\left\{\exp\left[ig\int_0^1 ds\, \frac{dx^\mu}{ds} A_\mu^a(x(s))\, t^\alpha\right]\right\}$$

$$= 1 + ig\int_0^1 ds\, \frac{dx^\mu}{ds} A_\mu^a(x(s))\, t^\alpha$$

$$- \frac{g^2}{2!}\, 2\int_0^1 ds \int_0^s ds'\, \frac{dx^\mu}{ds}\frac{dx^\nu}{ds'}\, A_\mu^\alpha(x(s))A_\nu^\beta(x(s'))\, t^\alpha t^\beta$$

$$+ \cdots$$



geometrical factor ...

Conclusion: $\langle W(R,T)\rangle \propto \langle \bar{Q}(0,0)\,U\,Q(R,0) \quad \bar{Q}(R,T)\,U\,Q(0,T)\rangle$

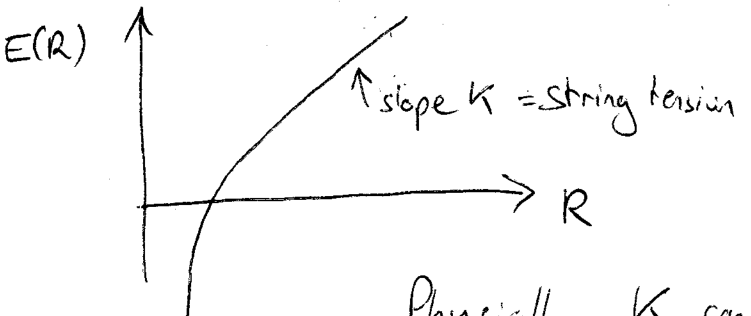
ie the correlator of a heavy quark meson $(c, b, t \ldots)$ with spatial separation R.

(alternatively, think of Wilson loop as the world-line of a $Q-\bar{Q}$ virtual pair)

From 1st lecture, we expect this to decay as $e^{-E(R)T}$ where $E(R)$ is energy of separation, ie the <u>static quark potential</u>

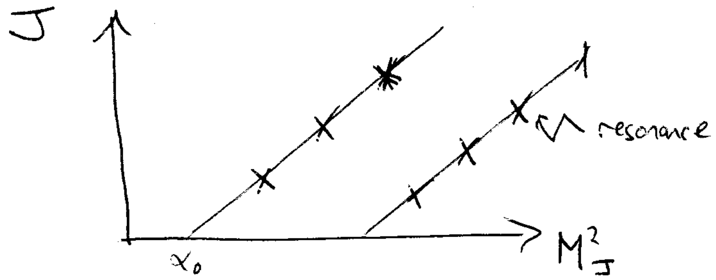
{ If you think all this a bit hand-waving, then consult a book on the <u>transfer matrix</u> in Euclidean field theory. Strictly, the correlator decays as $\sum_i c_i\, e^{-E_i(R)T}$, where E_i are the eigenvalues of the "lattice Hamiltonian". Only as $T \to \infty$ will the ground state energy dominate. }

So, the area law Wilson loop decay $\Rightarrow$ $E(R) = KR$



$\Rightarrow$ the area law is a mathematical signal for <u>color confinement</u>

Physically, K can be estimated from <u>Regge trajectories</u>



$$J = \alpha_0 + \alpha' M_J^2 .$$

with $\alpha' = \dfrac{1}{2\pi K}$

= weight of 3 elephants. $\quad K \sim 0.2\,(\mathrm{GeV})^2$ in string model of hadrons

ie $\sqrt{K} \sim 440$ MeV

Exercise: The string model of hadrons assumes that a meson consists of a $\bar{Q}Q$ pair rotating about a common centre & moving at speed c, connected by a string of flux of tension $\equiv$ rest energy per unit length K.

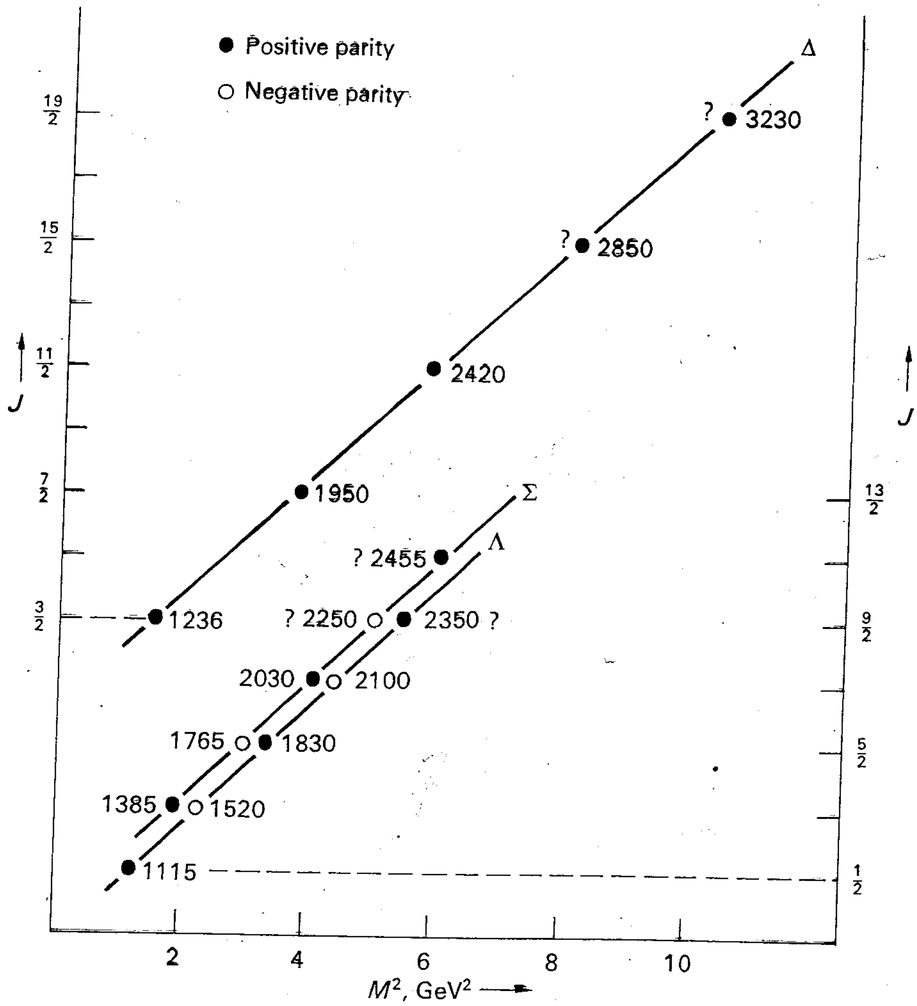


Fig. 4.40 Chew-Frautschi plots of fermion Regge trajectories. The trajectory marked Δ consists of the sequence $I = \frac{3}{2}$, $S = 0$, and $J^P = \frac{3}{2}^+, \frac{7}{2}^+, \frac{11}{2}^+, \ldots$; that marked Λ, of the sequence $I = 0$, $S = -1$, $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \ldots$; and that marked Σ, of the sequence $I = 1$, $S = -1$, $J^P = \frac{3}{2}^+, \frac{5}{2}^-, \frac{7}{2}^+, \ldots$. Resonances for which the spin-parity is not firmly established are indicated by a question mark.

ation is depicted for the 32^4 lattice at $\beta = 6.4$ in Fig. 1 where the corrected data points are plotted together with the interpolating fit $V(R) = V_0 + KR - e/R + f/R^2$, with fit parameters V_0, K, e, and f as given in Table II. Our potential fits yield $\chi^2/N_{\mathrm{DF}} < 1$ as long as the first two[3]

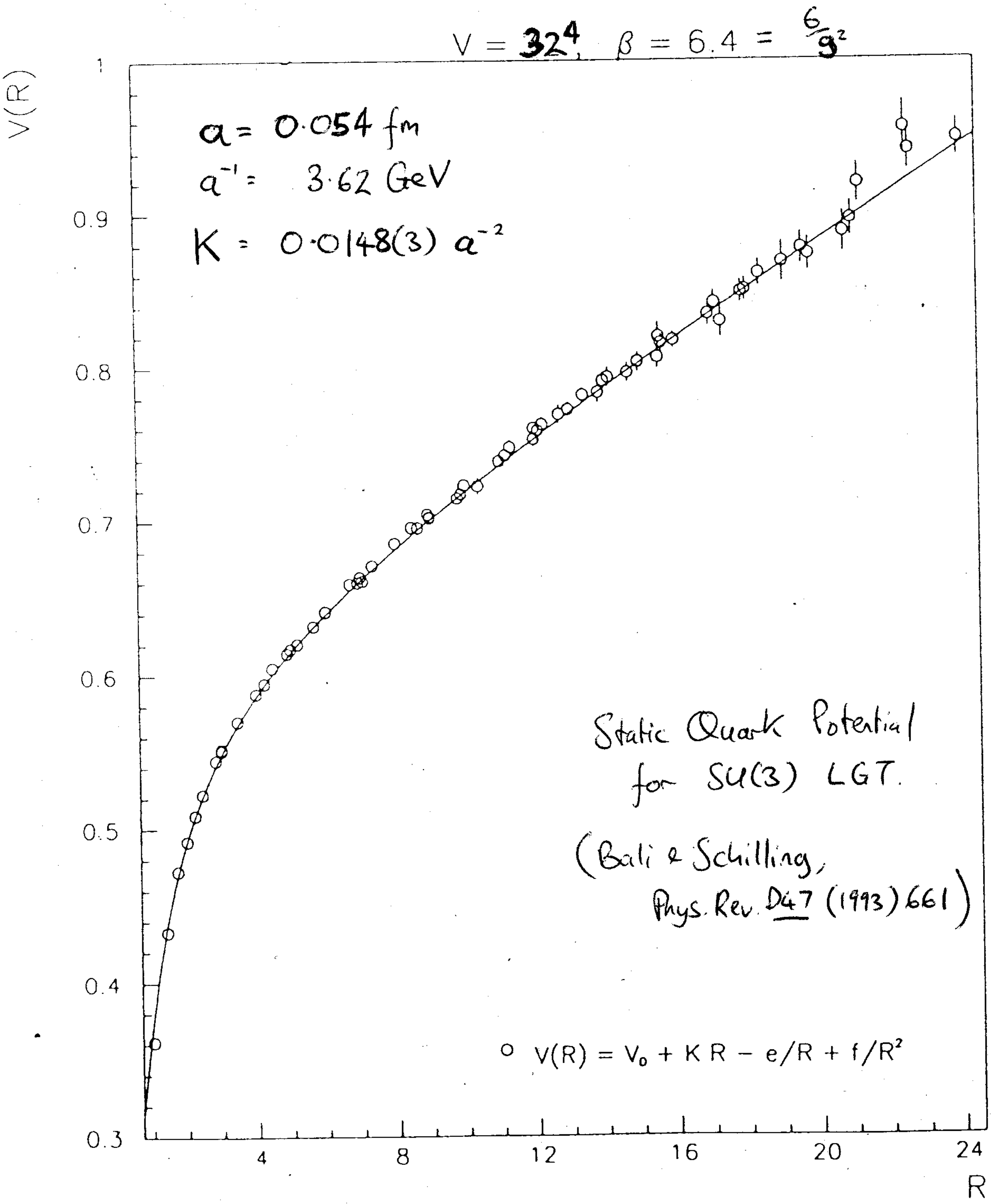


FIG. 1. The $q\bar{q}$ potential at $\beta = 6.4$ (in lattice units). The data points have been corrected for the lattice Coulomb propagator [Eq. (8)]. The fit parameters are contained in Table II.

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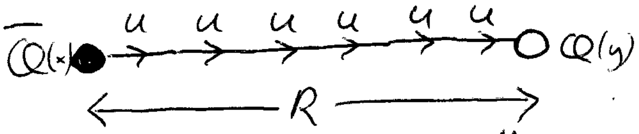
Physical Interpretation of String Tension :

So far we have not considered quarks. Suppose I introduce infinitely massive spinless quarks Q into the quantum theory. The dynamics of infinitely massive particles are not very interesting — they just sit there. In this case, however, the quark has gauge degrees of freedom, and undergoes local gauge transformations as follows:
↳ ie. Q is an N-vector, it transforms in fundamental rep' of $SU(N)$

quark $\quad Q(x) \mapsto \Omega(x)\, Q(x)$; anti-quark $\bar{Q}(x) \mapsto \bar{Q}(x)\, \Omega^\dagger(x)$

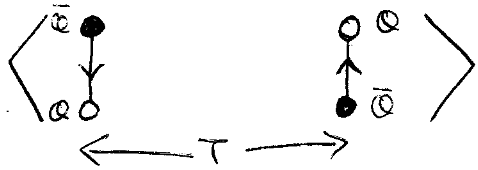
Now, by Elitzur's theorem, isolated Q's and $\bar{Q}$'s have vanishing quantum expectations (as they are gauge non-singlet). However, consider a $\bar{Q}\,Q$ system separated by distance R. A gauge-invariant operator which projects onto this is



product of U matrices $\simeq$ $P\exp \int_x^y ig\, \underline{A}\cdot \underline{dl}$ Schwinger line integral

(Peskin & Schroeder 15.56)

Now consider the expectation



This has a formal similarity with the amplitude to create a $\bar{Q}Q$ "meson" at time 0 and destroy it at time T. We can integrate out the Q fields by Wick contracting the $\bar{Q}\,Q$ pairs lengthwise

$$Q(0)\,\bar{Q}(T) \sim S(0,T) \quad \text{Euclidean propagator.}$$

Since the quarks are massive, we ~~take~~ take the <u>static limit</u> in which only motion along the Euclidean ~~time~~ time direction is retained. Then recalling that in gauge theories translations in gauge theories are generated by ~~exp~~ $i(p_\mu - gA_\mu)$, we arrive at

$$S(0,T) \sim e^{i(p_4 - gA_4)T} \sim e^{-MT}\, P\exp i\int_0^T g\, \underline{A}_4 \cdot dt$$

$$\sim P\prod U_4(0)\, U_4(0+\hat{t})\, U_4(0+2\hat{t}) \ldots U_4(T-1)$$

By evaluating relativistic expressions for the mass M and angular momentum J of the string, prove the relation

$$J = \frac{1}{2\pi K} M^2$$

By measuring K in lattice units (ie. by measuring the dimensionless quantity Ka^2) we callibrate the lattice

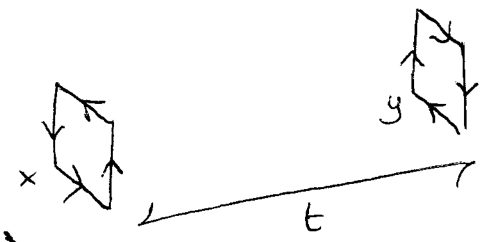
eg. for data at $\beta = 6.4$:
physical $K \sim 0.2\,(\mathrm{GeV}^2) \Rightarrow a^{-1} = 3.62$ GeV

$$\left(\text{ie. } (0.440\, a)^2 = 0.0148 \Rightarrow a^{-1} = \frac{.440}{\sqrt{0.0148}} = 3.62 \text{ GeV} \right)$$

$$1\,\mathrm{fm} \simeq (197\,\mathrm{MeV})^{-1} \Rightarrow a = 0.054\,\mathrm{fm}$$

Therefore, if we calculate the mass of any other observable in lattice units, we can use the value for a to convert into physical units

eg. the Glueball



Expect by transfer matrix arguments $\cdot \langle U_{\mu\nu}(x)\, U^{+}_{\mu\nu}(y) \rangle \sim \sum_i |\langle 0 | U_{\mu\nu} | i \rangle|^2 e^{-E_i t}$

where E_i is the energy of state $|i\rangle$

$\Rightarrow$ Timeslice propogator

$$\sum_{\underline{x}\,\underline{y}} \langle U_{\mu\nu}(\underline{x},0)\, U^{+}_{\mu\nu}(\underline{y},t) \rangle \sim \sum_i c_i\, e^{-m_i t}$$

For $t \to \infty$, correlation dominated by lightest state - sometimes called the mass gap of the theory.

Can use different combinations of gauge invariant operators to project onto states with different "quantum numbers"

$\Rightarrow \Rightarrow$ Results (to cut a long story short)
(from numerical simulations)

mass of scalar 0^{+} ~1.5–1.7 GeV
" " tensor 2^{+} ~ 2 GeV
pseudovector 1^{+} ~ 2.6 GeV

Another example (far too brief, unfortunately)

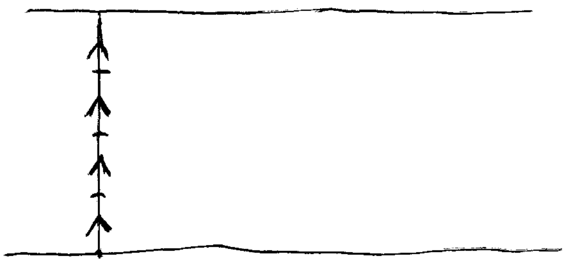
It can be shown that $Z = \mathrm{tr}\,(\mathcal{T}^{N_t}) = \mathrm{tr}\,(e^{-aN_t H})$
where N_t is the number of lattice spacings in the timelike direction, $\mathcal{T}$ is the transfer matrix, and H is a "Hamiltonian" operating on some Hilbert space which governs evolution through imaginary time:

$$\mathcal{T} \simeq \exp(-aH)$$

But in classical statistical mechanics, $Z = \mathrm{tr}\,(e^{-H/kT})$

$\Rightarrow$ if $N_t < \infty$, $T > 0$ ie non-zero temperature on finite lattice

$\Rightarrow$ can use LGT to explore thermal field theory.



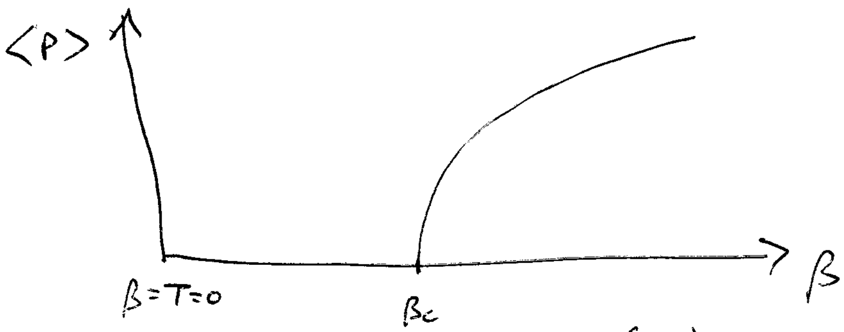
Now consider
$$P(\underline{x}) = \mathrm{tr}\,\mathcal{P}\prod_{t=1}^{N_t} U_4(\underline{x}, t)$$

known as Wilson line or Polyakov line

P is a loop which closes on itself around the time boundary.
Elitzur's theorem does not apply, so in general $\langle P \rangle \neq 0$.
We can interpret $\langle P \rangle \propto e^{-FN_t a}$ where F is free energy of an isolated quark, ie isolated color source.

$\langle P \rangle = 0 \Rightarrow F = \infty \Rightarrow$ color confinement

$\langle P \rangle \neq 0 \quad \Rightarrow \quad F$ is finite $\Rightarrow$ deconfinement. (quark/gluon plasma)



For a given N_t, find $a(\beta_c) \Rightarrow kT_c = N_t\, a(\beta_c)$

$\Rightarrow$ can estimate T_c for <u>deconfining phase transition</u>

Numerical simulation: $\dfrac{T_c}{\sqrt{K}} \sim 0{\cdot}6 \Rightarrow T_c \simeq 250$ MeV.

($\sim 10^{12}$K $\sim$ 1s after Big Bang!)