Spacetime : Lecture 3

Penrose Limits

$$x \longrightarrow x'$$

finite range

$$ds^2 \longrightarrow \Omega(x')\, \underbrace{ds'^2}_{\text{Minkowski space}}$$

"conformal factor" → diverge at the boundaries (at ∞)

Light-cones are always at 45°

Choose special coordinates — class of observers in de Sitter for which spatial sections — flat

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}\, d\hat{x}_i d\hat{x}_i \qquad i = 1 \ldots p+1$$

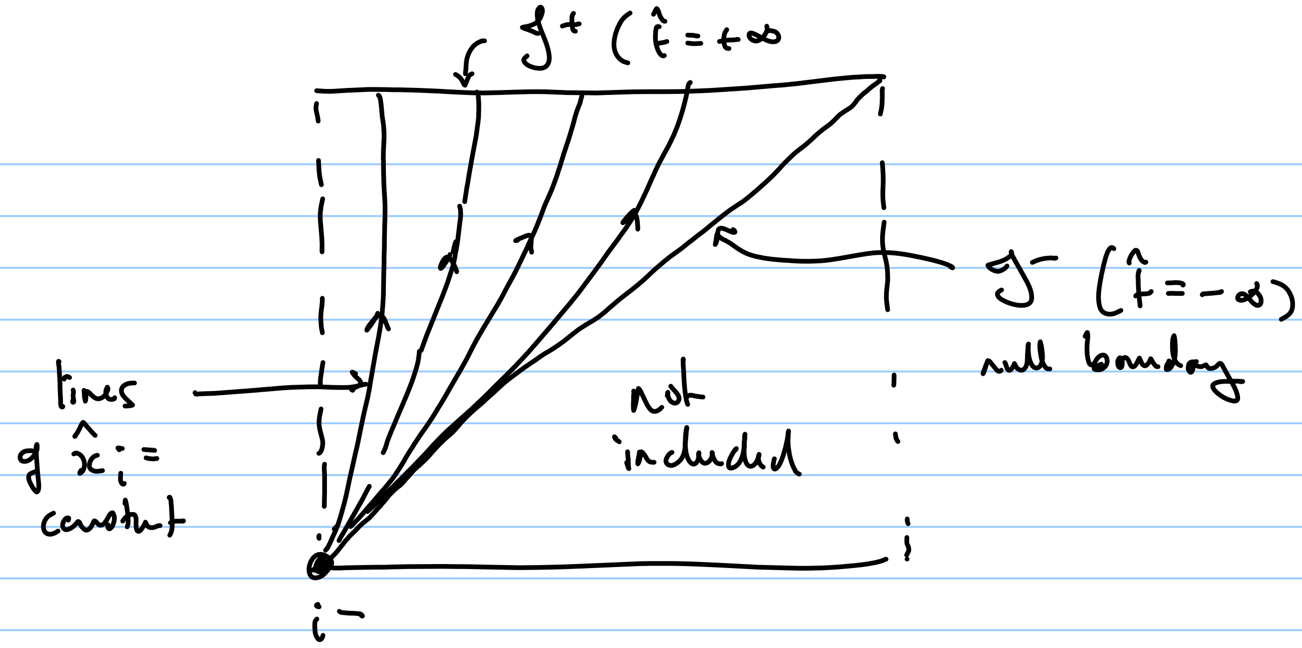
$$e^{\hat{t}/\alpha} = \frac{1}{2}\left(\underbrace{\sinh(t/\alpha) + \cosh(t/\alpha)\cos\chi}\right)$$

$$= \sec(t')\left(\sin(t') + \cos(\chi)\right)$$

$$\hat{x}_i = \frac{\alpha\, \Omega_i}{\sinh(t/\alpha) + \cosh(t/\alpha)\cos\chi} \qquad i = 1 \ldots p+1$$

S^{p+1}

but since $-\infty < \hat{t} \leq \infty$



"steady-state universe"

Anti-de Sitter $\quad AdS_{p+2}$

$\Lambda = -3/\alpha^2 < 0$

$$ds^2 = -\cosh^2(r/\alpha)\, dt^2 + dr^2 + \alpha^2 \sinh^2(r/\alpha)\, d\Omega_p^2$$

$-\infty \leq t \leq \infty \qquad 0 \leq r \leq \infty \qquad S^p$

when r is small

$dr^2 + r^2 d\Omega_p^2 \rightarrow$ $p+1$ sphere with r the polar angle

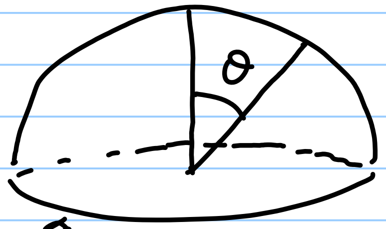
To study the causal structure we can define

$$\tan\theta = \sinh(r/\alpha)$$

$0 \leq \theta \leq \pi/2 \qquad 0 \leq r \leq \infty$

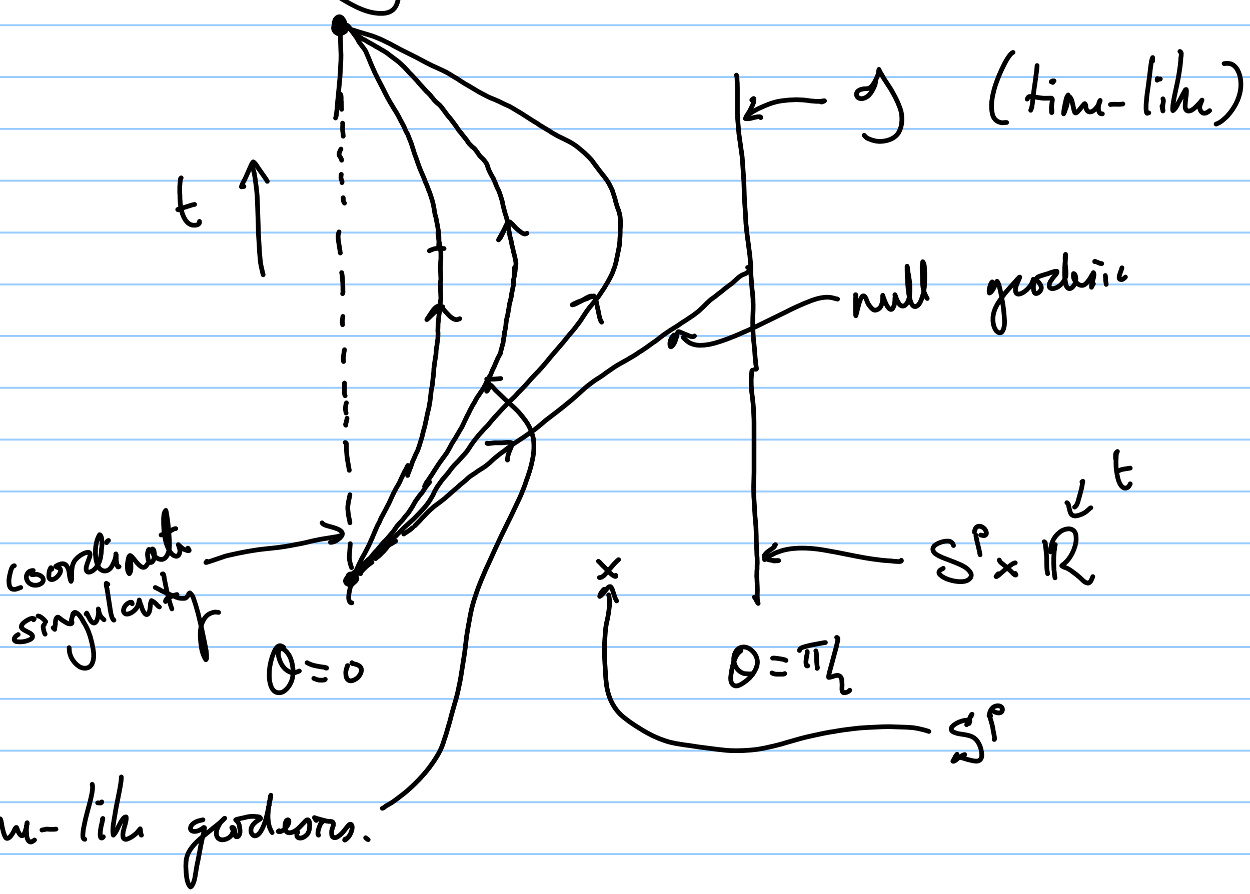
$$ds^2 = \frac{\alpha^2}{\cos^2\theta}\Big[-\frac{dt^2}{\alpha^2} + \underbrace{d\theta^2 + \sin^2\theta\, d\Omega_p^2}_{S^{p+1}}\Big]$$

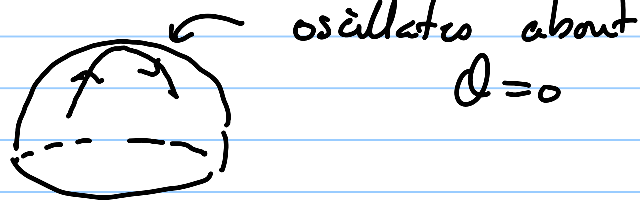
diverges at $\theta = \pi/2$



a boundary at $\theta = \pi/2$ the equator

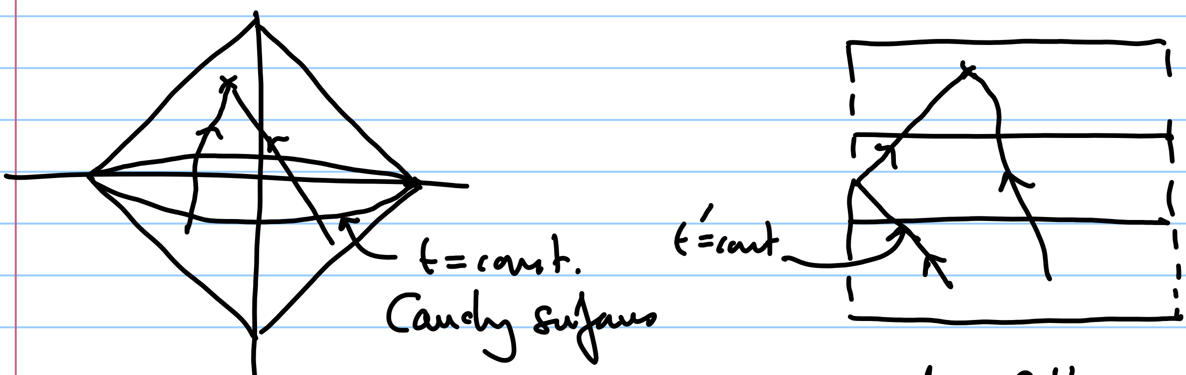
Penrose Diagram in (t, θ)





time-like geodesic can never reach the boundary.

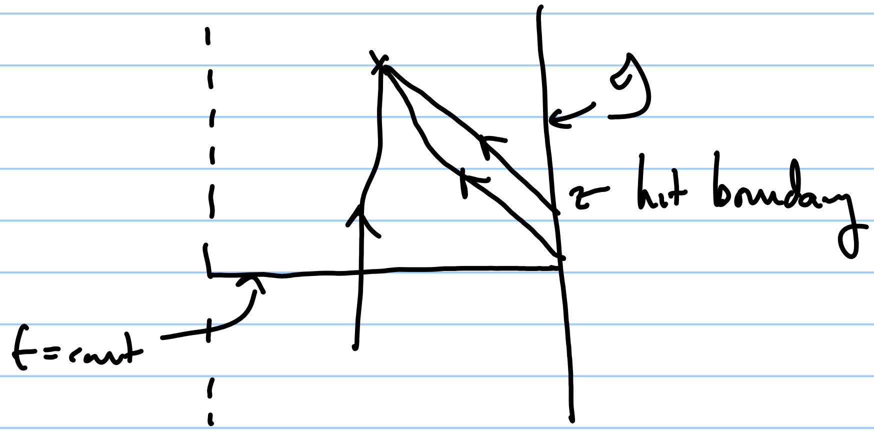
Cauchy surfaces:



Minkowski   de-Sitter

both have Cauchy surfaces

Anti-de Sitter



So in order to have a well-defined initial-value problem need to specify boundary conditions at $\mathscr{I}$. (c.f. AdS/CFT)

(t, r, Ω_p) – "global coordinates"
↳ cover the whole space

$r \updownarrow 0$ boundary $\mathscr{I}$ $S^p \times \mathbb{R}$

(u, τ, x^i) – "Poincaré coordinates"

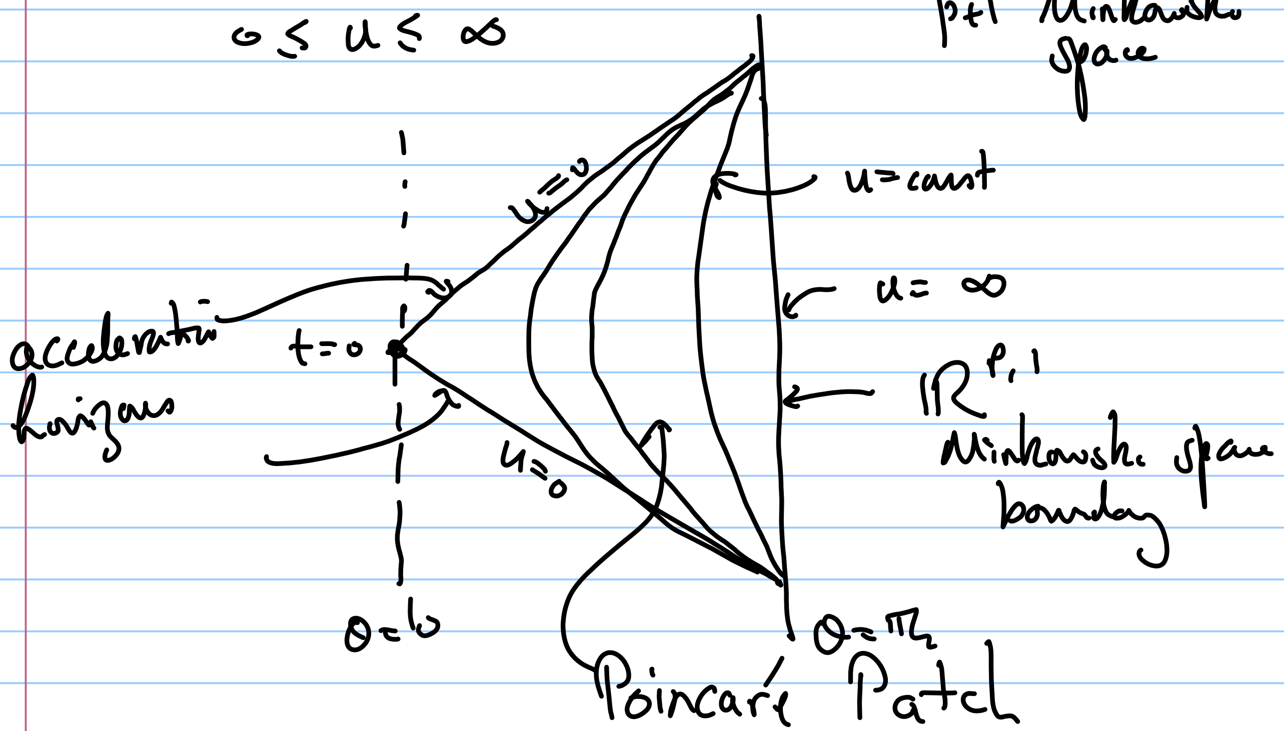
$$u x^i = \tan\theta\, \Omega_i \qquad i = 1 \ldots p$$

$$u \tau = \frac{\sin(t/\alpha)}{\cos\theta}$$

$$\frac{1}{2\alpha}\left(\frac{1}{u} + u\left(\alpha^2 + x^{i2} - \tau^2\right)\right) = \frac{\cos(t/\alpha)}{\cos\theta}$$

$$ds^2 = \alpha^2\left(\frac{du^2}{u^2} + u^2 \underbrace{\left(-d\tau^2 + dx^{i2}\right)}_{p+1 \text{ Minkowski space}}\right)$$

$0 \leq u \leq \infty$



Black hole Spacetimes

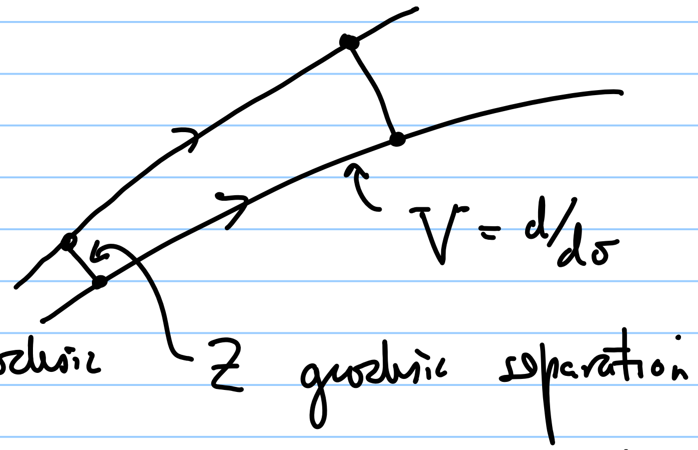
Schwarzschild in 4-d

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \quad \overset{S^2}{\downarrow}$$

need $r > 2m$ (Schwarzschild radius)

$r=2M$? Geodesics reach $r=2M$ in finite proper time.

is it a curvature singularity?



Can always find an orthogonal basis E_a $a=0,1,2,3$ with $E_0 = V$

$$g(E_a, E_b) = \eta_{ab}$$

Can show that

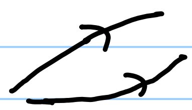
Geodesic deviation

$$\frac{d^2 Z^a}{d\sigma^2} = -\underbrace{R(V, Z, V)^a}_{\text{Riemann tensor}}$$

$a=1,2,3$

$$= -R^a{}_b Z^b$$

$\uparrow$ 3×3 matrix



In BH $R^a{}_b$ is finite as we cross $r=2M$. i.e. $r=2M$ is like the horizon in the Rindler wedge - find new coordinates to extend across.

Finkelstein coordinates

Kruskal "

:

$r > 2M$

$$\tan v = \sqrt{\frac{r}{2M} - 1}\; e^{\frac{r+t}{4M}}$$

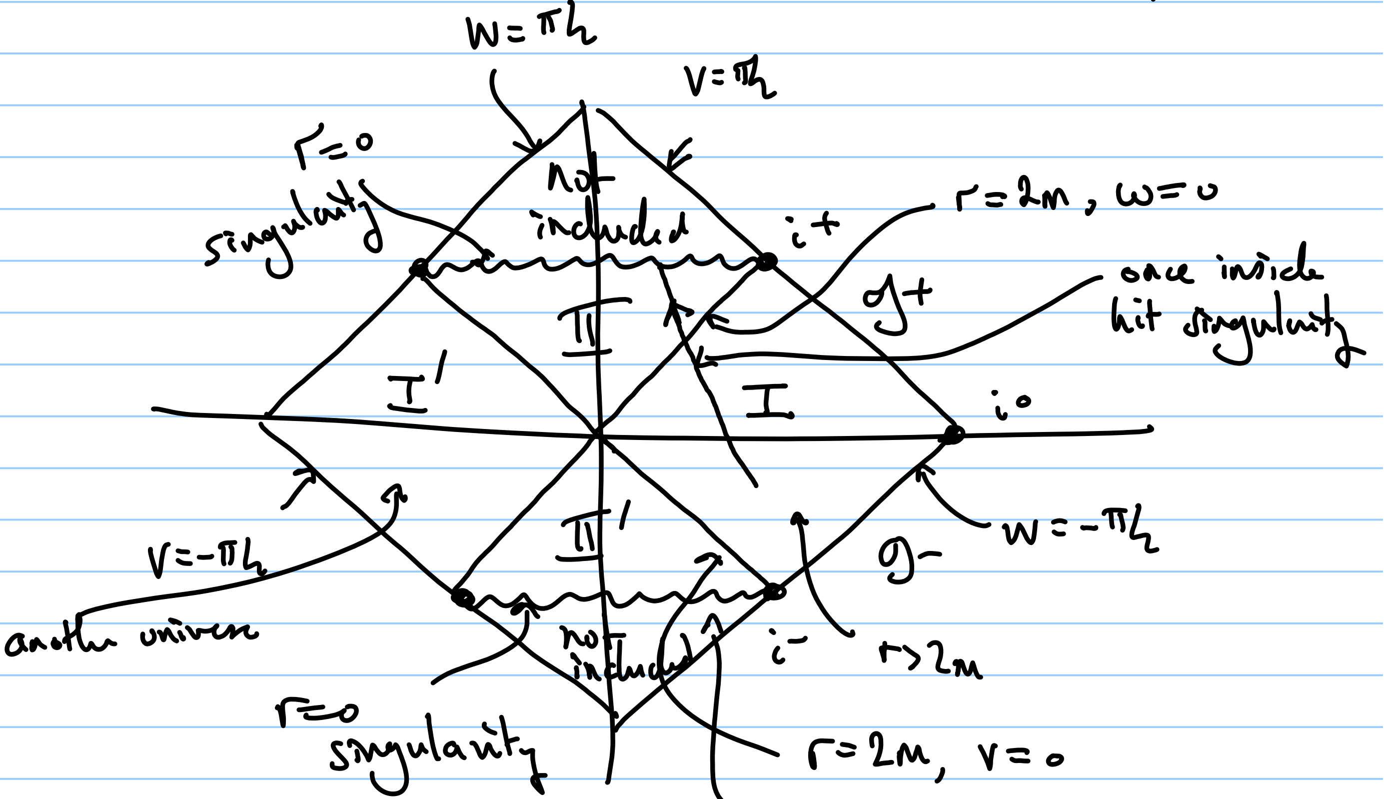
$$\tan w = -\sqrt{\frac{r}{2M} - 1}\; e^{\frac{r+t}{4M}}$$

$-\pi/2 \leq v \leq \pi/2 \qquad -\pi/2 \leq w \leq \pi/2$

$$ds^2 = -f(v,w)\, dv\, dw + r^2 d\Omega_2^2$$

conformal factor $\quad f = \frac{16M^3}{r} e^{-r/2M} \sec^2(v) \sec^2(w)$

$r = r(v,w)$



$$\tan v \tan w = -\left(\frac{r}{2M} - 1\right) e^{r/2M}$$

$r = 2M \qquad \tan v \tan w = 0 \Rightarrow v = 0 \text{ or } w = 0$

$r = 0 \qquad \tan v \tan w = 1$

Physically a black-hole formed by collapse

only this part is physically relevant

collapsing matter