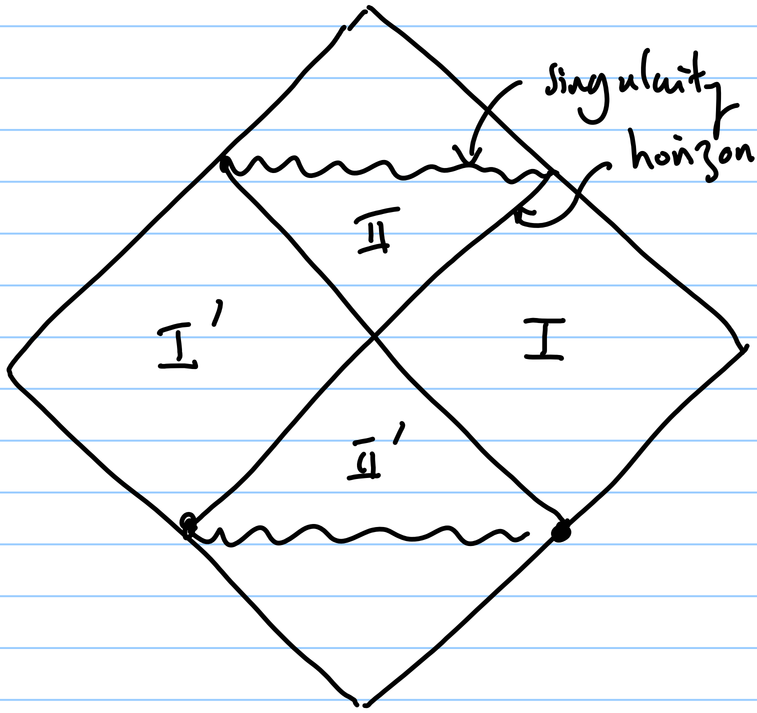
Spacetime : Lecture 4

Note Title 30/10/2009



Reissner - Nordström (Charged)

$$ds^2 = -\underbrace{\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)}_{g(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}} dt^2 + \left(\cdots\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

3 Cases :

$e^2 < M^2$ $\quad g(r) \overset{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}{=} 0$ has 2 real solutions

at $r = r_\pm$ $\quad r_+ > r_-$ two horizons

extremal $e^2 = M^2$ $\quad g(r) = \left(1 - \frac{r_0}{r}\right)^2$ One horizon

$e^2 > M^2$ $\quad g(r)$ has no real solutions

No horizons

$e^2 < M^2$

$$ds^2 = -g(r)\,dt^2 + \frac{dr^2}{g(r)} + \dots$$

$$= -g(r)\left(dt^2 - dr^{*2}\right)$$

$$r^* = \int^r \frac{dr}{g(r)} = \int^r \frac{dr}{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}$$

tortoise coordinate

$$= r + \frac{r_+^2}{r_+ - r_-}\log|r-r_+|$$

$$+ \frac{r_-^2}{r_- - r_+}\log|r-r_-|$$

initially $r > r_+ > r_-$

note $-\infty \leq r^* \leq \infty$

$r = r_+$

$$\tilde{v} = t + r^* \qquad ds^2 = -g(r)\,d\tilde{v}\,d\tilde{w}$$

$$\tilde{w} = t - r^*$$

$-\infty$ to ∞

$$v = \tan^{-1}\left(e^{\lambda \tilde{v}}\right) \qquad w = \tan^{-1}\left(-e^{-\lambda \tilde{w}}\right)$$

$-\infty \leq \tilde{v} \leq \infty$ $\qquad -\infty \leq \tilde{w} \leq \infty$

λ parameter

$0 \leq v \leq \pi/2$ $\qquad -\pi/2 \leq w \leq 0$

$$ds^2 = -\frac{g(r)}{\lambda^2}\, e^{-2\lambda r^*} \sec^2(v)\sec^2(w)\, dv\,dw$$

$$g(r)e^{-2\lambda r^*} = \frac{e^{-\lambda r}}{r^2}|r-r_+|^{1-\frac{2\lambda r_+^2}{r_+-r_-}}|r-r_-|^{1-\frac{2\lambda r_-^2}{r_+-r_-}}$$

In order that the metric is well-behaved at $r=r_+$ need an appropriate choice of λ

e.g. $\lambda = \dfrac{r_+-r_-}{2r_+^2}$

So that

$r=r_+$ is a coordinate singularity

$$g(r)e^{-2\lambda r^*} = \frac{e^{-\lambda r}}{r^2}|r-r_-|^{1+\frac{r_-^2}{r_+^2}}$$

$$\tan(v)\tan(w) = -e^{\lambda(\tilde v-\tilde w)} = -e^{2\lambda r^*}$$

$$= -e^{2\lambda r}|r-r_+|^{\frac{2\lambda r_+^2}{r_+-r_-}}|r-r_-|^{\frac{2\lambda r_-^2}{r_+-r_+}}$$

$$= -e^{2\lambda r}|r-r_+||r-r_-|^{-r_-^2/r_+}$$

so/

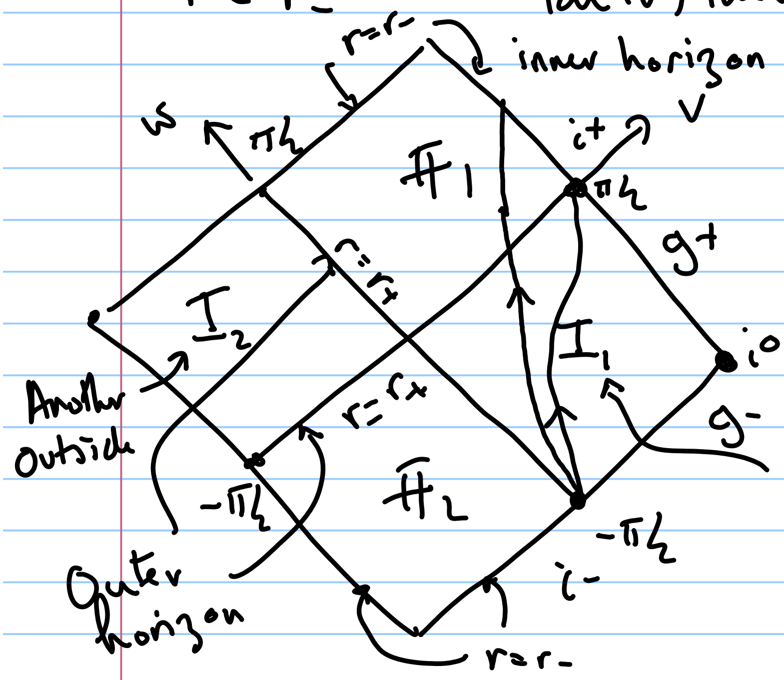
$r=r_+$ $\quad \tan(v)\tan(w)=0 \quad v=0$ or $w=0$

$r=r_-$ $\quad \tan(v)\tan(w)=\infty \quad v=\pi/2 \quad w>0$

or $\quad w=\pi/2 \quad v>0$

or $\quad v=-\pi/2 \quad w<0$

or $\quad w=-\pi/2 \quad v<0$



$r>r_+$ (outside)

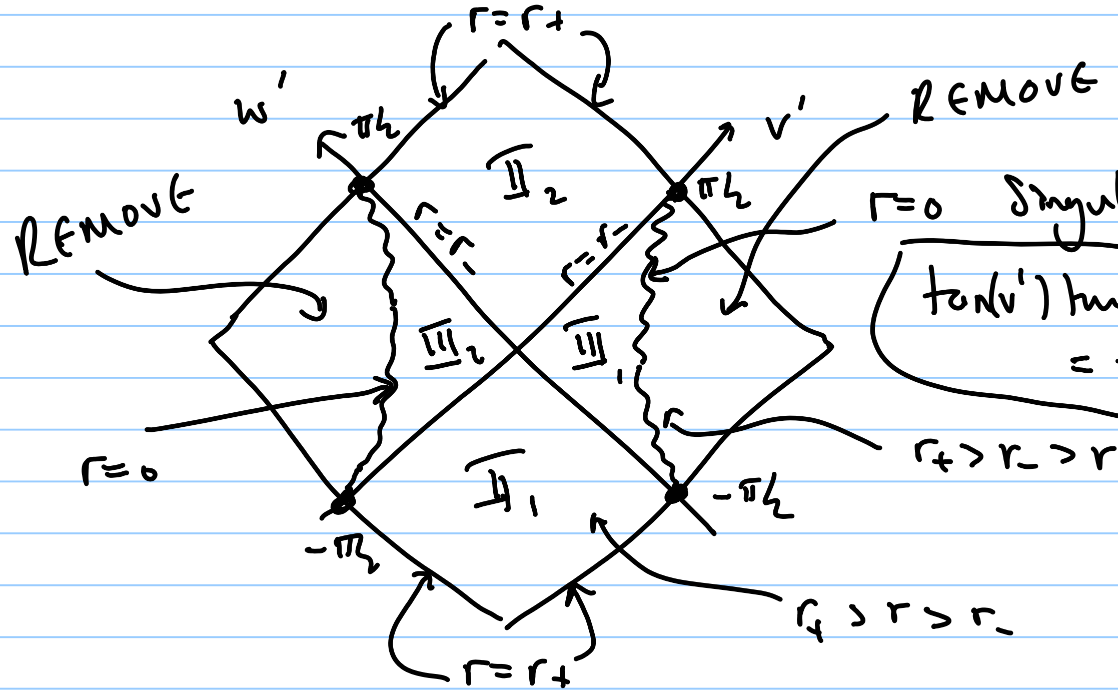
I. $r>r_+>r_-$

II. $r_+>r>r_-$

The coordinates (v, w), i.e. with $\lambda = \frac{r_+ - r_-}{2r_+^2}$, are not good at $r = r_-$. But we introduce good coordinates (v', w') by taking $\lambda = \lambda' = \frac{r_- - r_+}{2r_-^2}$. In these coordinates

$$e^{-2\lambda r^*} g(r) = \frac{e^{-2\lambda' r}}{r^2} |r - r_+|^{1 + r_+^2/r_-^2}$$

and $\tan(v') \tan(w') = -e^{2\lambda' r} |r - r_+|^{-r_+^2/r_-^2} |r - r_-|$

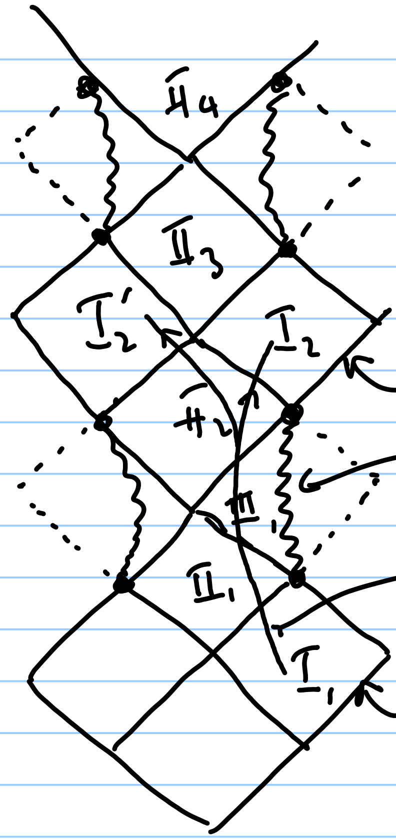


r = 0 Singularity

$\tan(v') \tan(w') = -r_+^{-r_+^2/r_-} r_-$

$e^2 < M^2$

Infinite sequence



another universe.

time-like singularity.

observer

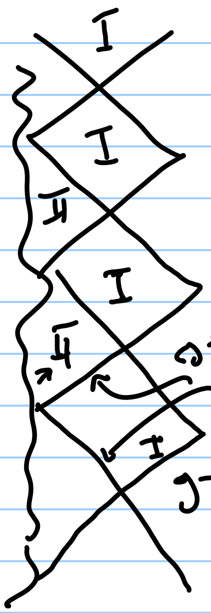
original universe

$e^2 = M^2$

$r_+ = r_- = r_0$

extremal case

inside $r < r_0$



$r = r_0$.

$e^2 > M^2$

no real solution to $g(r) = 0$

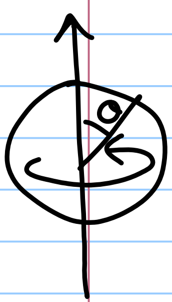
no horizon

– NAKED SINGULARITY

Kerr (Spinning black-hole)

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta\, d\phi^2$$

$$- dt^2 + \frac{2mr}{\rho^2}\left(a\sin^2\theta\, d\phi - dt\right)^2$$



$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2mr + a^2$

a = angular velocity parameter $\quad a \to 0$ Schwarzschild

Look along the axis of symmetry $\theta = 0$

$$ds^2 = -dt^2 \underbrace{\left(1-\frac{2Mr}{r^2+a^2}\right)}_{g(r)} + \frac{dr^2}{1-\frac{2Mr}{r^2+a^2}} + \dots$$

$g(r)=0$ when $\quad r = r_\pm = M \pm \sqrt{M^2-a^2}$

$a<M \quad$ 2 real roots - 2 horizons

$a=M \quad$ extremal one root $\quad r_+=r_-=M$ - 1 horizon

$a>M \quad$ no real root - 0 horizons

$$r^* = \int^r dr/g(r) = r + \frac{a^2+r_+^2}{r_+-r_-}\log|r-r_+| + \frac{a^2+r_-^2}{r_--r_+}\log|r-r_-|$$

Same structure as RN blackhole

→ Penrose diagrams are exactly the same