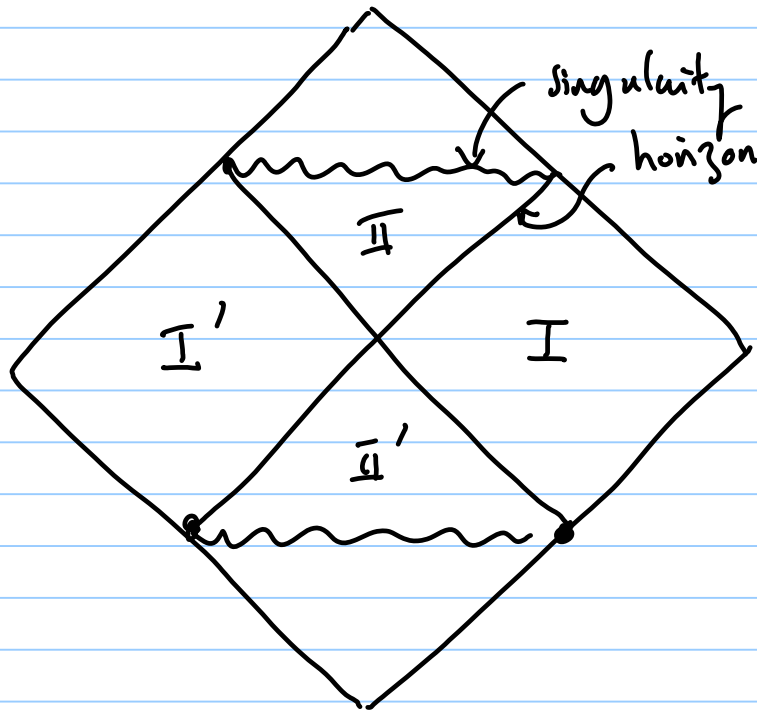


Spacetime : Lecture 4

Note Title

30/10/2009



Reissner - Nordström (Charged)

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} \right) dt^2 + \left(\dots \right)^{-1} dr^2 + r^2 d\Omega_2^2$$

\swarrow
 $g(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$

3 cases:

$$e^2 < M^2 \quad g(r) = \left(1 - \frac{r_+}{r} \right) \left(1 - \frac{r_-}{r} \right) \text{ has 2 real solutions}$$

at $r = r_{\pm}$ $r_+ > r_-$ two horizons

extremal $e^2 = M^2$

$$g(r) = \left(1 - \frac{r_0}{r} \right)^2 \quad \text{One horizon}$$

$e^2 > M^2$

$g(r)$ has no real solutions
No horizons

$$e^2 < M^2$$

$$ds^2 = -g(r) dt^2 + \frac{dr^2}{g(r)} + \dots$$

$$= -g(r) (dt^2 - dr^{*2})$$

$$r^* = \int^r \frac{dr}{g(r)} = \int^r \frac{dr}{(1-\frac{r_+}{r})(1-\frac{r_-}{r})}$$

to use
coordinate

$$= r + \frac{r_+^2}{r_+ - r_-} \log|r - r_+|$$

$$+ \frac{r_-^2}{r_- - r_+} \log|r - r_-|$$

initially $r > r_+ > r_-$

note $-\infty \leq r^* \leq \infty$
 \uparrow
 $r = r_+$

$$\tilde{v} = t + r^*$$

$$\tilde{w} = t - r^*$$

$$\uparrow \quad -\infty \text{ to } \infty$$

$$ds^2 = -g(r) d\tilde{v} d\tilde{w}$$

$$v = \tan^{-1}(e^{\lambda \tilde{v}})$$

$$w = \tan^{-1}(-e^{-\lambda \tilde{w}})$$

λ parameter

$$-\infty \leq \tilde{v} \leq \infty$$

$$-\infty \leq \tilde{w} \leq \infty$$

$$0 \leq v \leq \pi/2$$

$$-\pi/2 \leq w \leq 0$$

$$ds^2 = -g(r) \frac{e^{-2\lambda r^*}}{12} \sec^2(v) \sec^2(w) dv dw$$

$$g(r) e^{-2\lambda r^*} = \frac{e^{-2r}}{r^2} |r - r_+|^{1 - \frac{2\lambda r_+^2}{r_+ - r_-}} |r - r_-|^{1 - \frac{2\lambda r_-^2}{r_+ - r_-}}$$

In order that the metric is well-behaved at $r = r_+$ need an appropriate choice of λ

e.g. $\lambda = \frac{r_+ - r_-}{2r_+^2}$

So that

$$g(r) e^{-2\lambda r^*} = \frac{e^{-2r}}{r^2} |r - r_-|^{1 + \frac{r_-^2}{r_+^2}}$$

$r = r_+$ is a coordinate singularity

$$\tan(v) \tan(w) = -e^{2(\tilde{v} - \tilde{w})} = -e^{2\lambda r^*}$$

$$= -e^{2\lambda r} |r - r_+|^{\frac{2\lambda r_+^2}{r_+ - r_-}} |r - r_-|^{\frac{2\lambda r_-^2}{r_+ - r_+}}$$

$$= -e^{2\lambda r} |r - r_+| |r - r_-|^{-\frac{r_-^2}{r_+}}$$

So

$$r = r_+ \quad \tan(v) \tan(w) = 0 \quad v = 0 \quad \text{or} \quad w = 0$$

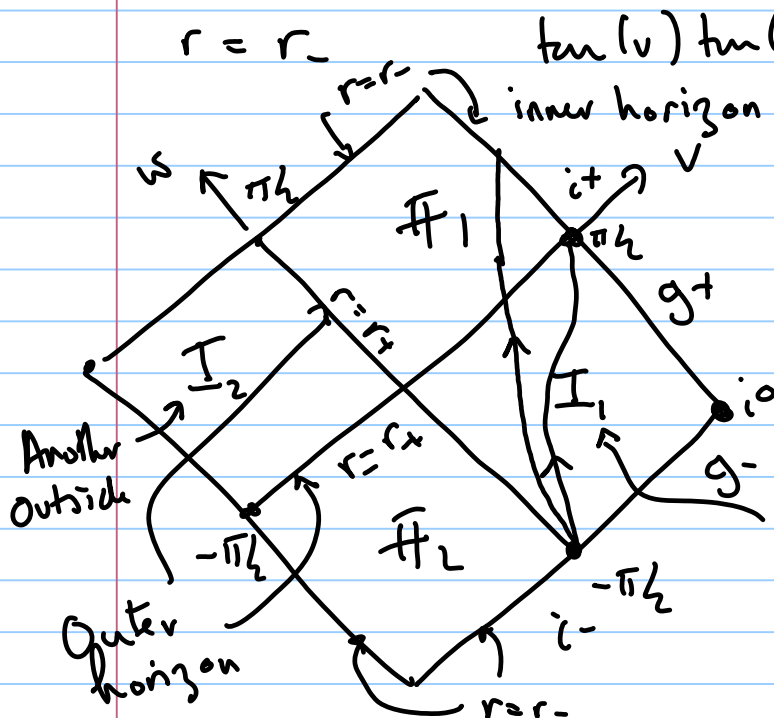
$$r = r_- \quad \tan(v) \tan(w) = \infty \quad v = \frac{\pi}{2} \quad w > 0$$

$$w = \frac{\pi}{2} \quad v > 0$$

$$w = \frac{\pi}{2} \quad v > 0$$

$$w = -\frac{\pi}{2} \quad v < 0$$

$$w = -\frac{\pi}{2} \quad v < 0$$



$r > r_+$ (outside)

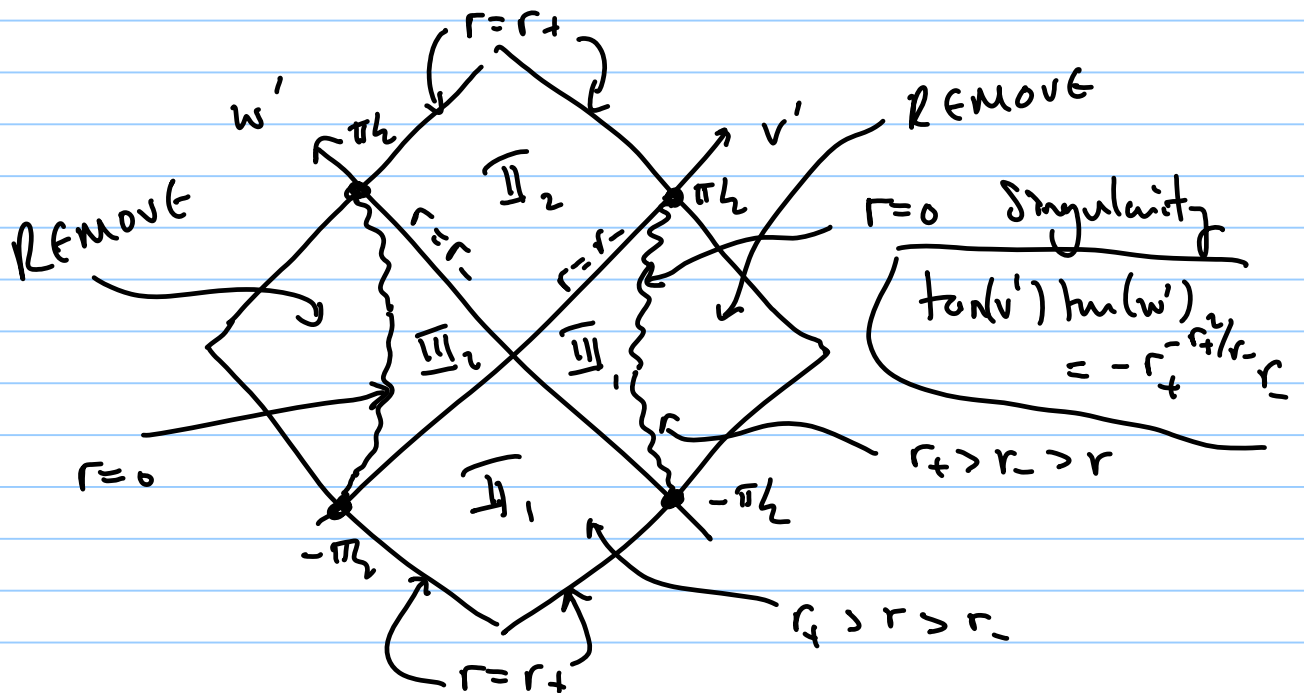
$$\text{I} \cdot \quad r > r_+ > r_-$$

$$\text{II} \cdot \quad r_+ > r > r_-$$

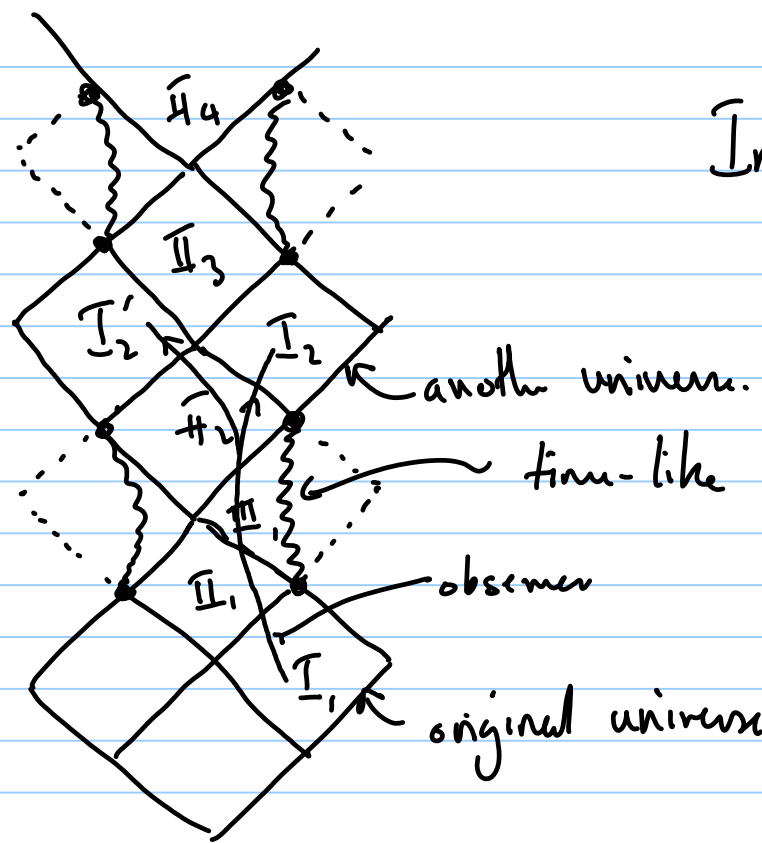
The coordinates (v, w) , i.e. with $\lambda = \frac{r_+ - r_-}{2r_+^2}$, are not good at $r = r_-$. But we introduce good coordinates (v', w') by taking $\lambda = \lambda' = \frac{r_- - r_+}{2r_-^2}$. In these coordinates

$$e^{-2\lambda r^*} g(r) = e^{-2\lambda' r} |r - r_+|^{1 + \frac{r_+^2}{r_-^2}}$$

$$\text{and } \tan(v') \tan(w') = -e^{2\lambda' r} |r - r_+|^{-\frac{r_+^2}{r_-^2}} |r - r_-|$$



$$e^2 < M^2$$



Infinite sequence

another universe.

time-like singularity.

observer

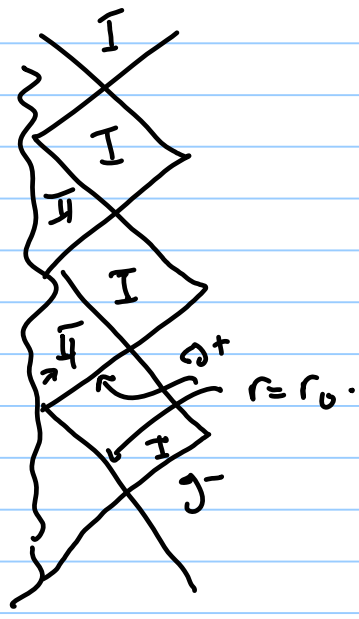
original universe

$$e^2 = M^2$$

$$r_+ = r_- = r_0$$

extremal case

inside $r < r_0$



$$e^2 > M^2$$

no real solution to $g(r) = 0$



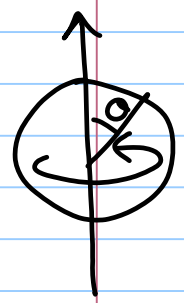
no horizon

- NAKED SINGULARITY

Kerr (Spinning black-hole)

$$ds^2 = \frac{\rho^2}{\Delta} (dr^2 + d\theta^2) + (r^2 + a^2) \sin^2 \theta d\phi^2$$

$$- dt^2 + \frac{2mr}{\rho^2} (a \sin^2 \theta d\phi - dt)^2$$



$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2mr + a^2$$

a = angular velocity parameter $a \rightarrow 0$ Schwarzschild

Look along the axis of symmetry $\theta = 0$

$$ds^2 = - dt^2 \underbrace{\left(1 - \frac{2Mr}{r^2 + a^2}\right)}_{g(r)} + \frac{dr^2}{1 - \frac{2Mr}{r^2 + a^2}} + \dots$$

$$g(r) = 0 \text{ when } r = r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

$a < M$ 2 real roots - 2 horizons

$a = M$ extremal one root $r_+ = r_- = M$ - 1 horizon

$a > M$ no real root - 0 horizons

$$r^* = \int^r \frac{dr}{g(r)} = r + \frac{a^2 + r_+^2}{r_+ - r_-} \log|r - r_+| + \frac{a^2 + r_-^2}{r_- - r_+} \log|r - r_-|$$

Same structure a RN blackhole

\rightarrow Penrose diagrams are exactly the

same

