Particle Physics and Cosmology: Experiment 1

Solar Models

Introduction

In the astrophysics course last year we discussed the equations that govern the structure of stars. These equations are summarised below, but you should remind yourself of the details and the assumptions that were made.

The pressure, P, mass inside radius r, M, temperature , T, density, ρ, and luminosity, L, are determined by

$$\frac{dP}{dr} = -\frac{\rho G M(r)}{r^2},$$

$$\frac{dM}{dr} = 4\pi r^2 \rho,$$

$$P = nkT/V = \rho kT/\mathcal{M},$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon(r)$$

and

$$L(r) = -\frac{16\pi a c r^2 T^3}{3\kappa\rho}\frac{dT}{dr},$$

where κ is the opacity and ϵ the energy production rate.

The aim of this experiment is to solve these equations numerically and determine the form of the main sequence section of the Hertzsprung-Russell diagram. You can use Mathematica or write your own turbobasic/C/fortran code.

Part A: Getting Started

Using your Mathematica notes from Level 1 or your PH204 notes and the notes at the end of this script, write a program to solve simple first order differential equations. Test your code by numerically solving equations that you can solve analytically and compare the results. There are sufficient differential equations for everyone to use a different one!

Part B: Solar Models

Once you are familiar with the numerical solution of simple differential equations, input the solar model equations. Test the solutions giving suitable opacity laws and energy production rates, i.e. make the equations soluble analytically and compare results.

When the code has been tested, construct some stars. You need to give the central pressure and central temperature of the star to fix the boundary conditions for the equations. If you guess values for both of these the solutions will be unphysical, but by varying one of these parameters you can search for a physical solution. A simple search strategy, such as bisection, will allow you to build a star. Once you have a physical solution, record the features of your star. Now change the second parameter and build another star.

You should examine the effects of varying opacity laws, energy production rates and compositions. Can our simple model reproduce a star of the same mass and luminosity as the Sun?

Brief Notes on Mathematica

Mathematica is a symbolic manipulation package. It is much more powerful than a standard number cruncher, although this all we are using it for here.

Basics to get you going:

x=y sets x to be the value of y now (just as in turbobasic)

x := y sets x to be the value of y every time x is needed, i.e. it uses the value of y at the time x is required.

(* comments are surrounded like this *)

Keywords have capital first letters - you should use lower case

Don't use statements like x=x+1, in Mathematic as in real maths this is nonsense

x == y sets up an equation (like x=y in real maths)

NDSolve[$\{y'[x] == x, y[0] == 0\}, \{y\}, \{x, 0, 10\}$] Numerical Solves the Differential equation $\frac{dy}{dx} = x$ for y with the boundary condition y(0)=0 in the range x=0 to x=10.

/. % uses the previous result

/. %% uses the last but one result

Plot[Sin[x], $\{x, 0, 4\}$] plots sin(x) between x=0 and x=4.

See the sample codes for examples.

Using Fortran or C

If you are using an unfamiliar language, try the 'hello world' approach: Using either sample code or a book as a guide to syntax, write code to do the following:

- write 'hello world' on the screen,
- write 'hello world' to a file,
- use a 'do' or 'for' loop to write the numbers 1 to 10 on the screen,
- use an 'if' statement to write the numbers between 1 and 10 that are more than 2 away from 5 on the screen,
- use a subroutine to apply the test used above,
- use the code to solve simple differential equations then do the star.

Opacity and Energy Production

From your astrophysics notes we know that the energy production rate per unit volume is of the from

$$\text{rate} = B\rho^2 T^{-2/3} \mathrm{e}^{-\alpha/T^{1/3}}$$

with

$$\alpha = 3\Big(\frac{q_1^2 q_2^2 e^4 \frac{m_1 m_2}{m_1+m_2}}{2^5 k \epsilon_0^2 \hbar^2}\Big)^{1/3}$$

where the q's are the charges of the participating nuclei (in terms of the electron charge) and the m's are their masses. The rate determining step is proton +proton $\rightarrow$ deuteron + positron + neutrino. For this reaction $q_1 = q_2 = 1$ and $m_1 = m_2$, giving

$$\alpha = 3.38 \times 10^3 K^{1/3}$$

The energy production rate per unit mass, ϵ, is simply the above rate divided by the density. Note that ϵ is proportinal to the density (if the density doubles there are **twice** as many particles in the volume each or which has **twice** as many particles to interact with). We can find the constant B from books, e.g. Bahcall 'Neutrino Astrophysics' , which gives $\epsilon/\rho = 1.2 \times 10^{-8}$ W/kg/(kg/m^3) at $T = 1.54 \times 10^7$K. If we set $T_{ref} = 1.54 \times 10^7$ we can write

$$\frac{\epsilon}{\rho} = 1.2 \times 10^{-8} \frac{T^{-2/3} \mathrm{e}^{-\alpha/T^{1/3}}}{T_{ref}^{-2/3} \mathrm{e}^{-\alpha/T_{ref}^{1/3}}}$$

Note that this has the correct dependence on T and matches the calibration value at $T = T_{ref}$.

For the opacity we use the three approximations discussed in the astrophysics lectures:

$$\kappa_1 = \text{constant} \quad , \quad \kappa_2 = \beta \frac{\rho}{T^{3.5}} \quad , \quad \kappa_3 = \gamma \sqrt{\rho} T^4.$$

To set the constants we again use a plot from Taylor's book (page 101). We can directly read off

$$\kappa_1 = 10^{-1.6}.$$

The approximations κ_2 and κ_3 are equal at $T = T_{ref} = 10^{4.57}$K for $\rho = \rho_{ref} = 0.1$kg m^{-3} where they both have the value 10^4 m^{2}kg^{-1}. Thus we

$$\kappa_2 = 10^4 \frac{\rho}{\rho_{ref}} \Big(\frac{T_{ref}}{T}\Big)^{3.5}$$

$$\kappa_3 = 10^4 \sqrt{\frac{\rho}{\rho_{ref}}} \Big(\frac{T}{T_{ref}}\Big)^4$$

The last thing is to decide which form is appropriate.

We want κ_3 if $\kappa_3 < \kappa_2$

We want κ_2 if $\kappa_2 < \kappa_3$ and $\kappa_2 > \kappa_1$

We want κ_1 if $\kappa_2 < \kappa_3$ and $\kappa_1 > \kappa_2$

We use mathermatica's Sign function to do this. Sign[positive number] =1 and Sign[negative number] =-1. Thus

$$\frac{1+\mathrm{Sign}[x]}{2} = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

The functions check1 and check2 are ± 1 depending on the relative values of the κ's. we can then use expressions like the one above to set kappa[r] equal to the appropriate approximation. Work through the algebra for all three cases if you don't believe me!

Chasing the Initial Conditions

For a fixed central temperature, if the central pressure is too low we have a low density gas cloud. The density profile will fall as r^{-2} at large distances and the temperature will become constant. The outer regions form an isothermal sphere. The only physical problem with this is that the gas cloud is of infinite extent. In reality there will be some limit to the size of the cloud and the edge will be in contact with cold empty space. The gas cloud will therefore loose energy and, according to the Virial theorem, contract and get hotter. Thus in some sense we have produced an embryonic star. We are however interested in stars on the main sequence, so we do in the code what nature would do, i.e. we increase the central pressure.

For a fixed central temperature, if the central pressure is too high it all goes horribly wrong! The temperature drops to zero while the pressure is finite. The ideal gas equation then produces an infinite density and the code falls over- look out for the 'singularity suspected' error message. This tells you that the pressure is too high.

Once you have a pressure that is too high and one that is too low, hunt for the ideal pressure by bisection. The closer you get to the ideal pressure the less dense the isothermal sphere surrounding the star becomes. For pressures below the ideal pressure (i.e. ones that don't give singularities) look at the mass profile. On large length scales the mass is dominated by the isothermal sphere and looks linear. However, on smaller scales you should see an initial accelerating growth followed by a slowing down. At the ideal pressure the mass profile would become a constant beyond the edge of the star. You don't have to get exactly the correct pressure to see the profile turning over. Once you have a clear turnover estimate the mass and luminosity of your star.