QCD and Supersymmetry

Lecture 2: What is "Orientifold Planar Equivalence ?"

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Motivation

Yesterday, we learnt about the non-perturbative structure of $\mathcal{N} = 1$ super Yang-Mills theory.

Today, we will see that we can copy some of yesterday's results to QCD !

What is planar equivalence ?

Statement:

In the limit of infinite number of colors, $N \to \infty$, and in a well-defined common sector

$SU(N)$ gauge theory + a Dirac fermion in the antisymmetric representation

$=$

$SU(N)$ gauge theory + a Majorana fermion in the adjoint representation

The first theory, often called "orientifold field theory" (due to a relation with string theory) contains an antisymmetric Dirac fermion $\Psi_{[ij]}$ ($\Psi_{ij} = -\Psi_{ji}$). It is a non-supersymmetric theory.

The other theory is $\mathcal{N} = 1$ SYM.

So, the claim is that in the limit $N \to \infty$, a supersymmetric theory and a non-supersymmetric theory become equivalent !

Why is it useful for QCD ?

The relation with QCD

For $SU(3)$ the antisymmetric representation is identical to the anti-fundamental representation

$$q^i = \frac{1}{2}\epsilon^{ijk}\Psi_{[jk]}$$

So, we can start with one-flavor QCD, think about the quark as an antisymmetric quark, generalize to an $SU(N)$ theory with an antisymmetric fermion and finally take the $N \to \infty$ limit.

By using the above procedure, and the relation with $\mathcal{N}=1$ SYM we will be able to approximate (one-flavor) QCD by SUSY YM and to copy results from the latter to QCD !

What is the large-N limit ?

In 1973 't Hooft suggested that a generalization of QCD from $SU(3)$ to $SU(N)$ may result in a simpler theory in the large N limit ('t Hooft, 1973).

More precisely, 't Hooft showed that when taking the limit $N \to \infty$, while keeping fixed the combination $\lambda \equiv g^2 N$ (now called the 't Hooft coupling), the theory is controlled by *planar diagrams.*

Let us focus on $U(N)$ pure Yang-Mills theory, namely on

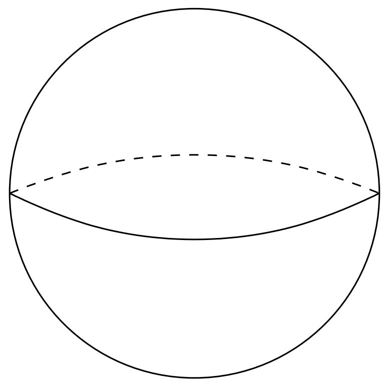
$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu}$$

't Hooft showed that Feynman diagrams are classified according to their topology.

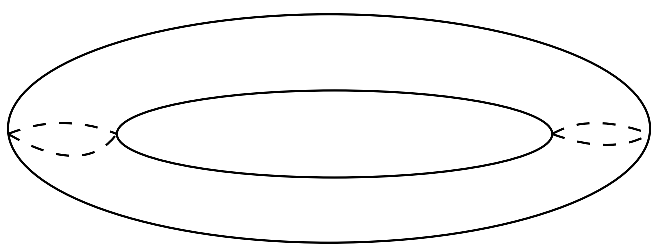
Large-N

For simplicity let us focus on 'vacuum diagram':

1. Planar diagram, admitting the topology of a sphere are $\mathcal{O}(N^2)$.



2. Non-planar diagrams carry a weight of $\mathcal{O}(N^{(2-2h)})$, where h is the number of holes in the diagram.



Thus, in the limit $N \to \infty$ only vacuum diagrams with sphere topology dominate.

For this reason the large-N limit is simpler: it selects only a sub-set of Feynman diagrams.

’t Hooft double-index notation

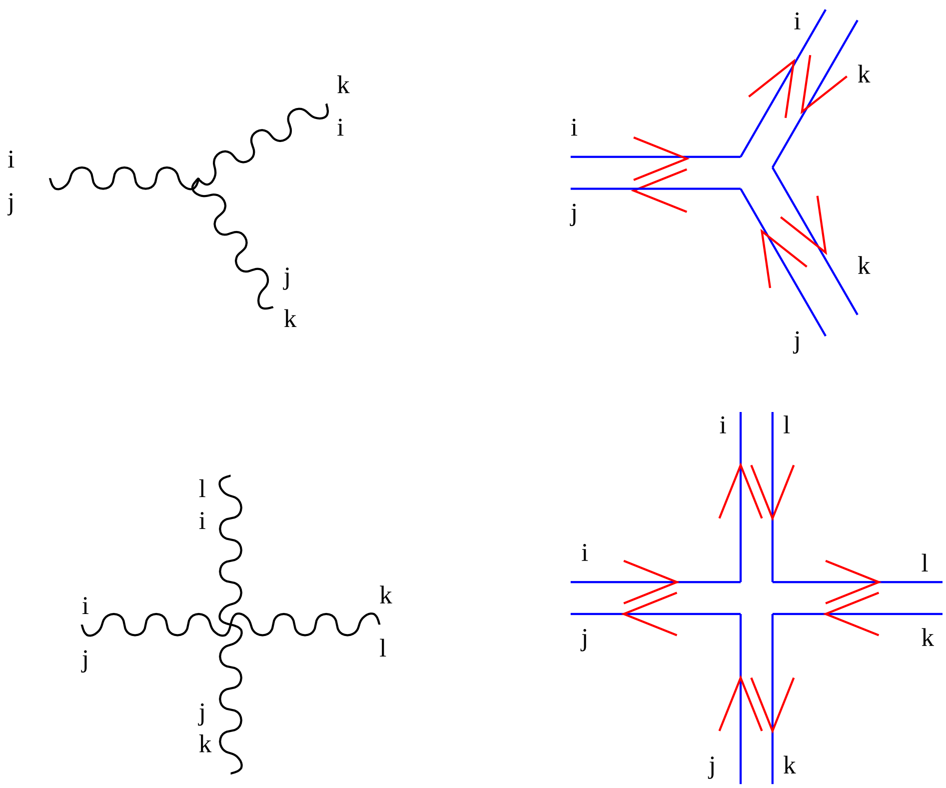
In order to understand ’t Hooft’s idea, let us introduce the ’t Hooft notation:

The gluon propagator is represented by two lines with arrows that point in opposite directions.

These lines represent the flow of color in the Feynman diagram. We should think about the adjoint representation as the tensor product of the fundamental and the anti-fundamental representations.

’t Hooft double-index notation

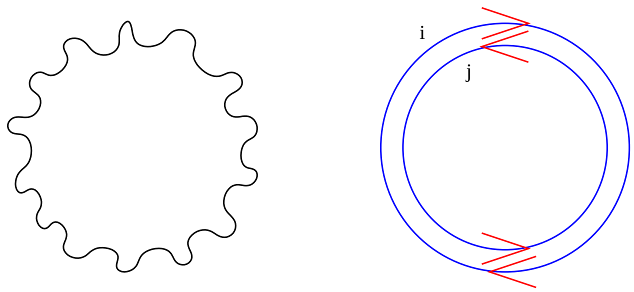
The three and four gluons vertices are depicted in the figure below



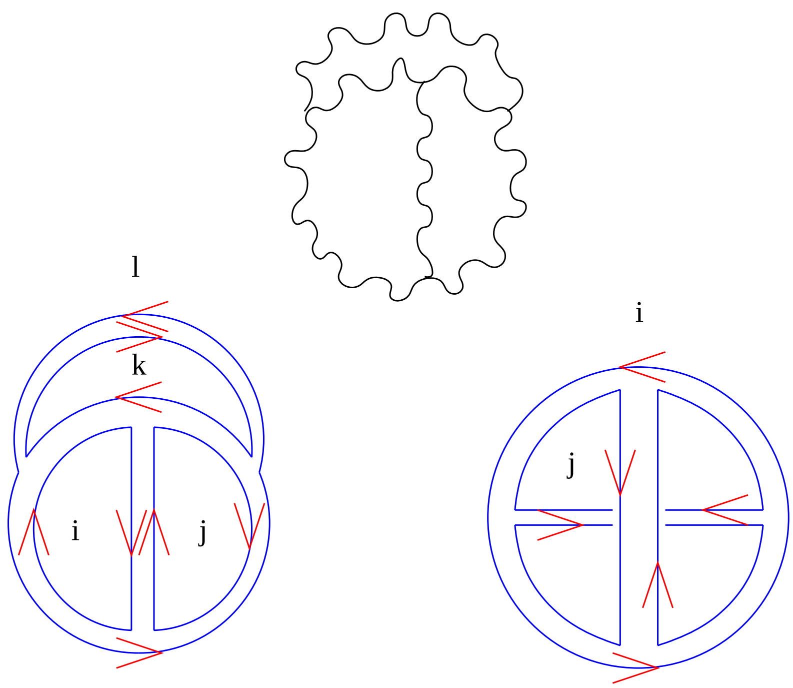
(I’ve ignored the ghosts).

Examples

Let us consider two examples. First, a one-loop vacuum diagram



Second, a three-loop vacuum diagram which contains both planar and non-planar* contributions



Topology: Sphere

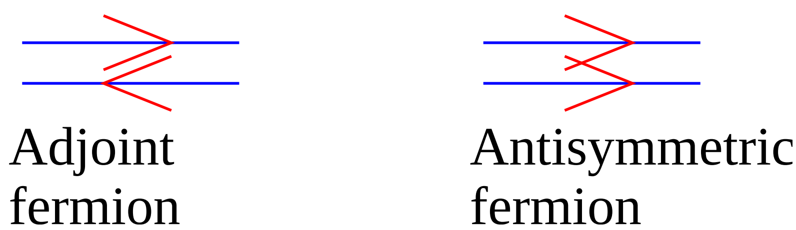
$g^4N^4 = (g^2N)^2N^2$

Topology: Torus

$g^4N^2 = (g^2N)^2$

The large N limit of $\mathcal{N} = 1$ SYM and the "Orientifold" theory

$\mathcal{N} = 1$ SYM contains in addition to the gluon a gluino. Since the gluino transform in the adjoint representation, its Feynman rules look exactly as the Feynman rules of the gluon



What about the antisymmetric fermion of the "orientifold theory" ?

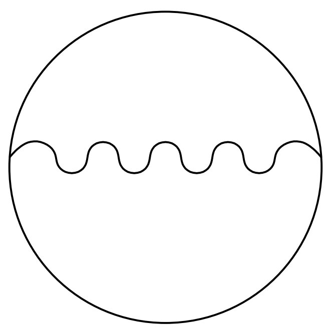
Since $\Psi_{[ij]}$ carries two fundamental indices, it is denoted by two lines with arrows that point in the same direction, since the two-index antisymmetric representation is obtained by the tensor product of the fundamental representation with itself.

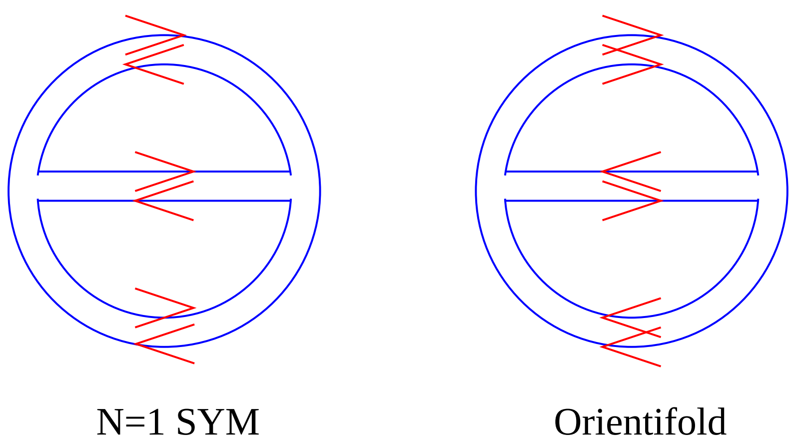
Perturbative planar equivalence

It was shown that all *planar diagrams* of $\mathcal{N}=1$ SYM and the "orientifold" theory are equivalent to each other (Armoni, Shifman, Veneziano, 2003).

Why ? in order to map a graph in $\mathcal{N}=1$ SYM to a graph in the "orientifold theory", simply reverse the orientation of one of the lines along each fermionic loop.

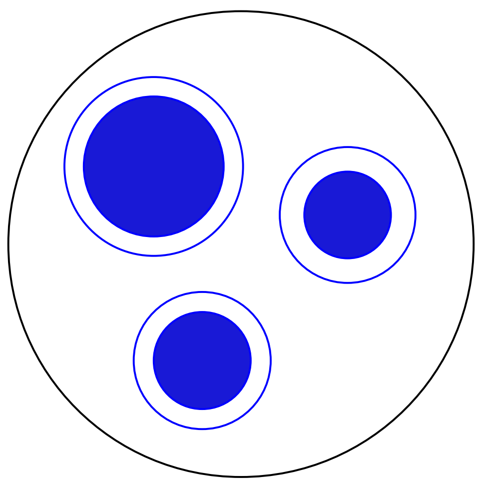
For example





Perturbative planar equivalence

The general rule is as follows: draw the planar diagram in the 't Hooft notation on a sphere. Fermionic loops will divide the sphere to two regions: "interiors" and "exteriors". In order to relate the two theories, simply reverse the arrows on all the lines in the "interiors" (or the "exteriors").



So there is a one-to-one correspondence between the planar graphs of the two theories. Hence the theories are equivalent at the planar level.

Non-perturbative planar equivalence

The full non-perturbative equivalence is more difficult to prove (Armoni, Shifman and Veneziano, 2003,2004),(Unsal and Yaffe,2006).

The idea is to show the equivalence of the partition functions of the two theories at the planar level.

The partition function of $\mathcal{N}=1$ SYM is

$$\mathcal{Z}_0 = \int \mathcal{D}A \, \exp\left(iS[A]\right) \det\left(i \not{\partial} + \not{A}^a T^a_{\mathrm{adj}}\right)$$

Next, we will make use of the fact that

$$T^a_{\mathrm{adj}} \sim T^a_{\square \times \overline{\square}} = T^a_{\square} \otimes 1 + 1 \otimes T^a_{\overline{\square}} \,,$$

to write the partition function as follows

$$\mathcal{Z}_0 = \int \mathcal{D}A \, \mathcal{D}B \, \delta(A-B) \, \exp -\tfrac{1}{2}\left(S[A] + S[B]\right)$$
$$\det\left(i \not{\partial} + \not{A}^a (T^a_{\square} \otimes 1) + \not{B}^a (1 \otimes T^a_{\overline{\square}})\right)$$

Non-perturbative proof

The determinant is gauge invariant and hence it can be written formally by using Wilson loops

$\mathcal{Z}_0 = \int \mathcal{D}A\, \mathcal{D}B\, \delta(A - B)\, \exp -\frac{1}{2}(S[A] + S[B])$
$\sum \prod \mathcal{W}(A)\mathcal{W}(B)$.

Or $\mathcal{Z}_0 = \sum \langle \prod \mathcal{W}\mathcal{W} \rangle$

The partition function of the "orientifold field theory" is similar, but with reversed orientation for the red Wilson loop $\mathcal{W}^\star$.

At large-N gauge invariant amplitudes factorise. For example

$\langle \mathcal{W}(A)\mathcal{W}(B) \rangle = \langle \mathcal{W}(A)\mathcal{W}^\star(B) \rangle = \langle \mathcal{W} \rangle \langle \mathcal{W} \rangle$

(We assume here that C-parity is not broken).

More generally, we proved that at large-N
$\langle \prod \mathcal{W}_{\mathrm{adj.}} \rangle_{\mathrm{conn.}} = \langle \prod \mathcal{W}_{\mathrm{anti-symm.}} \rangle_{\mathrm{conn.}}$

So, the partition functions of the two theories coincide.

Comments about C-parity

An alternative proof was given by Unsal and Yaffe. They have shown that:

“Non-perturbative planar equivalence between $\mathcal{N} = 1$ SYM and the orientifold theory holds if and only if charge conjugation is not broken”

Whether C-parity is broken or not in a given theory is a difficult question.

Surprisingly, C-parity is spontaneously broken in gauge theories coupled to fermions, when the theory is compactified on a small radius with *periodic boundary conditions* for the fermions.

Lattice simulations show (DeGrand and Hoffmann, 2006), (Lucini, Patella and Pica, 2007) that C-parity is restored above a critical radius, in a transition similar to the deconfinement/confinement.

Thus, provided that C-parity is not broken, planar equivalence holds.

Brief description of the implications

- Since the partition function of the two theories coincide it follows that the $\mathcal{O}(N^2)$ contribution to the vacuum energy of the non-supersymmetric vanishes.
- We can repeat the proof in the presence of sources to conclude that the value of the quark condensate is identical to the value of the gluino condensate.
- The bosonic color-singlet spectra is the same (however, there are no fermionic color singlets in the orientifold theory).
- Bosonic Green functions should also coincide.
- The centre of the gauge group of the orientifold theory is enhanced in the common sector from $\mathcal{Z}_2$ to $\mathcal{Z}_N$.
- A degeneracy between bosonic and fermionic open QCD-strings and an approximate meson-boson degeneracy.

The relation with string theory

The AdS/CFT (gravity/gauge theory) correspondence suggests a classical description of the large-N gauge theory. This is a modern version of the old idea that large-N theories should be controlled by a classical master field.

The supersymmetric theory is dual to the type IIB superstring.

Interestingly, the "orientifold theory" is dual to a non-supersymmetric string theory called type 0'B (Sagnotti, 1995).

The type 0'B closed string spectrum is the bosonic truncation of type IIB spectrum. This is in agreement with the fact that the color-singlet spectrum of the "orientifold gauge theory" is the bosonic truncation of the $\mathcal{N} = 1$ spectrum.

Conclusions

$\mathcal{N} = 1$ super Yang-Mills theory is equivalent in the large-N limit to the "orientifold theory" in a common sector of bosonic color-singlets.

Since the orientifold theory is a generalization from $SU(3)$ to $SU(N)$ of one-flavor QCD, we can use the results that were described in the first lecture to estimate non-perturbative quantities in QCD.

Tomorrow we will see that there are indeed "supersymmetry relics" in QCD.