Return of the Phantom Anomaly

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based on,
"Nonplanar Anomalies in Noncommutative Theories and the Green-Schwarz Mechanism",
A.A., Esperanza Lopez and Stefan Theisen,
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Outline

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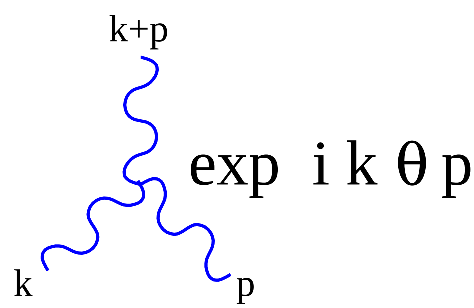
Introduction

Since the recent discovery of the relation between noncommutative field theories and string theory (Connes, Douglas, Schwarz and Seiberg, Witten) there was a lot of activity in this field.

Noncommutative theories are realized by replacing the ordinary multiplication by a $*$-product

$$f \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{(\xi)}\partial_\nu^{(\eta)}} f(x+\xi)g(x+\eta)|_{\xi,\eta\to 0}.$$

This modification of the Lagrangian affects the Feynman rules, by adding a momenta dependent phase to each vertex.

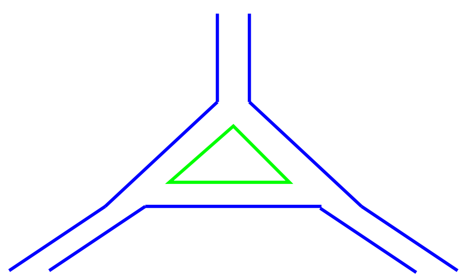


The change in the Feynman rules leads to the following interesting consequences:

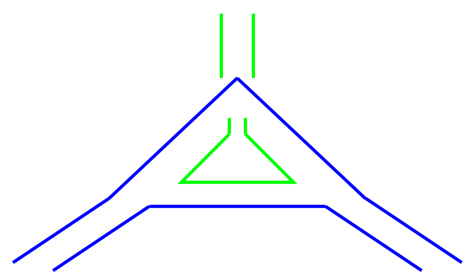
(i). Planar graphs of the noncommutative theory differ from the planar graphs of the commutative theory only by global phases and therefore contain the same UV divergences as of the commutative theory (Gonzalez-Arroyo et.al., Filk).

(ii). One loop nonplanar graphs are UV finite.

Accordingly, *anomalies* can be also classified as planar or nonplanar.



Planar Anomaly in 4d



Nonplanar Anomaly in 4d

(In a theory with bifundamental fermions the nonplanar graph is related to a mixed anomaly $U^2(N)U(M)$.)

Planar anomalies are well understood. The 'planar' axial current (related to the planar vertex in the planar graph) is anomalous

$$D_\mu j_A^\mu = -\frac{g^2}{8\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu}$$

The anomalies which are related to nonplanar graphs are less understood. In particular, it was claimed (Intriligator, Kumar) that this anomaly vanishes since the associated integral is UV finite and one can shift the integration variable.

In particular, if this analysis is correct, the "good" global anomaly which is responsible for the decay process $\pi^0 \to \gamma\gamma$ is forbidden.

Therefore noncommutativity (even at the GUT scale) is excluded in nature !

Our suggestion is different.

Indeed for any *non-zero* noncommutative momentum θq which flows through the nonplanar vertex

$$q_\mu j_A^\mu(-q) = 0$$

However, for $\theta q = 0$ we have a non-zero anomaly which takes the following form

$$\int d^2x_{NC}\ \partial_\mu j_A^\mu = -\frac{g^2}{8\pi^2}\int d^2x_{NC}\ F_{\mu\nu}\tilde{F}^{\mu\nu}$$

The whole anomaly is "concentrated" in the zero noncommutative momentum !

The axial charge is not conserved and therefore the pion decay is not forbidden.

Our suggestion for the solution of the pion decay issue raises an immediate paradox. Consider a noncommutative theory, that originates from a decoupling (Seiberg-Witten) limit of string theory. Suppose that some $U(1)$'s are anomalous. A theory with anomalous local symmetry is not consistent.

In the usual commutative case, string theory resolves the problem by giving a string scale mass to these $U(1)$'s such that they 'look' global to the low-energy observer. The gauge symmetry becomes $SU(N)$.

This kind of solution seems to be excluded in the noncommutative case since $SU(N)$ noncommutative theories are inconsistent.

Noether procedure and axial currents

Consider a noncommutative $U(1)$ gauge theory with a massless Dirac fermion in the fundamental representation

$$S = i \int d^4x \, \bar{\psi} \star \not{D} \psi \, ,$$

where $D_\mu \psi = \partial_\mu \psi + i g A_\mu \star \psi$. The action is invariant under global axial transformations $\delta_\alpha \psi(x) = i \, \alpha \, \gamma^5 \psi(x)$. We have two possible choices of currents (due to the cyclicity property $\int f * g = \int g * f$)

$$i) \; j^\mu_A = \bar{\psi} * \gamma^\mu \gamma^5 \psi \, , \quad ii) \; j'^{\,\mu}_A = -\psi^t (\gamma^\mu \gamma^5)^t * \bar{\psi}^t \, .$$

j_A is gauge invariant, whereas j'_A is gauge covariant. The two currents satisfy, classically,

$$i) \; \partial_\mu j^\mu_A = 0 \, , \quad ii) \; D_\mu j'^{\,\mu}_A = \partial_\mu j'^{\mu}_A + ig \, [A_\mu, j'^{\,\mu}_A]_* = 0 \, .$$

j'_A (the "planar" current) satisfies a local anomaly equation

$$D_\mu j'^{\,\mu}_A = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} * F_{\rho\sigma} \,.$$

The gauge invariant current j_A cannot satisfy a similar equation, since the r.h.s. is not gauge invariant ($F_{\mu\nu}$ is not gauge invariant even for the Abelian theory).

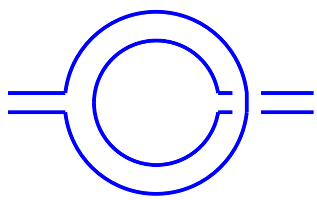
But, due to the cyclic symmetry of the $\star$-product under integration, both currents define the same gauge invariant charge

$$Q = \int \mathbf{dx}\, j^0_A = \int \mathbf{dx}\, j'^{\,0}_A \,.$$

The anomaly equation implies that Q is not conserved. If at the same time the divergence of j_A did vanish, we would encounter a contradiction. This problem does not arise when j_A satisfies instead the integral anomaly equation.

Nonplanar anomalies - 2d example

Consider the nonplanar anomaly diagram in 2d Euclidean space



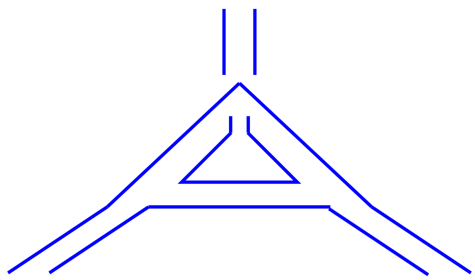
The 2d anomaly and the vacuum polarization are related to each other $A^{\rho\nu} = \epsilon^{\rho}_{\mu}\Pi^{\mu\nu}$ due to the special properties of the Dirac algebra in 2d.

$$g_{\mu\nu}\Pi^{\mu\nu}(q) =$$
$$4(d-2)\int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \; \frac{l^2 - x(1-x)q^2}{(l^2 + x(1-x)q^2)^2} \exp il\theta q$$

which vanishes, unless $\theta q = 0$.

Nonplanar anomalies - a perturbative calculation

Consider the nonplanar anomaly diagram in 4d



The expression is

$$\int \frac{d^4 l}{(2\pi)^4} \exp 2il\theta q \,\mathrm{tr}\, \left(\exp ik\theta p \; \gamma^5 \frac{\not l}{l^2} \gamma^\lambda \frac{(\not l + \not k)}{(l+k)^2} \gamma^\nu - \exp -ik\theta p \; \gamma^5 \frac{\not l}{l^2} \gamma^\nu \frac{(\not l + \not p)}{(l+p)^2} \gamma^\lambda \right)$$

with q^μ the momentum of the axial current and p^ν, k^μ the momenta of the vector currents. A second contribution with a similar form, but with (p, ν) interchanged with (k, μ), cancels the above expression.

So, it seems that indeed noncommutativity leads to a vanishing anomaly.

However, in arriving to the above expression, we shifted the integration variable l. This is legitimate only for $\theta q \neq 0$.

For $\theta q = 0$ we have the ordinary anomaly.

$$\partial_\mu j^\mu_A(-q) = -\frac{g^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}|_{\theta q = 0}$$

Nonplanar anomalies - a point-splitting calculation

The current j_A, regulated by point splitting, is

$$j^\mu_A(x) = \lim_{\epsilon\to 0} \bar{\psi}(x+\tfrac{\epsilon}{2})\,\gamma^\mu\gamma^5 * \mathcal{U}(x+\tfrac{\epsilon}{2}, x-\tfrac{\epsilon}{2}) * \psi(x-\tfrac{\epsilon}{2})$$

where a Wilson line has been introduced to preserve the gauge invariance of the regularized expression,

$$\mathcal{U}(x,y) = e^{ig\int_x^y dl\cdot A(l)}$$

It leads to

$$iq_\mu j^\mu_A(q) = -\frac{g^2}{2\pi^2}\,\lim_{\epsilon\to 0}\frac{\epsilon^\nu\,(\epsilon-\theta q)_\rho}{(\epsilon-\theta q)^2}\;\epsilon^{\mu\rho\alpha\beta}\times$$
$$\int\frac{d^4k}{(2\pi)^4}\;f(k,|\epsilon-\theta q|)\;F_{\mu\nu}(q-k)\;ik_\alpha A_\beta(k)\,e^{-\frac{i}{2}q\theta k}$$

where f is always non-zero and becomes 1 for $|\epsilon-\theta q|\to 0$. Again, the anomaly vanishes unless $\theta q = 0$.

A comment about analyticity

We found that the $\theta q = 0$ is different from the $\theta q \to 0$ result. Namely, the $\theta q = 0$ plane is singular.

This is not unusual in noncommutative theories.

In fact, in order to preserve gauge invariance, the limit $\theta \to 0$ cannot be smooth (A.A., Nucl.Phys.B593:229.)

For pure $U(N)$ noncommutative theory ($\theta \neq 0$)

$$\beta_0(SU(N)) = -\frac{g^3}{(4\pi)^2}\frac{11}{3}N$$
$$\beta_0(U(1)) = -\frac{g^3}{(4\pi)^2}\frac{11}{3}N$$

For the commutative theory ($\theta = 0$)

$$\beta_0(SU(N)) = -\frac{g^3}{(4\pi)^2}\frac{11}{3}N$$
$$\beta_0(U(1)) = 0$$

Global anomalies

Global anomalies have application both in phenomenology (pion decay) and in constraining effective descriptions ('t Hooft anomaly matching conditions).

Since we argue that mixed global anomalies do not automatically cancel, the pion decay is allowed.

In addition, if one considers, for example, a noncommutative version of Seiberg duality, the same set of global anomalies (apart from the baryon number which is now gauged) should be considered. The noncommutative theory is not less restrictive.

Local Anomalies - a paradox

Consider D3 branes placed on C^3/Z_3 orbifold singularity. The supersymmetric 4d field theory that 'lives' on the brane has the following chiral content

	$U_1(N)\times$	$U_2(N)\times$	$U_3(N)$
3 chirals	$\square$	$\overline{\square}$	1
3 chirals	1	$\square$	$\overline{\square}$
3 chirals	$\overline{\square}$	1	$\square$

The theory has mixed anomaly. For example, the anomaly $U_1^2(N)U_2(1)$. Diagrammatically, it is due to a nonplanar graph.

We have two $U(1)$'s which are anomalous and one $U(1)$ (the sum of the above $U(1)$'s) which is anomaly free. Local symmetries cannot be anomalous. String theory should resolve this problem.

In the ordinary case, the solution is very simple. String theory gives a mass $M^2 \sim \frac{1}{\alpha'}$ to the anomalous $U(1)'s$. The low energy theory becomes $SU(N)^3 \times U(1)$ theory instead of $U(N)^3$ theory.

Let us see the solution in detail. There are 2 sets of 0-forms and 2-forms (Hodge dual to each other) RR fields in the twisted sector that couple to the brane via WZ terms

$$S \sim \frac{1}{\alpha'^2} \int d^4x \; \Sigma_q C_q \wedge \mathrm{tr} \; \exp 2\pi\alpha' F$$

In terms of the dual 0-forms $C \equiv C_0$ we have the following action

$$S \sim \int d^4x \; \frac{1}{\alpha'} (\mathrm{tr} \; A_\mu + \partial_\mu C)^2 + C \, \mathrm{tr} \; F_{\mu\nu} \tilde{F}_{\mu\nu}$$

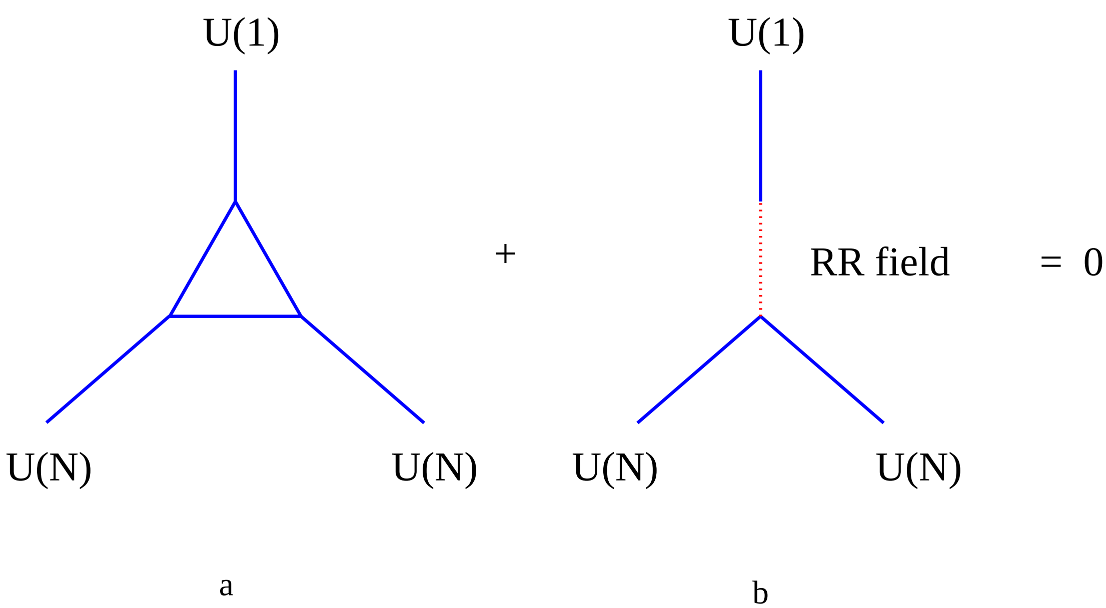
Under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \; , \;\; C \to C - \lambda .$$

In this way the $U(1)$'s get a gauge invariant mass of the string scale.

In addition, the anomaly cancels (as it should in string theory) due to the variation of the interaction term between the RR scalar and the gauge field.

Diagrammatically, we have a version of the Green-Schwarz mechanism



The triangle diagram is cancelled by a tree level diagram which involves a propagation of a closed string field.

The noncommutative Green-Schwarz mechanism

In the presence of a constant NS-NS 2 form, in the Seiberg-Witten limit, the field theory on the D3 branes becomes noncommutative.

What is therefore the mechanism that guarantees the consistency of the field theory ?

Clearly, the above mechanism should be modified.

First: the low energy theory cannot be an $SU(N)$ theory, since the $SU(N)$ algebra is not closed under noncommutative gauge transformation.

Second: the triangle diagram exists, in the noncommutative case, only for $\theta q = 0$. It cannot be be cancelled by the tree level diagram without a modification.

We suggest the following solution: The tree level diagram also exists only for $\theta q = 0$.

The reason is as follows. The intermediate RR field originates from a closed string. According to Seiberg and Witten, it 'feels' a different metric: the closed string metric. We need to translate the closed string metric to an open string metric by using

$$g^{-1} = G^{-1} - \frac{\theta G \theta}{(2\pi\alpha')^2},$$

Accordingly the RR field propagates in the open string metric as follows

$$\frac{1}{q^2 + \frac{n}{\alpha'}} \rightarrow \frac{1}{q^2 + \frac{(\theta q)^2}{(2\pi\alpha')^2} + \frac{n}{\alpha'}}$$

For $\theta q = 0$, the situation is as in the ordinary (commutative) theory. For $\theta q \neq 0$ the whole massive tower could (in principle) participate in the mechanism, but a direct calculation (and gauge invariance) lead to the vanishing of their contribution.

We therefore propose the Lagrangian

$$\mathcal{L} \sim \frac{1}{\alpha'}(\mathrm{tr}\, A_\mu + \partial_\mu C)(q) G^{\mu\nu} (\mathrm{tr}\, A_\nu + \partial_\nu C)(-q)|_{\theta q=0}.$$

This action is gauge invariant with respect to a noncommutative gauge transformation ! The reason is that for $\theta q = 0$ the gauge transformation

$$\mathrm{tr}\ \delta_\lambda A_\mu(q) =$$
$$\mathrm{tr}\ i q_\mu \lambda(q) + \mathrm{tr}\ \int d^4k A(k)\lambda(q-k)\sin k\theta q$$

becomes commutative.

The anomalous $U(1)$'s become massive only if they propagate with $\theta q = 0$, namely with momentum transverse to the noncommutative plane. Otherwise, the low-energy theory is $U(N)$.

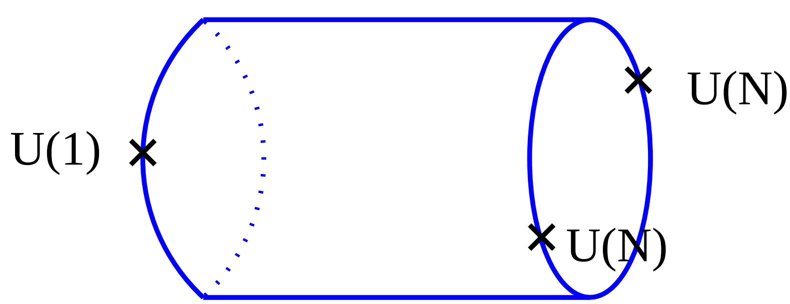
Concerning the anomaly cancellation:

Assuming, as in the commutative case, a term $C\,\mathrm{tr}\ F_{\mu\nu}\tilde{F}_{\mu\nu}$. We will obtain the following effective action after integration over the RR field

$$S_{eff} = \int d^4q\,(\frac{1}{\partial^2}\mathrm{tr}\,\partial A)(q)(\mathrm{tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu})(-q)|_{\theta q=0}.$$

Thus, the anomaly from the measure is cancelled by tree level contribution in a Green-Schwarz mechanism that involves only $\theta q = 0$ modes.

In a direct string theory calculation the mechanism originates from



Summary

- Nonplanar anomalies do not vanish.
- In the case of local nonplanar anomalies, anomaly cancellation can be achieved via a GS mechanism, which involves RR modes with $\theta q = 0$. The low-energy theory is a gauge invariant $U(N)$ theory and certain $U(1)$'s become massive only for $\theta q = 0$.
- Global anomalies do not automatically vanish and therefore noncommutativity is not excluded.