1/N Expansions in String and Gauge Field Theories

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Oberwoelz, September 2010

Motivation

It is extremely difficult to carry out reliable calculations in the strongly coupled regime of QCD.

In 1973 't Hooft suggested to consider a generalization of QCD where the gauge group $SU(3)$ is replaced by $SU(N)$.

't Hooft showed that the Feynman graphs of the $SU(N)$ theory are classified according to their topology. In the limit $N \to \infty$ planar graphs dominate, hence the large-N theory should be simpler than real world QCD ($SU(3)$).

Moreover, 't Hooft showed that the classification of the pure $SU(N)$ graphs matches a perturbative expansion of a *closed string theory* with a string coupling $1/N$. Since the seminal work of 't Hooft, there is a hope that QCD, or at least large-N Yang-Mills theory, will be described in terms of a weakly coupled string theory.

Motivation

25 years after 't Hooft, Maldacena (presumably inspired by Polyakov) taught us that $\mathcal{N} = 4$ super Yang-Mills can be described in terms of a closed string theory: type IIB string theory on $AdS_5 \times S^5$.

In this talk I will review various $1/N$ expansions in gauge field theories and their realization in string theory.

Outline

The outline of the talk is as follows

- ’t Hooft’s $1/N$ expansion of pure Yang-Mills theory
- Adding fundamental flavors
- The topological expansion (Veneziano)
- The “orientifold” large-N expansion
- The AdS/CFT correspondence
- Adding “quenched” matter (probe brane approximation)
- The worldline approach to fundamental matter
- The string dual to “orientifold” field theories

’t Hooft’s large-N_c limit

In 1973 ’t Hooft suggested that a generalization of QCD from $SU(3)$ to $SU(N)$ may result in a simpler theory in the large N limit (’t Hooft, 1973).

More precisely, ’t Hooft showed that when taking the limit $N \to \infty$, while keeping fixed the combination $\lambda \equiv g^2 N$ (now called the ’t Hooft coupling), the theory is controlled by *planar diagrams.*

Let us focus on $U(N)$ pure Yang-Mills theory, namely on

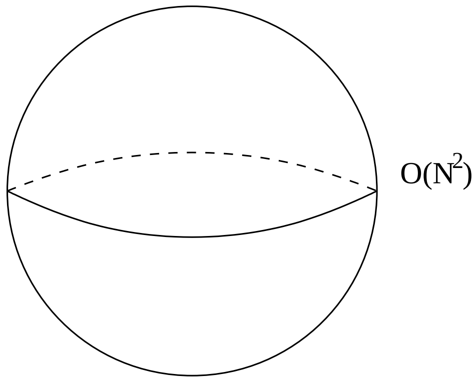
$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu}$$

’t Hooft showed that Feynman diagrams are classified according to their topology.

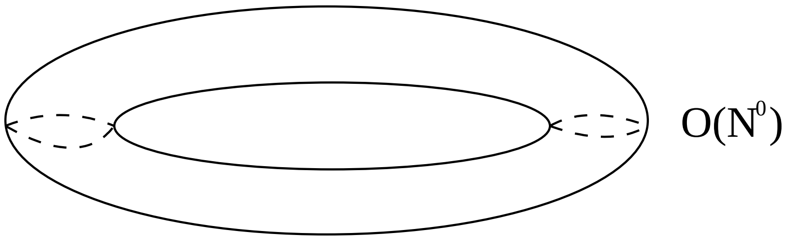
Large-N

For simplicity let us focus on 'vacuum diagram':

1. Planar diagram, admitting the topology of a sphere are $\mathcal{O}(N^2)$.



2. Non-planar diagrams carry a weight of $\mathcal{O}(N^{(2-2h)})$, where h is the number of holes in the diagram.



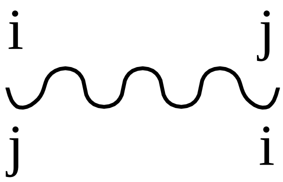
Thus, in the limit $N \to \infty$ only vacuum diagrams with sphere topology dominate.

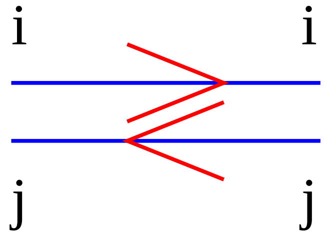
For this reason the large-N limit is simpler: it selects only a sub-set of Feynman diagrams.

’t Hooft double-index notation

In order to understand ’t Hooft’s idea, let us introduce the ’t Hooft notation:

The gluon propagator is represented by two lines with arrows that point in opposite directions.

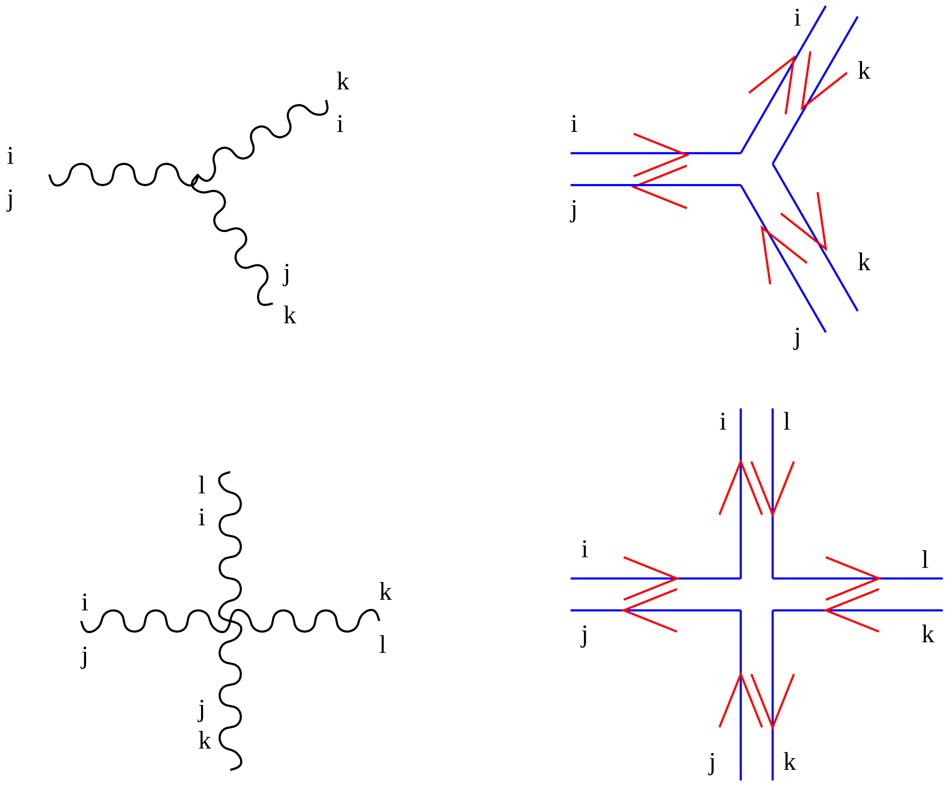




These lines represent the flow of color in the Feynman diagram. We should think about the adjoint representation as the tensor product of the fundamental and the anti-fundamental representations.

't Hooft double-index notation

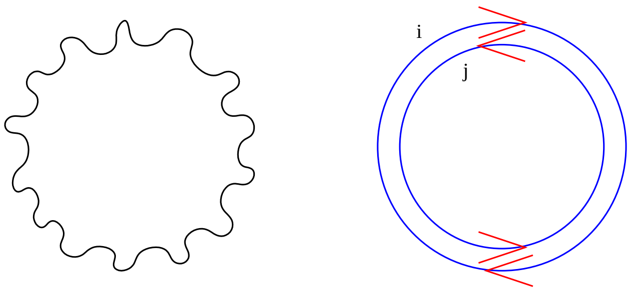
The three and four gluons vertices are depicted in the figure below



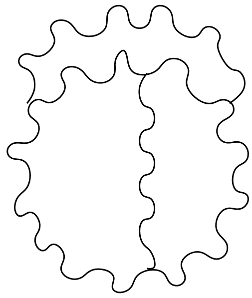
(I've ignored the ghosts).

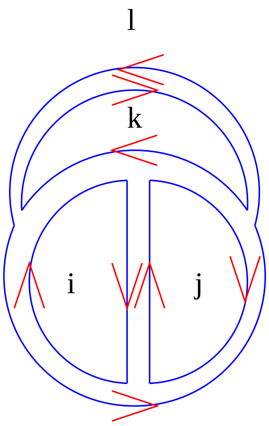
Examples

Let us consider two examples. First, a one-loop vacuum diagram



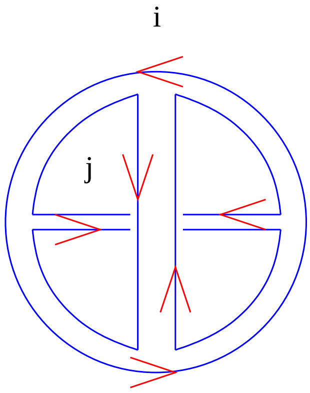
Second, a three-loop vacuum diagram which contains both planar and non-planar* contributions





Topology: Sphere

$g^4 N^4 = (g^2 N)^2 N^2$



Topology: Torus

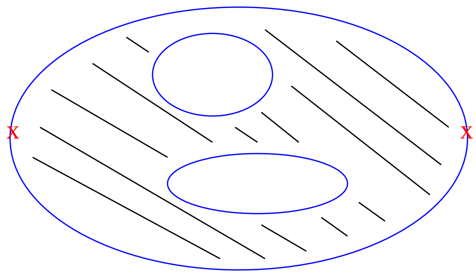
$g^4 N^2 = (g^2 N)^2$

Adding fundamental matter

Suppose that we wish to add N_f flavors of quarks, transforming in the fundamental representation

$$\mathcal{L} = \bar{\Psi}^k (i \not{D} - m) \Psi_k$$

Fundamental loops create "windows" in Feynman graphs



The resulting dependence on N_f and N_c of a Mesonic k-point function with w windows, h handles and b boundaries is

$$\langle M(x_1) M(x_2) ... M(x_k) \rangle \sim \left(\frac{N_f}{N_c}\right)^w N_c^{(2-\frac{k}{2}-2h-b)}$$

In the 't Hooft limit $N_c \to \infty$ while N_f is fixed diagrams with windows are sub-leading.

We can, however, consider another limit where both $N_c \to \infty$ and $N_f \to \infty$.

The topological expansion (Veneziano)

Consider again the expression

$$\langle M(x_1)M(x_2)...M(x_k)\rangle \sim \left(\frac{N_f}{N_c}\right)^w N_c^{(2-\frac{k}{2}-2h-b)}$$

In 1976 Veneziano suggested to keep $N_f/N_c \equiv \nu$ fixed while taking the limit $N_c \to \infty$. When ν is small, we can think about it an another coupling and expand in this parameter.

So while in the 't Hooft limit fermion loops are neglected, the Veneziano limit takes into account the back-reaction of dynamical quarks.

For example, the η' mass, given by the Witten-Veneziano formula, is a leading effect in the Veneziano limit

$$M^2_{\eta'} = \frac{N_f}{N_c}\chi$$

but it is not justified, as $\nu = 1$ and higher order contributions in ν should be taken into account.

The "orientifold" large-N expansion

(Armoni, Shifman, Veneziano, 2003)

There is another large-N generalization of QCD (introduced by Corrigan and Ramond in 1979 for a different reason).

Consider $SU(3)$ Yang-Mills with N_f flavors. Now generalize to $SU(N_c)$ with N_f flavors of fermions in the two-index antisymmetric representation $\Psi_{[ij]}$. For $SU(3)$ the antisymmetric is equivalent to the fundamental representation, so for $N_c = 3$ we simply obtain multi-flavor QCD.

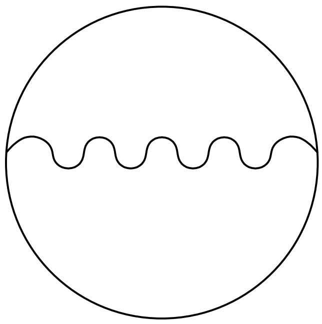
For large-N_c we obtain a theory which is equivalent in a well-defined bosonic sector to a theory with N_f adjoint fermions.

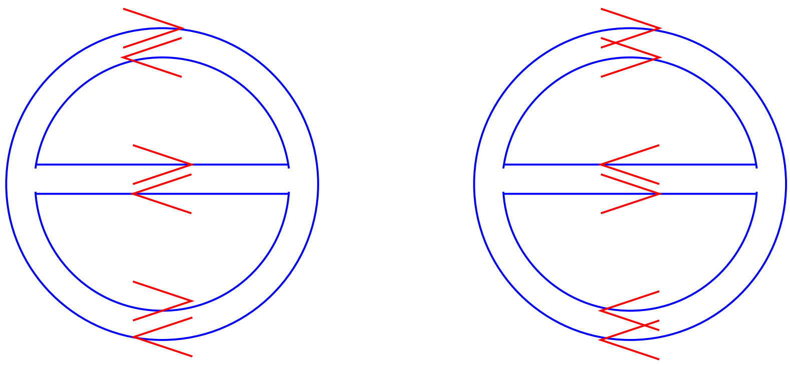
The reason is as follows: all we need to do in order to relate a theory with antisymmetric fermions to a theory with adjoint fermions is to reverse the arrow on one of the double lines that represent the fermion.

Adjoint fermion

Antisymmetric fermion

The planar graphs of theories with antisymmetric fermions and adjoint fermions are the same. For example





N=1 SYM

Orientifold

This is also true non-perturbatively, assuming that charge conjugation symmetry is not spontaneously broken. (Unsal and Yaffe, 2006)

When $N_f = 1$ the statement about planar equivalence relates the "orientifold" large-N_c limit of one-flavor QCD with $\mathcal{N} = 1$ super Yang-Mills theory (ASV, 2003).

It leads to several quantitative and semi-quantitative predictions, due to "SUSY relics in QCD", among them

- An approximate degeneracy between scalar and pseudo-scalar color-singlets
- A calculation of the quark condensate in one-flavor QCD from the gaugino condensate in $\mathcal{N} = 1$ SYM

  $$\langle \bar{q}q \rangle_{2\,\mathrm{GeV}} = -(270 \pm 30\,\mathrm{MeV})^3\,.$$

  This number has to be compared with the results of (DeGrand et.al. 2006)

  $$\langle \bar{q}q \rangle_{2\,\mathrm{GeV}} = -(269(9)\,\mathrm{MeV})^3\,.$$

- An NSVZ beta function for large-N_c QCD
- Same Regge slope for mesons and baryons

Large-N_c and String Theory

As we've already seen, the Feynman graphs of pure $SU(N)$ Yang-Mills theory (or any other theory with fields transforming in the adjoint representation) can be classified according to their topology.

This observation led 't Hooft to suggest that we should think about Feynman graphs as *string* worldsheets.

For pure Yang-Mills, the worldsheets correspond to a hypothetical *oriented closed string theory* with a string coupling $g_{st} = 1/N$.

But which closed string theory ? and what corresponds to the 't Hooft coupling $\lambda \equiv g^2 N$?

The answer, in the case of $\mathcal{N} = 4$ super Yang-Mills was given 25 years later by Maldacena !

The AdS/CFT correspondence

Maldacena conjectured that $\mathcal{N}=4$ SYM is equivalent to $D=10$ type IIB string theory on $AdS_5 \times S^5$.

The gauge theory 't Hooft coupling is related to the AdS (or sphere) radius by

$$\frac{R^2}{\alpha'} = \sqrt{4\pi\lambda}$$

Two-point functions of local operators are given by propagators of bulk operators.

For example: in large-N_c Yang-Mills we expect

$$\langle F^2(x) F^2(y) \rangle = \int d^4k \, (\exp ik(x-y)) G(k)$$

where $G(k)$ is the sum of all propagators of even-parity glueballs.

Similarly, in $\mathcal{N}=4$ SYM, $G(k)$ is the propagator of the dilaton, except that the propagation is in the bulk of the AdS space.

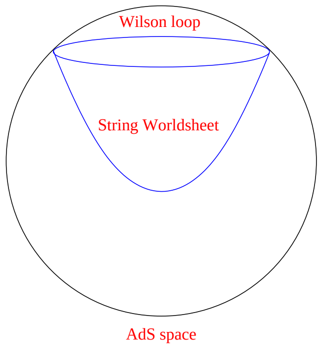
Calculating Wilson loops vev's

Another interesting example is the calculation of Wilson loops using the AdS/CFT correspondence.

The Wilson loop expectation value is given by

$$\langle \mathrm{tr}\, \exp i \int d\tau\, A_\mu \dot{x}^\mu \rangle = \exp -I_{\mathrm{N.G.}}$$

where I is the Nambu-Goto action of a string whose worldsheet extends to the bulk and whose boundary is given by the contour of the Wilson loop, as depicted in the figure below



In particular the expectation value of a rectangular Wilson loop yields an inter-quark potential

$$V(L) = \frac{\sqrt{\lambda}}{L}$$

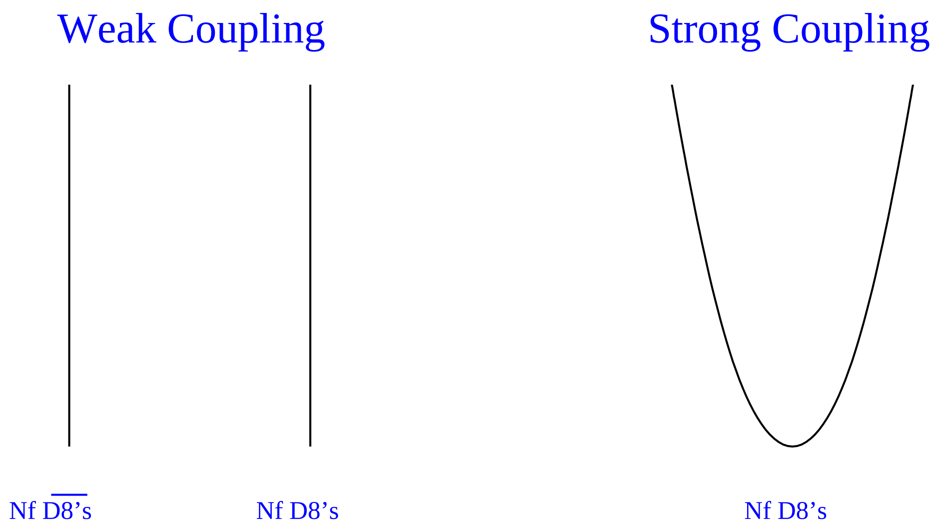
Adding fundamental matter

Adding fundamental matter in the 't Hooft limit of fixed N_f and $N_c \to \infty$ (quenched matter), corresponds to adding a "probe brane". The probe brane does not back-react on the geometry. The data which is obtained in this approach corresponds to the data that one can extract from lattice simulation that are made in the quenched approximation.

An example: the open strings on the probe brane correspond to mesons. Thus, by studying the small fluctuations of the brane one can find the meson spectra of a given theory (Sakai and Sonnenschein, 2003).

The Sakai-Sugimoto model

An interesting model, as close as possible to QCD, is the Sakai-Sugimoto model (Sakai and Sugimoto, 2004). In this model a set of N_f $D8$ and N_f anti-$D8$ branes are introduced. These branes correspond to the classical $U(N_f) \times U(N_f)$ flavor symmetry. At strong coupling, due to the topology of space, the branes merge to form a U-shape $D8$-branes.



So now only a sub-$U(N_f)$ symmetry is manifest. This is a geometric realization of chiral symmetry breaking !

Veneziano limit and string theory

When $\nu \equiv N_f/N_c$ is $\mathcal{O}(1)$, a description of the gauge theory in terms of a perturbative string theory is impossible, since the (would be string-)theory is necessarily strongly coupled.

The reason is that the effective string coupling which controls the perturbative expansion is $g_{eff} = N_f/N_c = g_{st}N_f \sim \mathcal{O}(1)$. Hence, a perturbative description might be possible only if $g_{st}N_f \ll 1$.

A certain approach, which might be useful for holographic description of the Veneziano limit is the worldline formalism (A.A. 2008, 2009).

$$(\det i \not{D})^{N_f} = \exp N_f \Gamma[A]$$

where Γ is a sum over all possible Wilson loops W. An observable in QCD can be written as

$$\langle \mathcal{O} \rangle_{\text{QCD}} = \langle \mathcal{O} \rangle_{\text{YM}} + \frac{N_f}{N_c} \sum \langle \mathcal{O}\mathcal{W} \rangle_{\text{YM}} + \left(\frac{N_f}{N_c}\right)^2 \sum \sum \langle \mathcal{O}\mathcal{W}\mathcal{W} \rangle_{\text{YM}} + \ldots$$

Veneziano limit and the worldline formalism

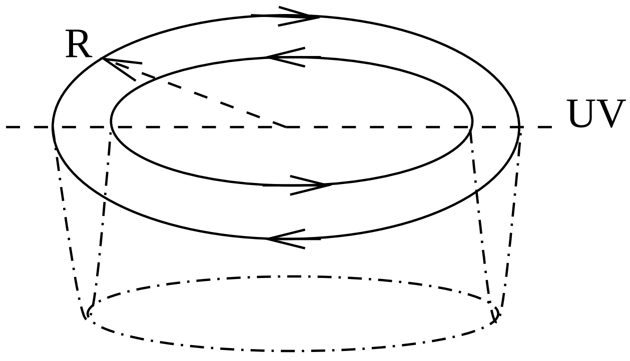
Since Wilson loops are minimal string worldsheets that terminate on the AdS boundary, it is clear how to incorporate fundamental matter into the AdS/CFT dictionary.

Let us consider an example. A calculation of the expectation value of a circular Wilson loop in QCD, namely in $SU(N_c)$ Yang-Mills theory with N_f massless quarks in the fundamental representation.

In the confining YM vacuum the Wilson loop admits an area law

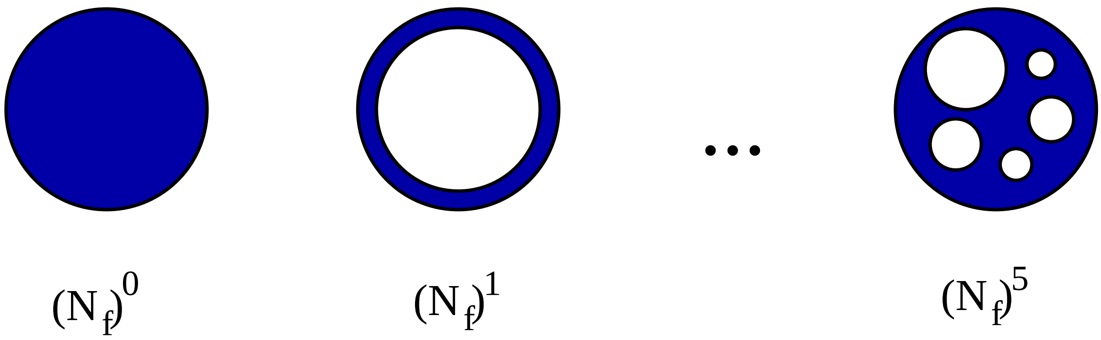
$$\langle\mathcal{O}\rangle_{\mathrm{YM}} = \exp -\sigma\mathcal{A}$$

In order to see the effect of dynamical quarks in the Veneziano limit we have to calculate the two-point function of the the circular Wilson loops $\mathcal{O}$ with all possible Wilson loops. The dominant Wilson loop will create a big hole in $\mathcal{O}$ as depicted in the figure below



AdS Space

Similarly, higher orders corrections in N_f, create w windows in the string worldsheet



The outcome is a correction to the above result

$$\langle \mathcal{O} \rangle_{\rm QCD} = \exp -\sigma \mathcal{A} + \frac{N_f}{N_C} \exp -\mu P + \ldots$$

Thus the fundamental quarks create a window in the string worldsheet to screen the external heavy quarks.

String dual to "Orientifold" field theories

The large-N_c limit based on theories with antisymmetric matter $\Psi_{[ij]}$ has a dual string description based on Sagnotti model (Sagnotti, 1995).

The dual string, also called *type 0'B string theory*, is a non-tachyonic non-supersymmetric theory. The closed string sector of the theory consists of the bosonic modes of the type IIB superstring.

Therefore any calculation made by using the AdS/CFT based on type IIB supergravity, which does not involve closed string fermionic modes, will be valid for the type 0'B string as well.

It means that all large-N_c results for the bosonic color-singlets of a supersymmetric gauge theory with adjoint matter is valid also for a theory with antisymmetric fermions. This is precisely the statement about planar equivalence, but now obtained via AdS/CFT.

AdS/CFT for “Orientifold” Theories

One of the predictions of planar equivalence is that scalars and pseudo-scalars become mass degenerate at large-N_c.

A careful gauge/gravity calculation (Armoni and Imeroni, 2005) suggests that the mass ratio of the η' and the σ in one-flavor QCD is

$$M_{\eta'}/M_\sigma \sim (N-2)/N = 1/3$$

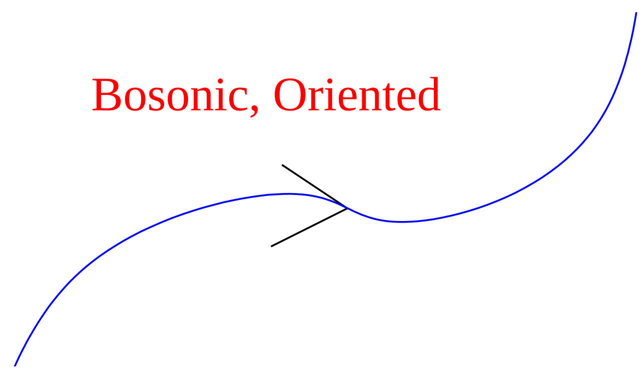
In order to check this prediction, a lattice simulation of one-flavor QCD was carried out by the Munster collaboration (Farchioni et.al. 2007). Their result for the masses of η' and σ, extrapolated to the chiral limit, are

$$M_{\eta'}/M_\sigma = 0.410(32)(25)\,,$$

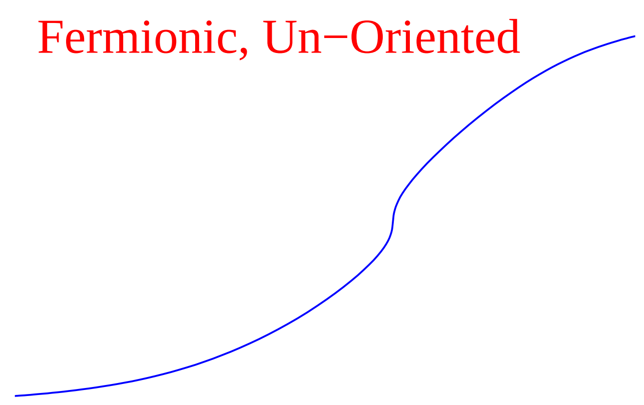
In a reasonable agreement with our estimate.

Open Strings

The Sagnotti model consists of two kinds of open strings. The bosonic oriented string



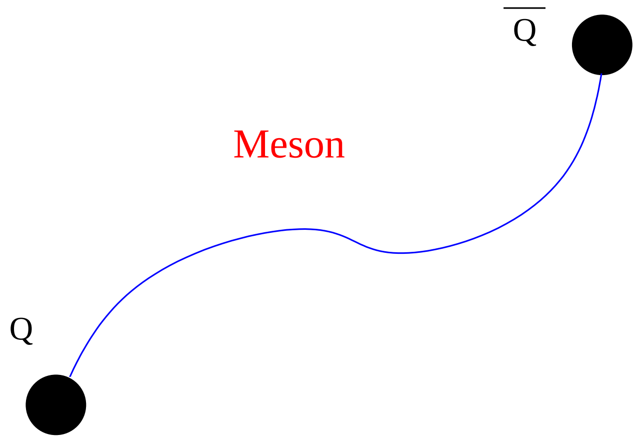
and the fermionic un-oriented string



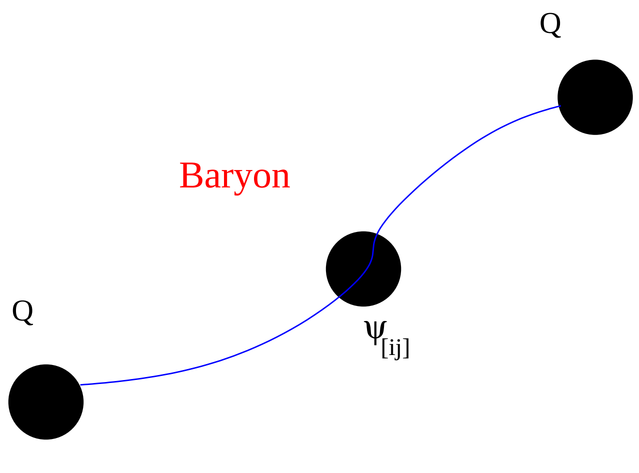
The two-kinds of open strings are mass degenerate, similarly to the open strings of type I, despite of the lack of supersymmetry in the type 0' case.

Meson and Baryons as open strings

The bosonic oriented string is interpreted as a Meson of the form $M = \bar{Q}^i Q_i$



The fermionic un-oriented string is interpreted as a *Baryon* of the form $B = Q^i \Psi_{[ij]} Q^j$



such a baryon has been advocated by Corrigan and Ramond a long time ago.

Open Strings

Type 0' string theory gives a realization of the CR baryon as an open string and predicts that it is mass degenerate with the meson, as if the mesons and the baryons are super-partners.

A careful field theory analysis (Armoni and Patella, 2008)reveals that for large radial or angular excitations the CR baryon and the meson become indeed degenerate, namely they admit the same *Regge slope*.

Note that this idea is similar (however different) to the idea of Wilczek that the baryon is made of a quark and a diquark.

Summary

In this talk I've discussed three $1/N$ expansions and their string theory realization

- 't Hooft's expansion
- The topological (Veneziano) expansion
- The "orientifold" large-N expansion

Each one of the above expansions has advantages and disadvantages. Which one should be used depends on the problem at hand.

Thank you!