

QFT HOMEWORK 1

DUE THURSDAY November 8

Note on normalization of states and creation and annihilation operators

In non-relativistic quantum mechanics, we normalize momentum eigenstates as follows

$$\langle \vec{k} | \vec{k}' \rangle = \delta^{(3)}(\vec{k} - \vec{k}')$$

If we consider these one particle states as being created from creation operator $|\vec{k}\rangle = a_{\vec{k}}^\dagger |0\rangle$, then the creation/annihilation operators satisfy the commutation relation

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$$

These states are not relativistically normalized in the sense that $\langle \Lambda \vec{k} | \Lambda \vec{k}' \rangle \neq \langle \vec{k} | \vec{k}' \rangle$. However, we can define relativistically normalized states as follows (we label them with the four vector k instead of the three vector \vec{k}):

$$|k\rangle = (2\pi)^{3/2} \sqrt{2\omega_{\vec{k}}} |\vec{k}\rangle$$

These satisfy $\langle \Lambda k | \Lambda k' \rangle = \langle k | k' \rangle$ and are created by creation operators $\alpha_k^\dagger = (2\pi)^{3/2} \sqrt{2\omega_{\vec{k}}} a_{\vec{k}}^\dagger$, such that $|k\rangle = \alpha_k^\dagger |0\rangle$.

We can then write the mode expansion of a scalar field $\phi(x)$ in to equivalent ways as follows:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_{\vec{k}}}} (a_{\vec{k}} e^{-ik \cdot x} + a_{\vec{k}}^\dagger e^{ik \cdot x}) = \int \frac{d^3k}{(2\pi)^3 (2\omega_{\vec{k}})} (\alpha_k e^{-ik \cdot x} + \alpha_k^\dagger e^{ik \cdot x})$$

Notice that in the first of these expressions in terms of $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$, the measure $\frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_{\vec{k}}}}$ is not Lorentz invariant and the creation annihilation operators are also not Lorentz invariant. In the second expression in terms of α_k and α_k^\dagger , the measure $\frac{d^3k}{(2\pi)^3 (2\omega_{\vec{k}})}$ is Lorentz invariant and so are the creation annihilation operators α_k and α_k^\dagger .

1) Consider a real free scalar field. Using the canonical commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}')$$

with other commutators being zero, show that the creation annihilation operators in the mode expansion of $\phi(\vec{x}, t)$ satisfy the only non-trivial

commutation relation

$$[\alpha_k, \alpha_{k'}^\dagger] = (2\pi)^3 2\omega_{\vec{k}} \delta^3(\vec{k} - \vec{k}'), \quad [a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta^3(\vec{k} - \vec{k}')$$

2) Using the commutation relations for the creation and annihilation operators given above, show that the state

$$|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle \equiv a^\dagger(\vec{k}_1) \cdots a^\dagger(\vec{k}_n)|0\rangle$$

is an eigenstate of the hamiltonian with eigenvalue $\omega_1 + \cdots + \omega_n$. The vacuum $|0\rangle$ is annihilated by $a(\vec{k})$, $a(\vec{k})|0\rangle = 0$, and take its energy to be zero.

3) Under Lorentz transformation, the scalar field transforms $U(\Lambda)\phi(x)U(\Lambda)^\dagger = \phi(\Lambda^{-1}x)$, where $U(\Lambda)$ is a unitary operator which implements the Lorentz transformations on the Fock space. Show that

$$\begin{aligned} U(\Lambda)^{-1}\alpha_{\vec{k}}U(\Lambda) &= \alpha_{\Lambda^{-1}\vec{k}} \\ U(\Lambda)^{-1}\alpha_{\vec{k}}^\dagger U(\Lambda) &= \alpha_{\Lambda^{-1}\vec{k}}^\dagger \end{aligned}$$

and

$$\begin{aligned} U(\Lambda)^{-1}a_{\vec{k}}U(\Lambda) &= \left(\frac{\omega_{\Lambda^{-1}\vec{k}}}{\omega_{\vec{k}}}\right)a_{\Lambda^{-1}\vec{k}} \\ U(\Lambda)^{-1}a_{\vec{k}}^\dagger U(\Lambda) &= \left(\frac{\omega_{\Lambda^{-1}\vec{k}}}{\omega_{\vec{k}}}\right)a_{\Lambda^{-1}\vec{k}}^\dagger \end{aligned}$$

and hence that

$$U(\Lambda)|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle = \left(\frac{\omega_{\Lambda^{-1}\vec{k}_1}}{\omega_{\vec{k}_1}}\right)\left(\frac{\omega_{\Lambda^{-1}\vec{k}_2}}{\omega_{\vec{k}_2}}\right) \cdots \left(\frac{\omega_{\Lambda^{-1}\vec{k}_n}}{\omega_{\vec{k}_n}}\right)|\Lambda\vec{k}_1, \Lambda\vec{k}_2, \dots, \Lambda\vec{k}_n\rangle$$

and

$$U(\Lambda)|k_1, k_2, \dots, k_n\rangle = |\Lambda k_1, \Lambda k_2, \dots, \Lambda k_n\rangle$$

4) Let $\Lambda(\lambda)^\mu{}_\nu$ be a one parameter family of Lorentz transformations under which a vector a^μ transforms as

$$a^\mu \rightarrow a^\mu(\lambda) = \Lambda(\lambda)^\mu{}_\nu a^\nu$$

Under this transformation, the scalar fields ϕ^a transform as

$$\phi^a(x) \rightarrow \phi^a(x, \lambda) = \phi^a(\Lambda(\lambda)^{-1}x).$$

Define a matrix $\epsilon^\mu{}_\nu$ as

$$D\Lambda^\mu{}_\nu \equiv \epsilon^\mu{}_\nu$$

i) Show that the condition that $\epsilon^\mu{}_\nu$ must satisfy given that $a^\mu b_\mu$ is invariant under Lorentz transformation is

$$\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$$

ii) What kind of transformations correspond to $\epsilon_{01} = -\epsilon_{10} = +1$ with all other components being zero.

iii) What kind of transformations correspond to $\epsilon_{12} = -\epsilon_{21} = +1$ with all other components being zero.

iv) Calculate $D\phi^a$.

v) Calculate $D\mathcal{L}$.

vi) Show that the Noether's current J^μ associated with Lorentz transformations is

$$J^\mu = \epsilon_{\lambda\sigma} \left(\sum_a \pi_a^\mu x^\lambda \partial^\sigma \phi^a - x^\lambda g^{\mu\sigma} \mathcal{L} \right)$$

This current must be conserved for all six independent antisymmetric matrices, $\epsilon_{\lambda\sigma}$, so the quantity in the parenthesis that is antisymmetric in λ and σ must be conserved, *i.e.* $\partial_\mu \mathcal{M}^{\mu\lambda\sigma} = 0$ where

$$\mathcal{M}^{\mu\lambda\sigma} = \sum_a \pi_a^\mu x^\lambda \partial^\sigma \phi^a - x^\lambda g^{\mu\sigma} \mathcal{L} - (\lambda \leftrightarrow \sigma).$$

vii) Show that

$$M^{\mu\lambda\sigma} = x^\lambda T^{\mu\sigma} - x^\sigma T^{\mu\lambda}$$

viii) Show that the charge $Q^{\lambda\sigma} = \int d^3x M^{0\lambda\sigma}$ is given by the expression

$$Q^{\lambda\sigma} = \int d^3x (x^\lambda T^{0\sigma} - x^\sigma T^{0\lambda})$$

ix) Consider a system of point particles with momentum density given by

$$T^{0i}(\vec{x}, t) = \sum_a p_a^i \delta^{(3)}(\vec{x} - \vec{r}^a(t))$$

Show that $J^i = \frac{1}{2} \epsilon_{ijk} Q^{jk}$ is the ordinary three dimensional angular momentum.

x) Using the fact that $\frac{d}{dt} Q^{0i} = 0$, show that

$$\frac{d}{dt} \left(\frac{\int d^3x x^i T^{00}}{\int d^3x T^{00}} \right) = \text{constant}$$

i.e. the relativistic analog of center of mass (we can call it center of energy) moves with a constant velocity.

5) Consider a real scalar field with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - g \rho(x) \phi(x)$$

where $\rho(x) \rightarrow 0$ as $x^\mu \rightarrow \infty$ in both the spatial and time-like direction.

i) Explicitly calculate all connected Wick diagrams.

ii) Show that if we start from the vacuum in the far past, the probability to create n particles in the far future p_n is of the form

$$p_n = e^{-|\alpha|} \frac{|\alpha|^n}{n!},$$

What is α ?

iii) What is the expectation value of the number of particles $\langle n \rangle$ in the final state?

iv) What is the expectation value of the Hamiltonian $\langle H \rangle$ in the final state?

v) What is the expectation value of momentum $\langle \vec{p} \rangle$ in the final state?