QFT HOMEWORK 1
DUE THURSDAY November 15

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The Scattering Amplitude in Non-Relativistic Quantum Mechanics

We start by considering the scattering of particles of mass m incident on a fixed scattering center which is represented by the potential $V(\vec{r})$. In the absence of the potential, the incident particles, with energy E and travelling in the z-direction, are represented by the plane wave

$$\psi_0(\vec{r}) = e^{ikz}$$

where the wave number k is given by

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

and $\psi_0(\vec{r})$ is normalized to one particle per unit volume. $\psi_0(\vec{r})$ is a solution of the free-particle Schrodinger equation

$$(\nabla^2 + k^2)\psi_0(\vec{r}) = 0.$$

In the presence of the potential $V(\vec{r})$, the plane wave $\psi_0(\vec{r})$ is distorted. The wave function describing the particles is now a solution of the Schrodinger equation

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r}), \tag{1}$$

satisfying the appropriate boundary conditions:

$$\psi(\vec{r}) \rightarrow \psi_0(\vec{r}) = e^{ikz}, \text{as } V(\vec{r}) \rightarrow 0. \tag{2}$$

Secondly, we require a solution which represents the plane incident wave $\psi_0(\vec{r})$ together with a scattered outgoing wave. We will assume that $V(\vec{r})$ is a short range potential; more precisely, either that it has a finite range a (i.e. $V(\vec{r}) = 0$ for $r > a$) or that it tends to zero sufficiently rapidly so that

$$rV(\vec{r}) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

in all directions. In this problem, we shall show that, for a potential satisfying this condition, the Schrodinger equation possesses solutions which are of the form

$$\frac{1}{r}e^{\pm ikr}f(\theta,\phi) \quad \text{as} \quad r \rightarrow \infty.$$

These wavefunctions represent spherical waves, centered on $\vec{r} = 0$; outgoing waves for the positive exponent, ingoing waves for the negative exponent. To see this,
i) Calculate the radial component of the probability current tensity $\vec{j}$. For a wave function $\psi(\vec{r})$, this is defined by

$$\vec{j} = \frac{-i\hbar}{2m}(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$$

Show that

$$j_r = \pm\frac{\hbar k}{m}|f(\theta, \phi)|^2 \frac{1}{r^2}$$

In our scattering problem, we require a wave function $\psi(\vec{r})$ which corresponds to the plane incident wave together with a scattered outgoing wave, i.e. with the asymptotic form

$$\psi(\vec{r}) \sim e^{ikz} + \frac{1}{r}e^{ikr}f(\theta, \phi) \quad \text{as} \quad r \to \infty.$$

$f(\theta, \phi)$ is called the scattering amplitude. From this, the cross-section can be easily found.
ii) Calculate the number of particles $n(\theta, \phi)d\Omega$ scattered into an element of solid angle $d\Omega$ in the direction (θ, ϕ) per unit time by first showing that

$$n(\theta, \phi)d\Omega = j_r r^2 d\Omega$$

and then using the result of (i).
iii) Using the fact that the incident wave is normed to one particle per unit volume, show that the incident flux I equals $\frac{\hbar k}{m}$.
iv) Using the result of (iii) show that the differential cross section becomes

$$\sigma(\theta, \phi)d\Omega = |f(\theta, \phi)|^2 d\Omega.$$

$\mathbf{f}(\theta, \phi)$

We will now get an expression for $f(\theta, \phi)$.
v) Show that the equation $(\nabla^2 + k^2)\psi(\vec{r}) = \delta^{(3)}(\vec{r})$ has a solution

$$\psi(r) = -\frac{e^{\pm ikr}}{4\pi r}$$

which are functions of the radial coordinate only.

We saw earlier that these solutions correspond to spherical outgoing and ingoing waves respectively. To represent scattered waves, we shall require the outgoing wave solutions only.
vi) Show that the solution to the equation $(\nabla^2 + k^2)\psi(\vec{r}) = F(\vec{r})$ is given by

$$\psi(\vec{r}) = -\frac{1}{4\pi}\int \frac{e^{ik|r-s|}}{|\vec{r}-\vec{s}|}F(\vec{s})d^3\vec{s}$$

vii) By taking $F\vec{s} = \frac{2m}{\hbar^2}V(\vec{s})\psi(\vec{s})$, show that the solution to (1) with boundary conditions (2)

$$\psi(\vec{r}) = e^{ikz} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{s}|}}{|\vec{r}-\vec{s}|}$$

viii) Show that for $r \gg s$,

$$|\vec{r}-\vec{s}| \sim r\Big[1 - \frac{1}{r^2}\vec{r}\cdot\vec{s} + \mathcal{O}(\frac{s^2}{r^2})\Big]$$

ix) Using this, show that

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int \exp(-i\vec{k}'\cdot\vec{s})V(\vec{s})\psi(\vec{s})d^3\vec{s} \tag{3}$$

where $\vec{k}' = k\vec{r}/r$. This is an exact expression for the scattering amplitude.

Born Approximation

We can easily derive the Born approximation from this expression for the scattering amplitude. If the potential $V(\vec{r})$ is weak so that it distorts the incident plane wave only slightly, then we can approximate the exact wave function in (3) by the plane wave

$$\psi_0(\vec{s}) = e^{i\vec{k}\vec{s}}$$

x) Show that with this replacement

$$f_{BA}(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int \exp(i\vec{\Delta}\cdot\vec{r})V(\vec{r})d^3\vec{r}$$

where $\Delta = \vec{k} - \vec{k}'$.

Scattering of Identical Particles

The elastic collisions of two particles interacting through a potential $V(r)$ which depends on their distance of separation r only is easily reduced to the potential scattering problem we have just discussed. Show that starting with the Hamiltonian

$$H = \frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

and defining the center of mass coordinate $\vec{R}$ and the relative coordinate $\vec{r}$ as follows:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

with m_1 and m_2 being the masses of the two particles, and $\vec{r}_1$ and $\vec{r}_2$ their position coordinates.

xi) Show that expressed in terms of $\vec{R}$ and $\vec{r}$, the Hamiltonian splits into a part describing the motion of the center of mass of the system and another part describing the motion of particles in a frame of reference in which their center of mass is at rest in terms of the relative motion of the two particles. Also show that this latter part is just the Hamiltonian for a particle of mass m equal to the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

in the potential $V(r)$. Hence our analysis and results for potential scattering at once apply to the elastic collisions of two particles in their center-of-mass frame.

For collisions of identical particles, our analysis must be modified. The states for identical particles must satisfy certain symmetry conditions under the interchange of particle labels and we must take this into account.

xii) Show that exchanging the two particles corresponds to replacing the $\vec{r} = (r, \theta, \phi)$ by $-\vec{r} = (r, \pi - \theta, \phi + \pi)$.

Hence by symmterizing the wavefunctions appropriately, show that the scattering amplitude is given by

$$f_{\text{identical}}(\theta, \phi) = f(\theta, \phi) \pm f(\pi - \theta, \phi + \pi)$$

where the $+$ sign corresponds to bosons and the $-$ sign corresponds to fermions.

2. <u>Scattering in model 3</u>

Compute the scattering amplitude in model 3 for $N + \bar{N} \to \phi + \phi$. Interpret the expression coming from each Feynman diagram in terms of their non-relativistic analogs.