Termodynamics and Transport in Improved Holographic QCD

Francesco Nitti

APC, U. Paris VII

Large $N$ @ Swansea
July 07 2009

Work with E. Kiritsis, U. Gursoy, L. Mazzanti, G. Michalogiorgakis
0707.1324, 0707.1349 0804.0899, 0812.0792, 0903.2895, 0906.1890
Phenomenological Holography and QCD

Bottom-up Holography:
Construct a gravity background such that, using holography rules, we can model the properties of the desired gauge theory (e.g. $AdS/QCD$, Pomarol and Da Rold ’05; Erlich et al ’05).
Bottom-up Holography:
Construct a gravity background such that, using holography rules, we can model the properties of the desired gauge theory (e.g. $AdS/QCD$, Pomarol and Da Rold '05; Erlich et al '05).

Minimal setup:
- 5 dimensions (4 space-time + radial (RG) direction)
- restrict to a small number of fields
- use hints from (what is known about) 5D non-critical string theory
Bottom-up Holography:
Construct a gravity background such that, using holography rules, we can model the properties of the desired gauge theory (e.g. $AdS$/QCD, Pomarol and Da Rold ’05; Erlich et al ’05).

Minimal setup:
- 5 dimensions (4 space-time + radial (RG) direction)
- restrict to a small number of fields
- use hints from (what is known about) 5D non-critical string theory

Aim: a realistic holographic model of the deconfined QGP (to compare with lattice and RHIC)
Outline
Outline

- Bottom-up holography
Outline

- Bottom-up holography
- Setup and Vacuum solutions
Outline

- Bottom-up holography
- Setup and Vacuum solutions
- Finite Temperature Solutions
  - Black hole solutions
  - Phase transition
Outline

- Bottom-up holography
- Setup and Vacuum solutions
- Finite Temperature Solutions
  - Black hole solutions
  - Phase transition
- Thermodynamics
Outline

- Bottom-up holography
- Setup and Vacuum solutions
- Finite Temperature Solutions
  - Black hole solutions
  - Phase transition
- Thermodynamics
- Transport: Shear and Bulk Viscosity
Outline

- Bottom-up holography
- Setup and Vacuum solutions
- Finite Temperature Solutions
  - Black hole solutions
  - Phase transition
- Thermodynamics
- Transport: Shear and Bulk Viscosity
- Heavy quark diffusion
Precursors

AdS/QCD (Pomarol and Da Rold ’05; Erlich et al ’05)

- the metric is the only bulk field in the gauge sector
- The 5D spacetime is $AdS_5$ truncated by a IR cut-off with no explicit dynamics
  $\implies$ confinement is imposed via boundary conditions

$e^{A(r)}$
**Precursors**

**AdS/QCD** (Pomarol and Da Rold ’05; Erlich *et al* ’05)
- the metric is the only bulk field in the gauge sector
- The 5D spacetime is $AdS_5$ truncated by a IR cut-off with no explicit dynamics
  $\Rightarrow$ confinement is imposed via boundary conditions

One step further: **Soft-Wall AdS/QCD** (Karch *et al* ’06)
- hard cut-off replaced by a dilaton that grows in the UV
- the metric is still exactly $AdS_5$ (No mass gap; gluons are not confined)
- 2 bulk fields now, but still no dynamics (no action, no field equations): cannot compute free energies consistently.
Improved Holographic QCD

Gursoy, Kiritsis, FN 0707.1324, 0707.1349

- Make the soft-wall **dynamical** and interpret it as the holographic dual of the glue sector.
- Treat metric-dilaton as a dynamical system (including full backreaction).
- Breaking of conformal symmetry, mass gap, confinement, and all non-perturbative dynamics driven by the dilaton dynamics (aka the Yang-Mills coupling).
**Improved Holographic QCD**

Gursoy, Kiritsis, FN 0707.1324, 0707.1349

- Make the soft-wall *dynamical* and interpret it as the holographic dual of the glue sector.
- Treat metric-dilaton as a dynamical system (including full backreaction).
- Breaking of conformal symmetry, mass gap, confinement, and all non-perturbative dynamics driven by the dilaton dynamics (aka the Yang-Mills coupling).

**Bulk Field Content ↔ Dimension 4 operators in Pure YM**

<table>
<thead>
<tr>
<th>4D Operator</th>
<th>Bulk field</th>
<th>Coupling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TrF^2$</td>
<td>$\Phi$</td>
<td>$\kappa N_c \int e^{-\Phi} TrF^2$</td>
</tr>
<tr>
<td>$T_{\mu\nu}$</td>
<td>$g_{\mu\nu}$</td>
<td>$\int g_{\mu\nu}T^{\mu\nu}$</td>
</tr>
</tbody>
</table>

$\lambda = \kappa N_c g^2_{YM} = e^\Phi$ (finite in the large $N$ limit).
The Setup

- only lowest dimension operators are considered
- action limited to 2-derivatives

\[ S_E = -M_p^3 N_c^2 \left( \int d^5 x \sqrt{-g} \left[ R - \frac{4}{3} (\partial \Phi)^2 - V(\Phi) \right] \right) \]
The Setup

- only lowest dimension operators are considered
- action limited to 2-derivatives

\[ S_E = -M_p^3 N_c^2 \left( \int d^5 x \sqrt{-g} \left[ R - \frac{4}{3} (\partial \Phi)^2 - V(\Phi) \right] \right) \]

- \( V(\Phi) \) fixed phenomenologically.
- \( N_c \) appears only as an overall factor.
  Effective Planck scale \( \sim N_c^2 \) is large.
- the parameter \( M_p \) can be fixed via thermodynamics in terms of \( V(-\infty) \)
Non-constant $V(\Phi)$: solutions have running coupling.

$$d s^2 = e^{2A(r)} \left( d r^2 + \eta_{\mu\nu} d x^\mu d x^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^{A(r)})$$
Vacuum Solution

Non-constant $V(\Phi)$: solutions have running coupling.

$$ds^2 = e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r))$$

UV: $e^A \to \infty$, $\lambda \to 0$, $V(\lambda) \sim \frac{12}{\ell^2} \left( 1 + v_0 \lambda + v_1 \lambda^2 \ldots \right)$

Asymptotically AdS solution; $\ell_{ads}^2 = 12/V(0)$ finite
Vacuum Solution

Non-constant $V(\Phi)$: solutions have running coupling.

$$ds^2 = e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r))$$

**UV:** $e^A \to \infty$, $\lambda \to 0$; $V(\lambda) \sim \frac{12}{\ell^2} \left( 1 + v_0 \lambda + v_1 \lambda^2 \ldots \right)$

Asymptotically AdS solution: $\ell_{ads}^2 = 12/V(0)$ finite

**IR:** $\lambda$ large, $e^A \to 0$; $V \sim \lambda^{4/3} (\log \lambda)^{1/2}$
Vacuum Solution

Non-constant $V(\Phi)$: solutions have running coupling.

$$ds^2 = e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r))$$

UV: $e^A \to \infty, \lambda \to 0, \quad V(\lambda) \sim \frac{12}{\ell^2} \left( 1 + v_0 \lambda + v_1 \lambda^2 \ldots \right)$

Asymptotically AdS solution; $\ell_{ads}^2 = 12/V(0)$ finite

IR: $\lambda$ large, $e^A \to 0; \quad V \sim \lambda^{4/3} (\log \lambda)^{1/2}$

Get asymptotic freedom, confinement, mass gap, linear glueball spectrum.
Vacuum Solution

Non-constant $V(\Phi)$: solutions have running coupling.

$$ds^2 = e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r))$$

UV: $e^A \to \infty, \lambda \to 0, \quad V(\lambda) \sim \frac{12}{\ell^2} \left(1 + v_0 \lambda + v_1 \lambda^2 \ldots \right)$

Asymptotically AdS solution; $\ell_{ads}^2 = 12/V(0)$ finite

IR: $\lambda$ large, $e^A \to 0; \quad V \sim \lambda^{4/3} (\log \lambda)^{1/2}$

Get asymptotic freedom, confinement, mass gap, linear glueball spectrum.

$$\beta(\lambda) \equiv \frac{d\lambda}{d \log E} = \frac{(d\lambda/dr)}{(dA/dr)}$$
Vacuum Solution

\[ e^A(r) \sim e^{-Cr^2}, \quad \lambda \sim e^{\frac{3}{2}Cr^2} \quad r \to \infty \]
Vacuum Solution

\[ e^A(r) \sim e^{-Cr^2}, \quad \lambda \sim e^{\frac{3}{2}Cr^2} \quad r \to \infty \]

In the string frame:
\[ S = \int \sqrt{g_s} \lambda^{-2}(R + \ldots), \]
\[ e^{A_s} \equiv \lambda^{2/3}e^A \]
Vacuum Solution

\[ e^{A(r)} \sim e^{-Cr^2}, \quad \lambda \sim e^{\frac{3}{2}Cr^2} \quad r \to \infty \]

In the string frame:

\[ S = \int \sqrt{g_s} \lambda^{-2} (R + \ldots), \]
\[ e^{A_s} \equiv \lambda^{2/3} e^{A} \]
Vacuum Solution

$$e^{A(r)} \sim e^{-Cr^2}, \quad \lambda \sim e^{\frac{3}{2}Cr^2} \quad r \to \infty$$

In the string frame:

$$S = \int \sqrt{g} \lambda^{-2}(R + \ldots), \quad e^{A_s} \equiv \lambda^{2/3} e^A$$

Wilson Loop area law with tension:

$$\sigma_c = \frac{1}{2\pi \ell_s^2} e^{2A(r_*)} \lambda^{4/3}(r_*)$$
Finite Temperature

4D Pure YM theory exhibits a first order deconfining phase transition around $T_c \sim 260 MeV$. 
Finite Temperature

4D Pure YM theory exhibits a first order deconfining phase transition around \( T_c \sim 260\, MeV \).

Can the holographic setup reproduce the phase transition and the YM equation of state found on the lattice?

Karsch, hep-lat/0106019
Finite Temperature Solutions

Gursoy, Kiritsis, Mazzanti, FN 0804.0899, 0812.0792

Two kinds of solutions with $\tau \sim \tau + 1/T$ and $R^3$ spatial sections
Finite Temperature Solutions

Gursoy, Kiritsis, Mazzanti, FN 0804.0899, 0812.0792

Two kinds of solutions with $\tau \sim \tau + 1/T$ and $R^3$ spatial sections

- Thermal gas (Confined phase):

$$ds^2 = e^{2A_0(r)} \left( dr^2 + d\tau^2 + dx_i^2 \right), \quad \Phi = \Phi_0(r), \quad 0 < r < +\infty.$$
Finite Temperature Solutions

Gursoy, Kiritsis, Mazzanti, FN 0804.0899, 0812.0792

Two kinds of solutions with $\tau \sim \tau + 1/T$ and $R^3$ spatial sections

- **Thermal gas (Confined phase)**:

  $$ds^2 = e^{2A_0(r)} \left( dr^2 + d\tau^2 + dx_i^2 \right), \quad \Phi = \Phi_0(r), \quad 0 < r < +\infty.$$  

- **Black hole (Deconfined phase - Gluon Plasma)**:

  $$ds^2 = e^{2A(r)} \left( \frac{dr^2}{f(r)} + f(r)d\tau^2 + dx_i^2 \right), \quad \Phi = \Phi(r), \quad 0 < r < r_h$$

  $$A(r) \xrightarrow{r \to 0} A_0(r), \quad f(r) \xrightarrow{r \to 0} 1, \quad f(r_h) = 0,$$

  $$T = \frac{|\dot{f}(r_h)|}{4\pi}, \quad S = (4\pi M_p^3 N_c^2 V_3) e^{3A(r_h)}$$
Deconfinement Transition

The gravity dual indeed displays a rich phase structure with a deconfinement phase transition:

- a minimum BH temperature $T_{\text{min}}$;
- Two BH branches (big and small) for $T > T_{\text{min}}$;
- Big BHs dominates above the critical temperature $T_c > T_{\text{min}}$. 

![Diagram showing $F$ vs $T/T_c$ for $N_c T_c^2 V_3$ with critical temperature $T_c$ marked.](image)
Thermodynamics of the deconfined phase

Thermodynamic observables are computed by geometric quantities:

- **pressure** \( p = -\mathcal{F}/V_3 \sim M_p^3 N_c^2 \int \sqrt{g} R + \ldots \)
- **entropy density** \( s \sim M_p^3 N_c^2 \text{Area}/V_3 \)
- **energy density** \( \epsilon = \text{Mass}/V_3 = -p + Ts \)
Thermodynamics of the deconfined phase

Thermodynamic observables are computed by geometric quantities:

- pressure \( p = -\mathcal{F}/V_3 \approx M_p^3 N_c^2 \int \sqrt{gR} + \ldots \)
- entropy density \( s \approx M_p^3 N_c^2 \text{Area}/V_3 \)
- energy density \( \epsilon = \text{Mass}/V_3 = -p + Ts \)

At high \( T \) we recover conformal behavior:

\[
\frac{\epsilon}{T^4} \approx \frac{3}{4} \frac{s}{T^3} \approx 3 \frac{p}{T^4} \rightarrow 3\pi^4 (M_p \ell)^3 N_c^2
\]

Requiring a free gas with \( N_c^2 \) d.o.f. fixes \( M_p \)

\( (M_p \ell) = 1/(45\pi^2)^{1/3} \)
An Explicit Model

Gursoy, Kiritsis, Mazzanti, FN 0903.2859

Phenomenological potential, fixed small-$\lambda$ and large-$\lambda$ behavior.
An Explicit Model

Gursoy, Kiritsis, Mazzanti, FN 0903.2859

Phenomenological potential, fixed small-$\lambda$ and large-$\lambda$ behavior.

\[ V(\lambda) = \frac{12}{\ell^2} \]

- $\ell$: Overall energy unit (e.g. fixed by lowest glueball state)
An Explicit Model

Gursoy, Kiritsis, Mazzanti, FN 0903.2859

Phenomenological potential, fixed small-$\lambda$ and large-$\lambda$ behavior.

\[ V(\lambda) = \frac{12}{\ell^2} \left(1 + \frac{8}{9}\beta_0\lambda\right) \]

- $\ell$: Overall energy unit (e.g. fixed by lowest glueball state)
- $\beta_0$ fixed by the one-loop $\beta$-function
An Explicit Model

Phenomenological potential, fixed small-\(\lambda\) and large-\(\lambda\) behavior.

\[
V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right)
\]

- \(\ell\): Overall energy unit (e.g. fixed by lowest glueball state)
- \(\beta_0\): fixed by the one-loop \(\beta\)-function
- \(V_1, V_3\) used for the fit to lattice data
An Explicit Model

Phenomenological potential, fixed small-$\lambda$ and large-$\lambda$ behavior.

\[
V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right)
\]

- $\ell$: Overall energy unit (e.g. fixed by lowest glueball state)
- $\beta_0$ fixed by the one-loop $\beta$-function
- $V_1, V_3$ used for the fit to lattice data

Other model parameters:

- $M_p$ fixed by high-$T$ regime
- $\ell_s/\ell$ fixed at zero-$T$ by matching $\sigma_c \sim (440 \text{ MeV})^2$. 
Matching Pure YM Thermodynamics

\[ V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right) \]
Matching Pure YM Thermodynamics

\[ V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right) \]

\[ V_1 \approx 14, \quad V_3 \approx 170 \]

lattice data:
Karsch, hep-lat/0106019
Matching Pure YM Thermodynamics

\[ V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right) \]

\[ V_1 \approx 14, \quad V_3 \approx 170 \]

\[ (L_h)_{HQCD} \simeq (L_h)_{lat} \simeq 0.31 N_c^2 T_c^4 \]

lattice data:
Karsch, hep-lat/0106019
Matching Pure YM Thermodynamics

\[ V(\lambda) = \frac{12}{\ell^2} \left( 1 + \frac{8}{9} \beta_0 \lambda + V_1 \lambda^{4/3} \left[ \log \left( 1 + V_3 \lambda^2 \right) \right]^{1/2} \right) \]

\[ V_1 \approx 14, \quad V_3 \approx 170 \]

\[ (L_h)_{HQCD} \simeq (L_h)_{lat} \simeq 0.31 N_c^2 T_c^4 \]

\[ (T_c)_{HQCD} \simeq 250 \text{MeV} \]

\[ (T_c)_{lat} \simeq 260 \text{MeV} \]

lattice data:
Karsch, hep-lat/0106019
Trace Anomaly and speed of sound

\[ \epsilon - 3p = \langle T^\mu_\mu \rangle_T \]

\[ c_s^2 = \frac{s}{c_V} \]
Trace Anomaly and speed of sound

\[ \epsilon - 3p = \langle T^\mu_\mu \rangle_T \]

\[ c_s^2 = \frac{s}{c_V} \]

\[ \frac{\epsilon - 3p}{T^4 N_c^2} \]

\[ \frac{c_s^2}{c_e^2} \]
In the long-wavelength limit:

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} \]

\[ -P^{\mu i} P^{\nu j} \left[ \eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{3} g_{ij} \partial \cdot u \right) + \zeta g_{ij} \partial \cdot u \right] \]
Hydrodynamic Transport

In the long-wavelength limit:

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} \]

\[ -P^{\mu i} P^{\nu j} \left[ \eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{3} g_{ij} \partial \cdot u \right) + \zeta g_{ij} \partial \cdot u \right] \]

\( \eta \) : Shear viscosity.

- RHIC data consistent with very small \( \eta/s, \sim 0.08 - 0.2 \)
- Closest match: Strong coupling holographic computation in \( \mathcal{N} = 4 \) SYM: \( \eta/s = (4\pi)^{-1} \approx 0.08 \)
- IHQCD setup: same result as in \( \mathcal{N} = 4 \) SYM (universal in 2-derivative models)
Hydrodynamic Transport

In the long-wavelength limit:

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} \]

\[ -P^{\mu i} P^{\nu j} \left[ \eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{3} g_{ij} \partial \cdot u \right) + \zeta g_{ij} \partial \cdot u \right] \]

\( \zeta \): Bulk viscosity

- Vanishes in a conformal fluid (like \( N = 4 \) plasma)
- How significant is \( \zeta \) close to \( T_c \)?
- Computed holographically by the scalar fluctuation of the metric-dilaton system (Gubser et al, 08)
Bulk Viscosity

- Indication from lattice: raise of $\zeta/s$ close to $T_c$ (Meyer ’08).
Bulk Viscosity

- Indication from lattice: raise of $\zeta/s$ close to $T_c$ (Meyer '08).
Bulk Viscosity

- Indication from lattice: raise of $\zeta/s$ close to $T_c$ (Meyer ’08).

- Buchel’s bound: $\zeta/\eta \geq 2(1/3 - c_s^2)$
Drag Force on a Heavy Quark

Trailing String picture:
Drag Force on a Heavy Quark

Trailing String picture:

Drag force:

\[ F = \frac{1}{v} \frac{dE}{dt} = -\frac{1}{2\pi \ell_s^2} \nu \ e^{2A(r_s)} \lambda(r_s)^{\frac{4}{3}}, \quad f(r_s) = \nu^2 \]
Drag Force on a Heavy Quark

Define diffusion time:

\[ F = -p/\tau \]
Drag Force on a Heavy Quark

Define diffusion time:

\[ F' = -\frac{p}{\tau} \]

- in \( \mathcal{N} = 4 \), \( \tau_{conf} \propto \frac{1}{\sqrt{\lambda T^2}} \)
Define diffusion time:

\[ F = -\frac{p}{\tau} \]

- in $\mathcal{N} = 4$, $\tau_{conf} \propto \frac{1}{\sqrt{\lambda T^2}}$
- in IHQCD $\tau = \tau(p)$; no need to fix $\lambda$;
Drag Force on a Heavy Quark

Define diffusion time:

\[ F = -p/\tau \]

- in \( \mathcal{N} = 4 \), \( \tau_{\text{conf}} \propto 1/\sqrt{\lambda T^2} \)
- in IHQCD \( \tau = \tau(p) \); no need to fix \( \lambda \);
- \( \tau(p) \approx \text{const.} \) for \( p \gg m_q \).
- We obtain \( \tau_{\text{charm}} \approx 4.5 \text{ fm}/c \) for \( p \approx 10 \text{ GeV} \), consistent with RHIC analysis.
To Summarize...

Positive features:

- Strict relationships between confinement, mass gap and phase transitions.
- Smooth connection to the correct UV of the theory
- An explicit model that can reproduce at the quantitative level
  - glueball spectrum
  - thermodynamics
  - several transport properties
- of large-$N$ pure Yang-Mills theory
To Summarize...

Positive features:

- Strict relationships between confinement, mass gap and phase transitions.
- Smooth connection to the correct UV of the theory
- An explicit model that can reproduce at the quantitative level
  - glueball spectrum
  - thermodynamics
  - several transport properties
  - of large-$N$ pure Yang-Mills theory

Some drawbacks:

- No explicit construction of gauge theory as decoupling limit.
- No derivation of the gravity side from a fundamental theory
- The UV needs some rethinking (hints that the AdS asymptotics must be in the string frame)
Perspectives

Toward QCD

- Include chiral sector
- Meson spectra and chiral condensate
- More thermodynamics (\(\chi_{SB/\text{restoration}}\))
- Finite baryon density
Perspectives

Toward QCD

- Include chiral sector
- Meson spectra and chiral condensate
- More thermodynamics (\(\chi\)SB/restoration)
- Finite baryon density

.... stay tuned.
Spectrum at $T = 0$

Glueball spectrum: spectrum of normalizable scalar and tensor fluctuations around the $T = 0$ background.
Spectrum at $T = 0$

Glueball spectrum: spectrum of normalizable scalar and tensor fluctuations around the $T = 0$ background.

Numerical computation in the explicit HQCD model shows good agreement with lattice data

<table>
<thead>
<tr>
<th></th>
<th>HQCD</th>
<th>Lat. $N_c = 3$ (Chen et al.)</th>
<th>Lat. $N_c = \infty$ (Lucini, Teper)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{m_{0^{*+++}}}{m_{0^{++}}}$</td>
<td>1.61</td>
<td>1.56(11)</td>
<td>1.90(17)</td>
</tr>
<tr>
<td>$\frac{m_{2^{++}}}{m_{0^{++}}}$</td>
<td>1.36</td>
<td>1.40(4)</td>
<td>1.46(11)</td>
</tr>
<tr>
<td>$\frac{T_c}{m_{0^{++}}}$</td>
<td>0.167</td>
<td>-</td>
<td>0.177</td>
</tr>
</tbody>
</table>
Vacuum Solution - UV

Non-constant $V(\Phi)$: solutions have running coupling.

$$ds^2 = e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \quad \lambda(r) = e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r))$$
Vacuum Solution - UV

Non-constant $V(\Phi)$: solutions have running coupling.

$$
\begin{align*}
   ds^2 &= e^{2A(r)} \left( dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right), \\
   \lambda(r) &= e^{\Phi(r)}, \quad (E_{4D} \sim e^A(r)) \\
   \text{UV:} \quad e^A &\rightarrow \infty, \lambda \rightarrow 0; \quad r \rightarrow 0
\end{align*}
$$

$$
V(\lambda) \sim \frac{12}{\ell^2} \left( 1 + v_0 \lambda + v_1 \lambda^2 \ldots \right)
$$

$$
e^{A(r)} = \frac{\ell}{r} \left( 1 + \ldots \right), \quad \lambda \sim -\frac{1}{b_0 \log r \Lambda} + \ldots, \quad b_0 = \frac{9}{8} v_0
$$

$$
\beta(\lambda) \equiv \frac{d\lambda}{d \log E} = -b_0 \lambda^2 + \ldots
$$

$$
\ell = \sqrt{V(0)/12} \text{ is the asymptotic } AdS \text{ radius; } \Lambda \text{ is an integration constant: the dynamically generated strong coupling scale.}
$$
Vacuum Solution - IR

- $V(\lambda)$ Monotonic in the range $\lambda \in (0, \infty)$ (no fixed points)
Vacuum Solution - IR

- $V(\lambda)$ Monotonic in the range $\lambda \in (0, \infty)$ (no fixed points)
- $\lambda(r)$ grows and $A(r)$ shrinks as $r$ increases
Vacuum Solution - IR

- $V(\lambda)$ Monotonic in the range $\lambda \in (0, \infty)$ (no fixed points)
- $\lambda(r)$ grows and $A(r)$ shrinks as $r$ increases

For large $\lambda$ take:

$$V \sim \lambda^{4/3} (\log \lambda)^{1/2}$$
Vacuum Solution - IR

- $V(\lambda)$ Monotonic in the range $\lambda \in (0, \infty)$ (no fixed points)
- $\lambda(r)$ grows and $A(r)$ shrinks as $r$ increases

For large $\lambda$ take:

$$V \sim \lambda^{4/3} (\log \lambda)^{1/2}$$

- Confinement (Wilson Loop area law)
- Spectrum with mass gap and linear glueball trajectories $m_n^2 \sim n$