Large *N* Limits & Equivalences

Laurence G. Yaffe
University of Washington

Based on work done in collaboration
with Pavel Kovtun and Mithat Ünsal

Refs: RMP 54, 407 (1982), hep-th/0311098, hep-th/0411177, hep-th/0505075, hep-th/0608180, hep-th/0702021, arXiv:0803.0344

Simplifications as $N \to \infty$

Rank of gauge group

- Topological diagrammatic expansion

  $\Rightarrow$ planar diagrams dominate

- Factorization: $\langle AB \rangle \to \langle A \rangle \langle B \rangle$
- Closed loop equations: $W_\Gamma = a_\Gamma^{\Gamma'} W_{\Gamma'} + b_\Gamma^{\Gamma'\Gamma''} W_{\Gamma'} W_{\Gamma''}$
- Vanishing meson, glueball widths
- Scattering amplitudes $\sim (N)^{2\text{-}\#\text{particles}}$
- Baryons $\sim$ solitons
- Volume independence

Why?

Large N limit = Thermodynamic limit

Phase transitions, spontaneous symmetry breaking, coexisting equilibrium states:

- Possible in large volume limit.

  Cluster decomposition:*

  $$\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/V)$$

  volume averaged operators

  Diagnostic of extremal (pure) equilibrium state in large volume limit

- Possible in large N limit.

  Factorization:

  $$\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N)$$

  “decent” gauge invariant operators

  Diagnostic of extremal (pure) state in large N limit

*Assuming finite correlation length

Large N limit = Classical limit

$N \to \infty$: quantum dynamics → classical dynamics

- Large N coherent states $\{|u\rangle\}$ $\sim$ classical phase space
- Quantum operators → classical observables

  $a(u) = \lim_{N\to\infty} \langle u|A|u\rangle$

- Vanishing overlaps: $\langle u|u'\rangle \sim e^{-N^2 f(u,u')}$

  ➡ $\lim_{N\to\infty} \langle u|AB|u\rangle = \lim_{N\to\infty} \langle u|A|u\rangle\langle u|B|u\rangle = a(u)\, b(u)$

- Classical action: $S_{\rm cl} = \lim_{N\to\infty} \frac{1}{N^2} \int dt\, \langle u|i\partial_t - \hat{H}|u\rangle$

  ➡ ground state properties, spectrum, scattering amplitudes, ...

Coherent states

Produced by action of "coherence group" $\mathcal{G}$: $|u\rangle = U|0\rangle, \; U \in \mathcal{G}$

Coherence group $\mathcal{G}$ generated by invariant "coordinates" & "momenta":

Point particles	$\{q_\alpha, \, p_\beta\}$
N-component vectors	$\{\vec{\phi}_\alpha \cdot \vec{\phi}_\beta, \, \vec{\pi}_\alpha \cdot \vec{\phi}_\beta\}$
U(*N*) gauge theory	$\{\mathrm{tr}\, U_\Gamma, \; \mathrm{tr}\, E_\alpha U_\Gamma\}$

Electric field

Wilson loop

Fermions at large N

Fundamental representation fermions:

- fermion loops ➨ $O(1/N)$ suppression in Feynman diagrams
- generate subleading $O(1/N)$ corrections to S_{cl}
- leading large N gauge dynamics unaffected

True

- quenched approximation [no det(D)] in lattice gauge theory OK ?
- no difference between μ_B and μ_I at large N ?

Depends...

Factorization & the sign problem

Dirac operator

$$\langle \mathcal{O} \rangle_{\mathrm{QCD}} = \frac{\langle \mathcal{O} \det D \rangle_{\mathrm{YM}}}{\langle \det D \rangle_{\mathrm{YM}}} \stackrel{?}{=} \frac{\langle \mathcal{O} \rangle_{\mathrm{YM}} \langle \det D \rangle_{\mathrm{YM}}}{\langle \det D \rangle_{\mathrm{YM}}} = \langle \mathcal{O} \rangle_{\mathrm{YM}}$$

Large *N* factorization

- Det D = |Det D| $e^{iN\Theta}$
  - phase Θ can be non-zero, $O(1)$ when $\mu_B \neq 0$
  - $\langle \Theta \rangle = 0$ but $\langle e^{iN\Theta} \rangle \neq 1$
- Quenched approximation:
  - Wrong in hadronic phase, $T<T_c^{\mathrm{YM}}$, with unbroken $Z(N)$ center symmetry, large phase fluctuations
  - OK in deconfined phase, $T>T_c^{\mathrm{YM}}$, with broken $Z(N)$ center symmetry, small phase fluctuations
- Perturbation theory can mislead!

Reality check

Above d=2:

- Can't sum planar diagrams
- Can't solve $N=\infty$ loop equations
- Can't analytically minimize S_{cl} on infinite-dimensional phase space
- Difficult to formulate useful finite-dimensional truncation

But...

- Can use loop equations, or coherent state dynamics, to compare large N limits of differing theories

Large N equivalences

Differing finite N gauge theories can have identical* large N limits:

Gauge group independence $U(N)$ vs. $O(N)$ vs. $Sp(N)$	Lovelace 1982
Volume independence	Eguchi & Kawai 1982, Bhanot, Heller & Neuberger 1982, Gonzalez-Arroyo & Okawa 1983, ...
Orbifold projections	Bershadsky & Johansen 1998, Schmaltz 1998, Strassler 2001, KUY 2003, ...
Orientifold projections	Armoni, Shifman & Veneziano 2003, ...

*With important caveats...

Orbifold projections

"Parent" theory:

Choose discrete symmetry $P \subset$ (gauge $\otimes$ spacetime $\otimes$ flavor)

operators, states invariant under $P \equiv$ "neutral", non-invariant $\equiv$ "non-neutral" or "twisted"

Eliminate degrees of freedom not invariant under P

"Daughter" theory:

May have "emergent" non-gauge symmetry $\mathcal{Q}$ not present in parent

operators, states invariant under $\mathcal{Q} \equiv$ "neutral", non-invariant $\equiv$ "non-neutral" or "twisted"

Operator mapping:

$\{$ neutral single-trace operators $\}_{\text{parent}} \Leftrightarrow \{$ neutral single-trace operators $\}_{\text{daughter}}$

Neutral sector equivalence

"Invertible" projections

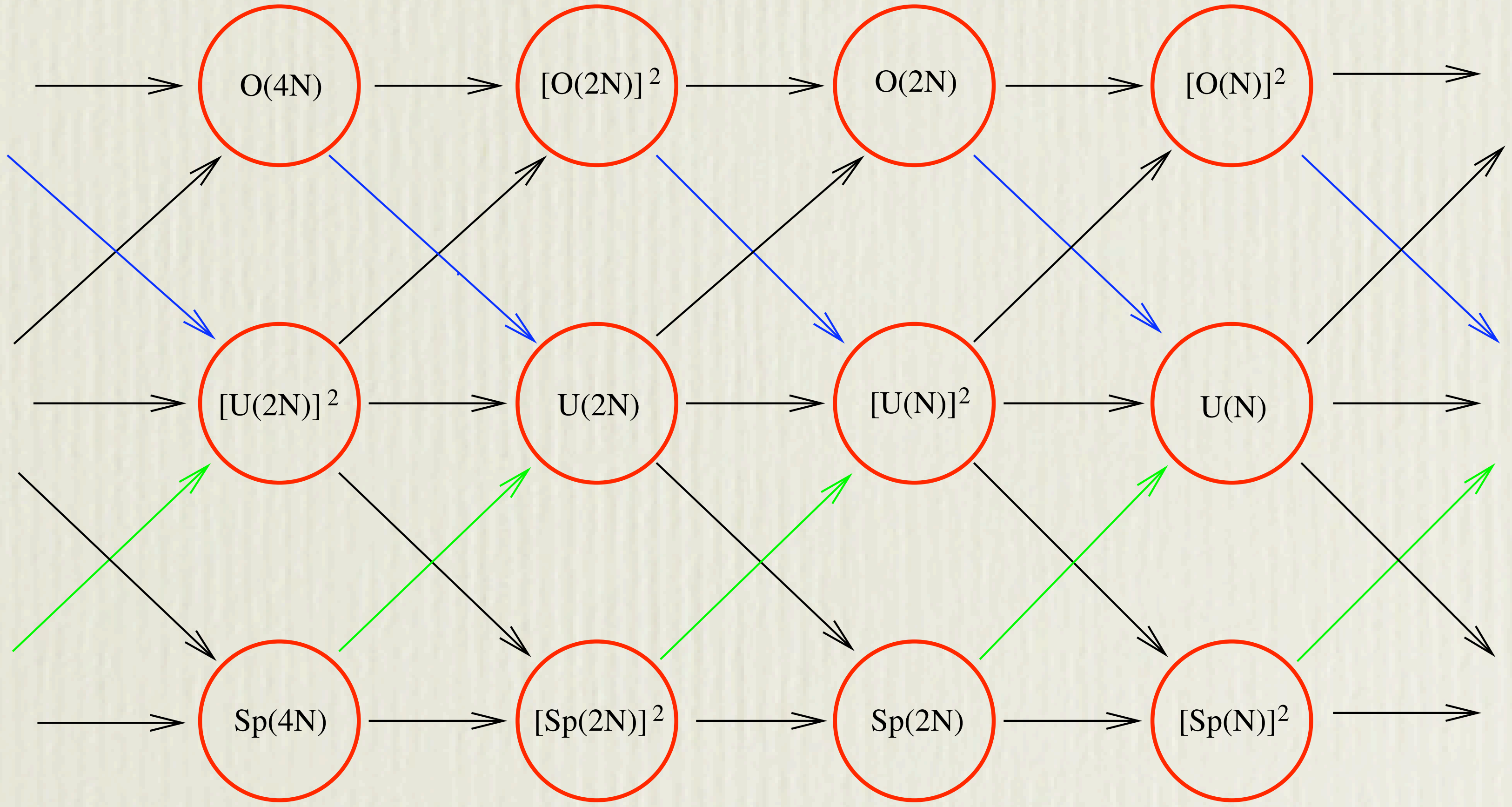
➡ non-perturbative equivalence of dynamics within neutral sectors

➡ non-perturbative equivalence of leading large N behavior of connected correlators of neutral operators *provided* symmetries defining neutral sector not spontaneously broken

➡ directly relate leading large N behavior of free energy, as well as spectrum, partial decay widths & scattering amplitudes of neutral glueballs & mesons

Z_2 projection "duality" web

projection	gauge group
$K = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}_{\text{gauge}}$	$U(2N) \longrightarrow U(N)^2$
$C = \text{charge conj.}$	$U(2N) \longrightarrow O(2N)$
$J = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}_{\text{gauge}}$	$O(2N) \longrightarrow U(N)$
$C \times J$	$U(2N) \longrightarrow Sp(2N)$



Some specific examples

projection	parent theory → daughter theory	emergent daughter theory symmetry
orbifold	$U(2N)$ SYM → $U(N)^2$ YM w. bifund. ferm.	$U(N)_1 \leftrightarrow U(N)_2$
orientifold	$SO(2N)$ SYM $\xrightarrow{J}$ $U(N)$ SYM; $SO(2N)$ SYM $\xrightarrow{J\,(-1)^F}$ $U(N)$ QCD(AS)	charge conjugation
volume reduction	$U(N)$ YM on $(KL)^d$ → $U(N)$ YM on $(L)^d$	$(Z_N)^d$ center

Symmetry realization engineering

Example: Z_N center symmetry in compactified $U(N)$ YM

"deconfinement" = failure of original Eguchi-Kawai proposal

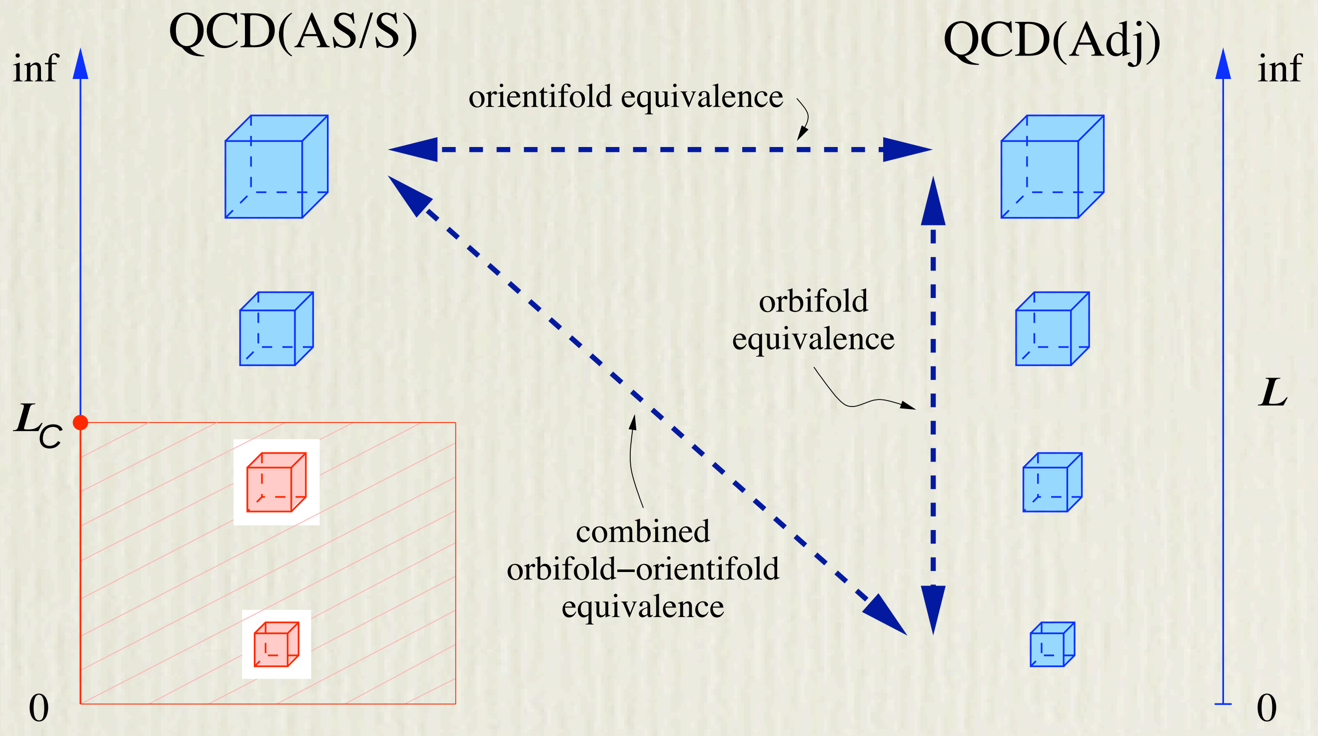
"Fixes":

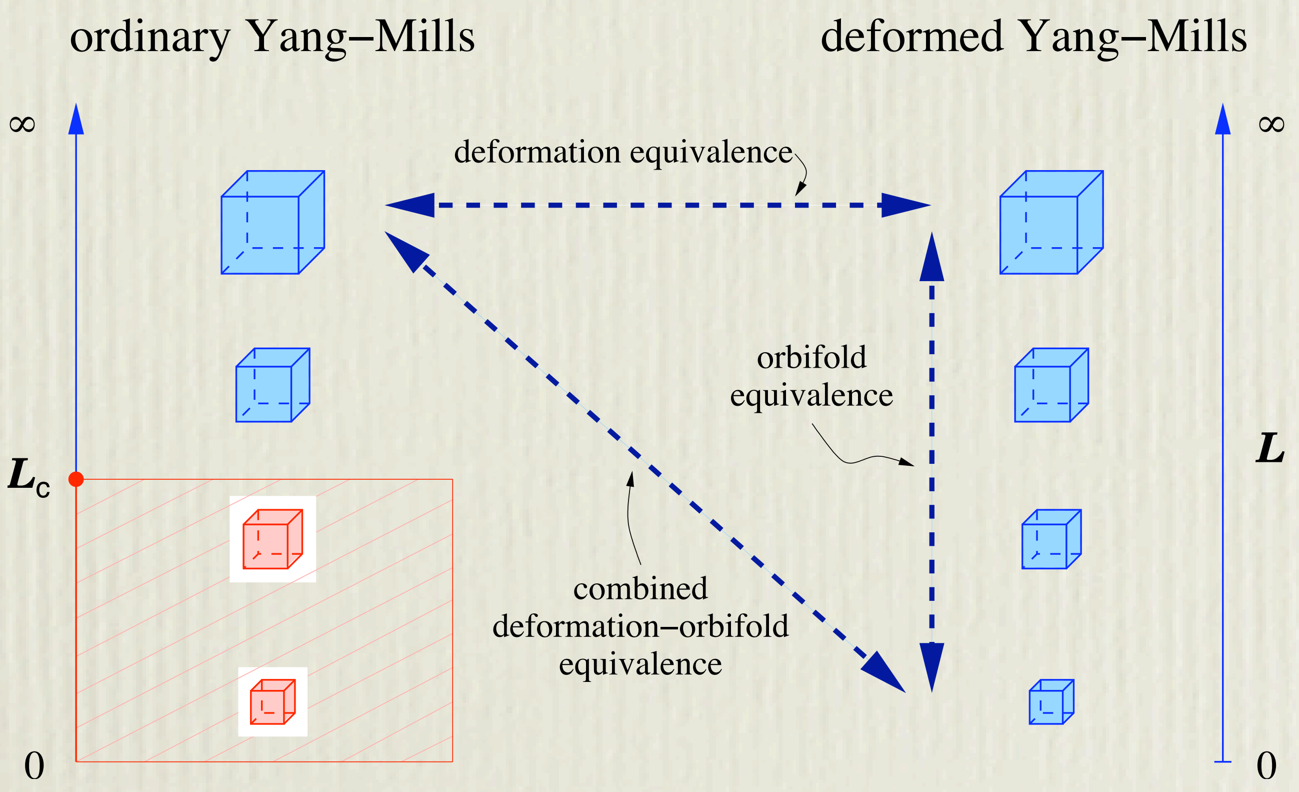
	quenched EK	✘
	twisted EK	✘
	YM → QCD(Adj)	✓
	YM → center-stabilized YM	✓

$$S^{\rm YM}_{\rm center-stabilized} = S^{\rm YM}_{\rm Wilson} + \sum_{n=1}^{\lfloor N/2 \rfloor} c_n \left|\mathrm{tr}\, L^n\right|^2 , \qquad c_n \text{ sufficiently positive}$$

Numerical utility?

Reproduce *C*-even properties of large volume, large-*N* QCD(AS) using single-site $SU(N)$ matrix model with light adjoint fermions





Reproduce properties of large volume, large-*N* YM using single-site matrix model with center-stabilizing terms

Some open questions

- Does $U(N)^2$ Yang-Mills with sufficiently light bifundamental fermions spontaneously break the gauge group interchange symmetry?
- Does QCD(AS) in large volume ever spontaneously break charge conjugation symmetry?
- How large must N be in QCD(Adj), or center-stabilized YM, for accurate volume independence down to a single-site?
- Is numerical simulation cost of single-site center-stabilized YM manageable (relative to large volume simulations) or prohibitive?
- Can large N equivalences improve understanding of phenomenologically interesting models of new strong dynamics sectors?
- Can one formuate accurate finite-dimensional truncations of dynamics on infinite dimensional large N phase space of gauge theories?