The Schwarzschild Solution

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Introduction

- We use the following conventions: Greek coordinate indices run from 0 to 3, while latin ones run from 1 to 3. The signature of the metric is mostly minus. Three-vectors and three-dimensional coordinates are denoted as $\mathbf{X}, \mathbf{x}$, while their four-dimensional equivalents are X, x.
- The material presented here is based on the books by Carroll and Weinberg, [1, 2].

Derivation of the solution

- Recall the field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{1}$$

Contracting with $g^{\mu\nu}$ gives

$$R - 2R = -8\pi G T^{\mu}_{\mu} \tag{2}$$

- We shall search for a solution of (1) describing a point-like mass at rest. Physically such a solution can be used to describe the gravitational field of objects with approximate spherical symmetry such as planets, stars, black holes.
- Our solution should satisfy the following criteria
  1. Far away from the mass the gravitational field should be trivial. I.e. the space should be asymptotically flat
  2. The solution needs to respect the spherical symmetry. Recall that the spherical symmetry group in three dimensions is

$$\mathcal{SO}(3) = \left\{ O \in \mathcal{GL}(3,\mathbb{R}) \middle| \, O^T = O^{-1} \right\} \tag{3}$$

     Where $\mathcal{GL}(3,\mathbb{R})$ is the group of invertible 3×3 matrices. As an example of an element of $\mathcal{SO}(3)$ recall the form of a rotation around the x^3 axis

$$O = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

     One says that a solution respecting rotational symmetry is *isotropic*.
  3. The solution should be *static*; that is it should be invariant with respect to translations in time.
  4. The solution should break the translational symmetry. It is thus *non-homogeneous*.
- We choose a set of coordinates $x^1, x^2, x^2, x^0 \equiv t$. Note that these are not the same as the coordinates of flat Minkowski space, yet as the space is supposed to be asymptotically flat we require

$$ds^2 \xrightarrow{\text{asymptotically}} -dx^{02} + dx^{12} + dx^{22} + dx^{32} \tag{5}$$

- Under rotations, that is under the action of $O \in \mathcal{SO}(3)$, the $\mathbf{x}$ transform as
$$x^i \mapsto O^i_j x^j \qquad dx^i \mapsto O^i_j dx^j \tag{6}$$
Or more succintly
$$\mathbf{x} \mapsto O\mathbf{x} \qquad \mathbf{dx} \mapsto O\mathbf{dx} \tag{7}$$
Now $\sum_i x^i x^i = \mathbf{x}.\mathbf{x}$ transforms as
$$\mathbf{x}.\mathbf{x} \mapsto (O\mathbf{x}).(O\mathbf{x}) = \mathbf{x}O^T O\mathbf{x} = \mathbf{x}.\mathbf{x} \tag{8}$$
and is therefore invariant. The same holds for $\mathbf{x}.\mathbf{dx}$ and $\mathbf{dx}.\mathbf{dx}$.

- The property that the gravitational field should be static means that $g_{\mu\nu}$ should be independent of the *time*-coordinate $t = x^0$. Note that this is highly heuristic however, as the notion of time in curved spaces is a very subtle issue – recall the definition of proper-time.

- There is another constraint on the metric which also follows from the homogeneity with respect to time. We shall state it here without proof: There are no off-diagonal terms such as $dtdx^i$. (However there may easily be terms of the form $dx^i dx^j$).

- Writing $r = \sqrt{\mathbf{x}.\mathbf{x}}$, we may summarise our discussion up to this point by making the following Ansatz for the metric:
$$ds^2 = -F(r)dt^2 + D(r)\left(\mathbf{x}.\mathbf{dx}\right)^2 + C(r)\mathbf{dx}^2 \tag{9}$$

- It is appropriate to introduce spherical coordinates
$$x^1 = r\sin\theta\cos\phi \tag{10}$$
$$x^2 = r\sin\theta\sin\phi \tag{11}$$
$$x^3 = r\cos\theta \tag{12}$$
One obtains
$$dx^3 = \cos\theta dr - r\sin\theta d\theta \tag{13}$$
As an exercise you might want to calculate dx^1 and dx^2 and verify that
$$ds^2 = \underbrace{-Fdt}_{(1)} + \underbrace{r^2 Ddr^2 + Cdr^2}_{(2)} + \underbrace{C\left(r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)}_{(3)} \tag{14}$$

- To simplify the ansatz (9) further, we use the diffeomorphism-invariance[1] of GR to define a new radial coordinate
$$r'^2 \equiv C(r)r^2 \tag{15}$$
We now look at the seperate parts of (14) to investigate how they change under this transformation.

  1. $F(r) \equiv B(r')$

[1] Diffeomorphism-invariance is just a fancy way of saying that we are free to use new coordinates $x' = x'(x)$ as long as the functions $x'(x)$ are differentiable, i.e. smooth.

2. First note that taking the total differential d on both sides of (15) leads to

$$dr^2 = \frac{4r'^2 dr'^2}{2C(r)r + r^2C'(r)} \tag{16}$$

Pluggint this into (2) in (14) gives

$$\left(r^2D(r) + C(r)\right) dr^2 = \underbrace{\frac{r^2D(r)/C(r) + 1}{\left(1 + r/2C'(r)\right)^2}}_{A(r')} dr'^2 \equiv A(r')dr'^2 \tag{17}$$

3. $C(r)r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) = r'^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$

Dropping the primes, the new metric reads

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{18}$$

- To make life easier for us we assume $A(r) = B(r)^{-1}$. There is no physical or mathematical reason to do so, except that we will see that we get the correct result. It will however make our calculations a bit easier.

- Except for the point-like mass we assume our space to be empty. That is, there are no further masses or electromagnetic fields etc. It follows that Apart from the point at $r = 0$ where we assume the piont-like mass to be, the energy-momentum tensor vanishes, $T_{\mu\nu} = 0$. As we explained earlier, it follows that $R = 0$ and therefore the gravitational equations of motion are

$$R_{\mu\nu} = 0 \tag{19}$$

- We shall calculate some Christoffel symbols. Recall that

$$\Gamma^\mu_{\nu\rho} = g^{\mu\lambda}\frac{1}{2}\left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}\right) \tag{20}$$

Now as in with the Ansatz (18)

$$g_{tt} = -B \qquad g_{rr} = \frac{1}{B} \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2\sin^2\theta \tag{21}$$

it follows that

$$\begin{aligned}
\Gamma^t_{\mu\nu} &= -\frac{1}{2B}\begin{pmatrix} \partial_t g_{tt} & \partial_r g_{tt} & \partial_\theta g_{tt} & \partial_\phi g_{tt} \\ \partial_r g_{tt} & -\partial_t g_{rr} & 0 & 0 \\ \partial_\theta g_{tt} & 0 & -\partial_t g_{\theta\theta} & 0 \\ \partial_\phi g_{tt} & 0 & 0 & -\partial_t g_{\phi\phi} \end{pmatrix} \\
&= -\frac{1}{2B}\begin{pmatrix} 0 & -B' & 0 & 0 \\ -B' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{22}$$

One may similiarly compute the other components of $\Gamma^\mu_{\nu\rho}$. (See e.g. chapter 8 of [2]. Note however that Weinberg's metric is of opposite signature. Also he does not make the simplifying assumption $A = B^{-1}$.)

- One calculates the Ricci tensor to be

$$R_{\mu\nu} = \begin{pmatrix} \frac{1}{2}B\left(\frac{2B'}{r} + B''\right) & 0 & 0 & 0 \\ 0 & -\frac{2B'+rB''}{2rB} & 0 & 0 \\ 0 & 0 & 1 - B - rB' & 0 \\ 0 & 0 & 0 & -\sin^2\theta\left(-1 + B + rB'\right) \end{pmatrix} \tag{23}$$

By staring at this for a sufficiently long time, one sees that there are in fact two differential equations for $B(r)$ to satisfy so that $R_{\mu\nu} = 0$ holds. These are

$$0 = rB'' + 2B' \tag{24}$$

$$0 = rB' + B - 1 \tag{25}$$

The second is only a first-order equation and therefore referred to as a *constraint*. Thus we are looking for a solution of the second-order equation that also satisfies the first-order equation. However taking the differential of the second equation with respect to r,

$$\frac{d}{dr}\left(rB' + B - 1\right) = rB'' + 2B' \tag{26}$$

we note that the first-order equation actually entails the second-order one. It is sufficient for us to find a solution to the easier first-order equation. While there are sophisticated methods of finding the solution, we simply use the world-famous & convenient approach known as *guess & check*.

- The trivial solution is $B(r) = 1$. The metric is now given by

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{27}$$

This however is simply the metric of flat Minkowski space written in spatial-polar coordinates. It does not break the translationally symmetry we demanded initially. We continue therefore to look for a different solution.

- Anothe solution is

$$B(r) = r_c - \frac{r_h}{r} \tag{28}$$

The constant of integration r_c is fixed by the first-order solution to be $r_c = 1$. There are no constraints on the second constant of integration r_h. We shall assume it to be positive however. Therefore the metric reads

$$ds^2 = -\left(1 - \frac{r_h}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_h}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{29}$$

This is the famous Schwarzschild solution.

Asymptotics

- Recall the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \tag{30}$$

We want to find solutions of the form

$$x^\mu = (\tau, r(\tau), \pi/2, 0) \tag{31}$$

Recalling that the only non-vanishing coefficients of $\Gamma^t_{\mu\nu}$ are Γ^t_{tr} and Γ^t_{rt} gives

$$0 = \ddot{t} = -2\Gamma^t_{tr}\dot{t}\dot{r} \tag{32}$$

Similiarly for r, one obtains after calculating the relevant Christoffel symbols,

$$\ddot{r} = -\frac{1}{2}\left(BB'\dot{t}^2 - \frac{B'}{B}\dot{r}^2\right) \tag{33}$$

- Assuming that we are very far away from the mass, we want to consider r to be large. One needs to ask though large in comparison to what? The only other quantity with dimensions of length though is the constant r_h. Hence we consider $\frac{r}{r_h}$ to be large and therefore perform a Taylor expansion in $\frac{r_h}{r} \equiv \chi$. We start with BB'

$$\begin{aligned} BB' &= \left(1 - \frac{r_h}{r}\right)\frac{r_h}{r^2} \\ &= \frac{1}{r}\frac{\chi}{1-\chi} \\ &= \frac{1}{r}\left(\chi + \chi^2 + \mathcal{O}\left(\chi^3\right)\right) \\ &= \frac{1}{r}\left(\frac{r_h}{r} + \frac{r_h^2}{r^2} + \ldots\right) \end{aligned} \tag{34}$$

Similiarly

$$\frac{B'}{B} = \frac{1}{r}\left(\frac{r_h^2}{r^2} + \ldots\right) \tag{35}$$

- Thus to zeroth order in r_h/r the geodesic equation (32) is trivially satisfied, while (33) becomes

$$\ddot{r} = 0 \tag{36}$$

which is the equation of motion for a particle in empty space – the particle will continue to move with constant velocity. Hence very far away from the mass, space is flat, as it should be, and masses moving very far away are not influenced by gravitation.

- When moving closer to the point mass at $r = 0$, we need to consider first order effects in χ. Equation (32) is still trivially satisfied, while (33) becomes

$$\ddot{r} = \frac{r_h}{2r^2} \tag{37}$$

This is actually a highly interesting result. To see why, consider ordinary Newtonian gravitation, where the gravitational force on a test-particle of mass m is $F(r) = \frac{GMm}{r^2}$. Combining this with $F(r) = m\ddot{r}$ gives

$$\ddot{r} = \frac{GM}{r^2} \tag{38}$$

Comparing the two results we see that the Schwarzschild solution reproduces classical Newtonian mechanics iff

$$r_h = 2GM \tag{39}$$

where M is the mass of the mass at $r = 0$. So we finally re-write the metric in it's most common form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{40}$$

The black hole[2]

- To understand the relation between the Schwarzschild solution and black holes - or rather to understand black holes in general - we need to study null geodesics, that is curves $y^\mu(\tau)$ satisfying $\dot{y}^2 = 0$. Making again an Ansatz which is trivial in the the spherical coordinates, $\dot{y}^\mu = (t(\tau), r(\tau), 0, 0)$

$$0 = -\left(1 - \frac{r_h}{r}\right) \dot{t}^2 + \left(1 - \frac{r_h}{r}\right)^{-1} \dot{r}^2 \tag{41}$$

which leads us to

$$\frac{dt}{dr} = \frac{\frac{dt}{d\tau}}{\frac{dr}{d\tau}} = \frac{\dot{t}}{\dot{r}} = \pm \left(1 - \frac{r_h}{r}\right)^{-1} \tag{42}$$

As $r \to r_h$, $\frac{dt}{dr} \to \pm\infty$. Ifft appears as if null geodesics cannot cross from $r > r_h$ into the area defined by $r < r_h$.

- As it is often the case in general relativity, the notion that null geodesics - and therefore light rays - cannot reach r_h is due to a bad choice of coordinates. The equation for $\frac{dt}{dr}$ can be integrated to give

$$t = \pm \left[r + r_h \ln\left(\frac{r}{r_h} - 1\right)\right] + \text{const.} \equiv r^* \tag{43}$$

- If we define new coordinates (t, r^*, θ, ϕ), null geodesics are now defined by

$$t + r^* = \text{const.} \qquad \text{infalling} \tag{44}$$
$$t - r^* = \text{const.} \qquad \text{outgoing} \tag{45}$$

- Finally defining a coordinate $v = t + r^*$ and using it to replace t in the original Schwarzschild solution, the metric becomes

$$ds^2 = -\left(1 - \frac{r_h}{r}\right) dv^2 + (dv dr + dr dv) + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{46}$$

These are the *Eddington-Finkelstein coordinates*. Infalling null geodesics are characterised by $v = \text{const.}$, which means $\frac{dv}{dr} = 0$. Outgoing ones are defined by

$$t - r^* = 0 \Leftrightarrow v = 2r^* \tag{47}$$

[2]See chapter 5.6 of [1].

which leads to

$$\frac{dv}{dr} = \begin{cases} 2\left(1 - \frac{r_h}{r}\right)^{-1} & \text{(outgoing)} \\ 0 & \text{(infalling)} \end{cases} \tag{48}$$

The point is that for $r < r_h$ all future-directed null geodesics are in the direction of decreasing r. It follows that once past r_h, light cannot escape the region $r < r_h$. As the light-cone bounds the space of time-like curves, the same holds true for massive objects. The hypersurface defined by $r = r_h$ denotes an *event horizon*.

- There is one note of caution on black holes and event horizons. There are many other solutions to the field equations describing black objects than only the Schwarzschild solution.[3] For most of these it is true that the event horizon is given by a hypersurface where $g_{tt} = 0$ and $g_{rr} = \pm\infty$, as in the orgiginal Schwarzschild metric. However as the Eddington-Finkelstein coordinates show, this does not have to be the case. A more appropriate definition of a black hole and its event horizon is an area from where null- & timelike-geodesics may not escape to spatial infinity. Or in the words of Sean Carroll, *a black hole is simply a region of spacetime separated from infinity by an event horizon.*

References

[1] S. M. Carroll, "Spacetime and geometry: An introduction to general relativity," *San Francisco, USA: Addison-Wesley (2004) 513 p*

[2] S. Weinberg, "Gravitation and cosmology: Principles and applications of the general theory of relativity," *New York, USA: John Wiley & Sons (1972) 657 p*

[3] See e.g. the Reissner-Nordstrom describing electrically charged black holes or the Kerr solution, which describes rotating black holes. In string theory & supergrvity one considers also extended objects such as black rings and branes.