PH-103 Syllabus - Spring 2007

Notation and Advice

In print, vectors are usually denoted with bold face, like $\mathbf{E}$, whereas scalars are denoted normally, like $E = |\mathbf{E}|$. In this note, unit vectors (vectors of length one) are denoted by a hat, $\hat{\mathbf{r}}$. In writing the bold face is usually replaced by a bar $\bar{E}$ or an arrow $\vec{E}$.

In answering exercises (exams), be precise, i.e. Gauss' law deals with a dot-product, not a multiplication. *Explain* why you can simplify it in case you think you can!

Electrostatics

Ref. [1], chapter 21,22.1-22.5, 22.8-22.9, 23, 24.1-24.8,24.10-24.12, 25.1-25.7.

Electrostatics covers aspects of electro-magnetism that deals with charges (or distributions) that produce constant electric fields (i.e. typically charges at rest).

The Coulomb Force

The force $\mathbf{F}_C$ on a charge q_1 exerted by a charge q_2 with respective position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is

$$\mathbf{F}_C = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathbf{r}_{12}^2}\hat{\mathbf{r}}_{12} \tag{1}$$

where $\mathbf{r}_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$,$\hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/|\mathbf{r}_{12}|$ and $\epsilon_0 = 8.85\times10^{-12}$ F/m is the permitivity of vacuum.

The Electric Field

The electric field at position $\mathbf{r}$ from a charge q is defined by the Coulomb force a charge Q would experience at this position

$$\mathbf{F}_C(\mathbf{r}) = Q\mathbf{E}(\mathbf{r}) \tag{2}$$

Field lines of the electric field originate on positive charges and terminate on negative charges (when due to charges).

The Superposition Principle

The superposition principle states that contributions are independent and can be added. It is valid for the Coulomb force and therefore the electric field, as well as for the electric potential.

For a collection of charges q_i with positions $\mathbf{r}_i$ the total field at position $\mathbf{r}$ can thus be calculated as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{(\mathbf{r}-\mathbf{r}_i)^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|} \tag{3}$$

Gauss' Law

The flux of a vector field (in this case the electric field $\mathbf{E}$) through a surface is defined as

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} \tag{4}$$

where $d\mathbf{a}$ is the normal vector to an infinitesimal surface segment (the magnitude is the area of this segment).

Gauss' law states that the total flux through a closed surface is proportional to the total charge inside the volume the surface encloses.

$$\oint \mathbf{E} \cdot d\mathbf{a} = Q_{enc}/\epsilon_0 \tag{5}$$

where $d\mathbf{a}$ is an infinitesimal surface area element and Q_{enc} is the total charge enclosed by the (closed) surface. The closed surface is also called a Gauss' surface. If the enclosed charge is better described by a continuous charge distribution we can write

$$Q_{enc} = \int_V \rho d\tau \tag{6}$$

where V is the volume enclosed, ρ is the charge per unit volume, and $d\tau$ is a volume element. This is a three dimensional integral.

Using Gauss' law we can find the field for some typical systems.

Sphere

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \ , \quad r \geq a \tag{7}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\hat{\mathbf{r}} \begin{cases} 0 & ; \text{ (charge on surface)} \\ \frac{r}{a^3} & ; \text{ (homogeneous charge distribution)} \end{cases} \quad ; \ r < a \tag{8}$$

where Q is the total charge on the sphere, a the radius of the sphere and r the distance from the sphere. If the sphere is a conductor the charge will distribute itself homogeneously on the surface.

Infinite straight line

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{\mathbf{r}} \tag{9}$$

where λ is the charge per unit length [C/m] and r the perpendicular distance from the wire.

Infinite plane

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \tag{10}$$

where σ is the surface charge density [C/m^2], and $\hat{\mathbf{n}}$ is the unit normal vector to the side of the infinite plane in question. The plane is assumed to have no thickness.

Surface of plane conductor

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \tag{11}$$

as equation (10). The factor of two vanishes as there is no field inside the conductor.

Using the divergence theorem[1] we find that Gauss' law can also be written

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{12}$$

where ∇ is a vector operator, the gradient, which in cartesian coordinates is given by

$$\nabla = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) \tag{13}$$

when the gradient is dotted onto a vector field (like $\mathbf{E}$) we say that we calculate the *divergence* of the vector field.

Electric Potential

When the line integral of the E-field along a closed path is zero, we can define an electric potential V which is a scalar. This is always true in electrostatics (i.e. when the E-field arises from non-moving charges).

$$V_b - V_a = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \tag{14}$$

where $V_b - V_a$ is the potential difference between the positions given by $\mathbf{a}$ and $\mathbf{b}$. When we work with point charges, or finite charge distributions we traditionally put our zero point for the potential at infinity.

The potential of a point charge q is given by

$$V = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}dr' = \frac{q}{4\pi\epsilon_0 r} \tag{15}$$

where we have defined zero at infinity. Note that positive charges have (per definition) positive potential.

If we have the electric potential the electric field is given by the gradient

$$\mathbf{E} = -\nabla V \tag{16}$$

Poisson's and Laplacs's Equations

Combining equation (16) and (12) we find

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{17}$$

which is called the *Poisson* equation. If $\rho = 0$, as in vacuum, the equation reduced to a zero on the right hand side and is called the *Laplace* equation. In cartesian coordinates the Laplace equation can be written

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0 \tag{18}$$

from which we conclude that unless all the second derivatives are zero, at least one must be of opposite sign to the others. This means that no *electrostatic* field can trap charges and is called Earnshaw's theorem.

[1] The divergence theorem states that the volume integral of the divergence of a vector field is equal to the integral over the vector field on the enclosing surface $\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{E}) d\tau$

Work and Energy in electrostatics

The work done when moving a charge from $\mathbf{a}$ to $\mathbf{b}$ in an electric field can be calculated from the definition of work

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})] \tag{19}$$

note that the work done by the electric field is *path independent*, exactly like for the gravitational field. This means that the Coulomb force is a *conservative* force.

The energy required to move a charge Q from infinity, where the potential is zero, to a position $\mathbf{r}$ where the potential is $V(\mathbf{r})$ is thus $QV(\mathbf{r})$.

The work needed to assemble a point charge distribution can thus be found to be

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i) \tag{20}$$

note that there is no superposition principle for work done, as it scales with charge squared.

Boundary Conditions

At a surface the perpendicular component of the $\mathbf{E}$ field, $E_\perp$ is discontinuous if there is a surface charge density σ

$$\Delta E_\perp = \frac{1}{\epsilon_0} \sigma \tag{21}$$

whereas the tangential component $E_{||}$ always is continuous.

In vector notation this can be written

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \tag{22}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the surface.

Conductors

Conductors are special in that they contain free charges. Because of this the electric field inside will be zero ($\mathbf{E} = \mathbf{0}$) as the charges move almost instantly if any field is present. This also causes all charges on a charged conductor to be evenly distributed on its surface(s). A conductor will always be an equipotential as $\mathbf{E} = \boldsymbol{\nabla} V = \mathbf{0}$, i.e. the potential being constant throughout. The electric field will, because of this always be perpendicular to the surface right outside the surface. Finally the field in any empty cavity completely enclosed by a conductor will be zero (Faraday cage).

Capacitance

A system of two isolated conductors is said to have capacitance. If the total charge of the system is zero, then the initial potential is zero. By moving charges from one conductor to the other we build up charge on both of them, and thus an electric field, and therefore a potential. The relationship between charge "moved" Q and the potential difference V between the conductors is given by

$$Q = CV \tag{23}$$

where C is the capacitance (unit Farad), which is a purely geometrical constant and defined to be positive.

The capacitance of two parallel plates assuming that their dimensions are much larger than their separation is (derivation, see [1])

$$C = \epsilon_0 \frac{A}{d} \quad , \quad A \gg d \tag{24}$$

where A is the area of each plate and d is the distance between the plates.

The energy stored in a capacitor is given by the work needed to charge it

$$W = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C} \tag{25}$$

Capacitors can be connected in series and parallel, and the effective capacitance of such assemblies can be calculated by

$$\frac{1}{C_{series}} = \sum \frac{1}{C_i} \qquad , \qquad C_{parallel} = \sum C_i \tag{26}$$

Dielectrics

Dielectrics are insulators and can be either *polar* or *non-polar*, in that they are made up from molecules/atoms that have a dipole-moment or not. Imposing an external electric field on a dielectric will polarise it either by aligning the dipole-moments of the dielectric or by polarising the atoms/molecules it is made up from. The *polarization* of a medium is defined as

$$\mathbf{P} = < \text{the dipole moment per unit volume} > \tag{27}$$

The torque $\boldsymbol{\tau}$ on a dipole of moment $\mathbf{p}$ in an electric field $\mathbf{E}$ is given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \tag{28}$$

If a linear dielectric is inserted in a capacitor it has the effect of changing the capacitance by the relative permittivity of the dielectric

$$C = \epsilon_r C_{vacuum} \tag{29}$$

the relative permittivity of air is above but close to one. Most materials have permittivities larger than one.

Circuits, current and resistance

Ref. [1], chapter 26, 27

Current and Resistance

Current is defined as charge per unit time and is measured in *Ampères* A

$$I = \frac{dQ}{dt} \tag{30}$$

and current density J is charge per unit time per unit area, such that

$$I = \int_S \mathbf{J} \cdot d\mathbf{a} \tag{31}$$

where S is the surface through which the current flows.

From charge conservation we derive Kirkhoff's current law, which states that the sum of all currents going to a certain point in space should be zero.

To make charges move in a conductor an electric field is necessary. In most conductors the relationship between the imposed electric field and the current density is Ohm's law

$$\mathbf{J} = \sigma \mathbf{E} \tag{32}$$

where σ is the conductivity of the conductor, related to its resistivity by $\sigma = 1/\rho$.

In a bulk of material, where a current I passes through this can be rewritten in terms of the potential drop U across the material to the more familiar form of Ohm's law :

$$U = RI \tag{33}$$

where the constant R is called the resistance of the material, which for a material of uniform resistivity is related to the resistivity by $R = L\rho/A = L/(\sigma A)$, where A is the cross sectional area through which the current is flowing and L the length of the resistor.

The current is carried by free charges moving in the material. The typical drift velocity is

$$v_d = \frac{I}{nAq} = \frac{J}{nq} \tag{34}$$

where A is the surface area of the material in question, n is number of charge carriers per unit volume and q their charge (typically e).

To drive a current work needs to be done. The work per unit time (for resistive dissipation) is

$$\frac{dW}{dt} = P = UI = I^2 R \tag{35}$$

which is called the Joule heating law.

Simple circuits and EMF

In a simple circuit the so-called *loop-rule* can be applied. It states that the sum of all potential changes around the loop in a circuit are zero, i.e. that the potential at any point is well-defined

$$\sum V_i = 0 \tag{36}$$

To drive a current we need a potential difference that makes the electrons flow, this is called an EMF, or electro-motive force (although it is not a force!) $\mathcal{E}$. In a simple circuit with a resistor and an EMF the relation between these and the current is

$$\mathcal{E} = IR \tag{37}$$

When resistors are connected in series and parallel the effective resistance they will show to a circuit are

$$R_{series} = \sum R_i \;\; ; \;\; \frac{1}{R_{parallel}} = \sum \frac{1}{R_i} \tag{38}$$

In a circuit of multiple loops, the loop rule can be applied on each loop separately, and the resulting system of equations solved.

Transients

When charging a capacitor C through a resistor R (connected in series) with an EMF $\mathcal{E}$ the potential across the capacitor builds up according to

$$V_C = \mathcal{E}\left(1 - e^{-t/RC}\right) \tag{39}$$

where $\tau = 1/RC$ is the time constant of the charging, given the time at which point the capacitor is ~63 % charged.

Discharging the capacitor looks similar

$$V_C = V_0 e^{-t/RC} \tag{40}$$

where V_0 is the starting potential across the capacitor (at $t = 0$).

Magnetostatics

Ref. [1], chapter 28.1-28.8, 29.1-29.5.

Magnetostatics covers aspects of electro-magnetism, where the currents can be described as *steady*, such that the magnetic fields are constant.

The Lorentz Force

The force on a charged particle moving in a magnetic field is called the Lorentz Force and is given by

$$\mathbf{F}_B = Q(\mathbf{v} \times \mathbf{B}) \tag{41}$$

where Q is the charge, $\mathbf{v}$ the velocity and $\mathbf{B}$ the magnetic field. Note that the magnetic force will always work perpendicular to the direction of motion, for this reason magnetic forces do no work, as

$$W_B = \int \mathbf{F_B} \cdot d\mathbf{l} = 0 \tag{42}$$

Cyclotron Motion

The Lorentz force is perpendicular to the direction of motion. This is also the case for the centripetal force. It can be shown that charged particles move on circular orbits in a homogeneous magnetic field. The radius of this circular path is

$$r = \frac{mv}{QB} \tag{43}$$

where m is the mass of the particles, v its velocity and Q its charge.

The angular velocity of this motion is

$$\omega = \frac{QB}{m} \tag{44}$$

where $\omega = 2\pi\nu = 2\pi/T$, where T is the period and ν is frequency. Note that these values are all independent of the particle's kinetic energy.

The Hall Effect

A charged particle moving in a material across which a magnetic field is imposed will also experience the Lorenz force. If a current I runs through a slap of material in a direction perpendicular to a magnetic field B, charges will be deflected. A steady state will arise, in which charges will accumulate on each side of the slap such that the electric field they give rise to compensates for the deflecting magnetic forces. This charge distribution results in a potential across the slap perpendicular to both the magnetic field and the direction of the current. This potential is called the Hall potential or Hall voltage

$$V_H = v_d d B \tag{45}$$

where v_d is the drift velocity of the charges, and d the width of the slap. The sign of the potential will depend on the sign of the charge carriers.

Using equation (34) we find that the density of charge carriers can be found as

$$n = \frac{IB}{qV_H l} \tag{46}$$

where I is the current, q the charge of the carriers and l the thickness of the slap.

Force on a current carrying wire

The Hall effect describes the balance of forces inside a slap of current carrying material. However, the magnetic force is not compensated on the material as a whole, and thus a current carrying wire in a magnetic field experiences a force given by

$$\mathbf{F}_{wire} = I\mathbf{L} \times \mathbf{B} \tag{47}$$

where L is a vector whose magnitude is equal to the length of the wire, and who points in the direction of the wire. A more general form is

$$\mathbf{F}_{wire} = I \int d\mathbf{L} \times \mathbf{B} \tag{48}$$

Biot-Savart's Law

The magnetic field from a piece of wire at position $\mathbf{r}$ relative to the wire segment $d\mathbf{l}$ carrying current I may be calculated by Biot-Savart's law

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \tag{49}$$

where $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 is the permeability of free space. The field from a wire can thus be calculated by integration.

The field from an infinite straight current carrying wire can be found to be

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\mathbf{r}} \tag{50}$$

where r is the perpendicular distance from the wire (cylindrical coordinates).

The magnitude of the force between two parallel current carrying wires is thus

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi d} \tag{51}$$

where we find that if the currents are parallel the wires attract, and if antiparallel they repel. d is the distance between the wires and L is the length.

Ampère's Law

Ampères law states that the integral of the magnetic field along a closed path is proportional to the current flowing through the surface enclosed by the path.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \tag{52}$$

where the path is called an Amperian loop. This is sometime referred to as Ampères circuital theorem.

Applying this law makes some problems easy, like for a long solenoid (i.e. the length much larger than the radius) where we find that the magnetic field outside is zero, and that inside it is parallel to the axis, and homogeneous

$$\mathbf{B}_{sol} = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & , \text{ inside} \\ 0 & , \text{ outside} \end{cases} \tag{53}$$

where $\hat{\mathbf{z}}$ is the direction of the axis of the solenoid, n is the number of windings per unit length and I is the current through the wire.

A toroid is a solenoid closed on itself to form a donut like shape. The field inside the toroid can be found, using Ampères law to be

$$\mathbf{B} = \frac{\mu_0 I N}{2\pi} \frac{1}{r} \hat{\boldsymbol{\phi}} \tag{54}$$

where N is the total number of windings and r is the distance from the center (cylindrical coordinates) and $\boldsymbol{\phi}$ is the azimuthal direction.

Induction

Ref. [1], chapter 30

Faraday's and Lenz's Laws

Faraday's law states that a changing magnetic flux induces an electric field. Lenz's law essentially gives the direction of the induced electric field (also included in Faraday's law). Magnetic flux is defined in the same way a electric flux (equation (4))

$$\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a} \tag{55}$$

where, if S is a closed surface, the total flux is zero. This is a way of saying that there are no magnetic monopoles (magnetic charges).

Faraday's law states that

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \tag{56}$$

where Lenz's law, which gives the minus sign can be expressed as "nature abhors a change in flux", which means that the induced electric field will be in a direction to try to compensate for the change in magnetic flux.

In the special case where the flux is changing through a loop made up of a conductor, the induced electric field gives rise to an EMF, and we can express Faraday's law as

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{57}$$

where Φ_B is the total flux through the loop. In the special case of a solenoid the total flux is the sum of the flux through each winding and if we assume that this is the same for each winding we can write

$$\mathcal{E}_{sol} = -N\frac{d\Phi_{B,loop}}{dt} \tag{58}$$

Self and mutual inductance

If we have a conducting loop, we can send current through and a magnetic field will arise. However, Faraday's law tells us that when we do this a counter-EMF will be induced which tries to oppose this change of flux. The magnetic field is proportional to the current, and the flux is therefore proportional to the current. We can write this as

$$\Phi_B = LI \tag{59}$$

where the proportionality constant L is called the self-inductance, measured in Henry (H). The EMF induced will be given by

$$\mathcal{E} = -L\frac{dI}{dt} \tag{60}$$

If the current through one system induced a change in flux through another system, a current change will induce an EMF in the second system. This phenomenon is called mutual inductance, and the constant of proportionality is called the mutual inductance M. It is easy (from Biot-Savart's law) to show that the mutual inductance is the same independent of which loop is inducing in what loop.

The self-inductance of a solenoid is given by (using equation (53) and (59))

$$L_{sol} = N\frac{\Phi_{loop}}{I} = \mu_0 n^2 l A \tag{61}$$

where n is the number of windings per unit length, l the length and A is the cross sectional area of the solenoid. Inductance, like capacitance, is a purely geometrical issue.

The work needed to build up a current I in an inductor can be found to be

$$W = \frac{1}{2}LI^2 \tag{62}$$

which is the equivalent of equation (25) for the inductor. This is thus the energy stored in the inductor.

Inductors in a circuit

If we build up a current in an inductor using an EMF $\mathcal{E}$ through a resistor R in series the current will build up from zero as

$$I = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right) \tag{63}$$

where $\tau = L/R$ is the time constant of the build-up.

AC Theory with Complex Numbers

Ref. [1], chapter 31.1-31.10 + notes

AC and Passive Linear Components

Alternating currents (AC) are generated by generators rather then batteries which supply direct current (DC). A general form describing an AC source is

$$V = V_0 \cos(\omega t) \tag{64}$$

where V_0 is the amplitude and ω is the angular frequency.

The current in simple circuits with an AC source and one linear passive element can be found

$$Resistance : I = I_0 \cos(\omega t) \quad ; I_0 = \frac{V_0}{R} \quad \text{(current and voltage in phase)} \tag{65}$$

$$Capacitance : I = I_0 \cos(\omega t + \pi/2); \; I_0 = V_0 \omega C \quad \text{(current leads voltage)} \tag{66}$$

$$Inductance : I = I_0 \cos(\omega t - \pi/2); \; I_0 = \frac{V_0}{\omega L} \quad \text{(current lags voltage)} \tag{67}$$

The elements are called passive and linear because we in all cases have that the current is proportional to the voltage, and that they only "respond", i.e. they do not generate anything on their own (like a generator).

Complex Numbers

In complex numbers we define the special number $j = \sqrt{-1}$, and we can write

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{68}$$

called Euler's formula.

Using this notation we can define a complex voltage

$$\tilde{V} = V_0 e^{j\omega t} \tag{69}$$

such that the real part is our physical voltage

$$V = Re\left[\tilde{V}\right] \tag{70}$$

A.C. Circuits with complex numbers

From the correlation with the passive linear components we define a complex impedance $\tilde{Z}$ such that we can write

$$\tilde{V} = \tilde{Z}\tilde{I} \tag{71}$$

where, we find that the impedances of the passive linear components above are

$$\tilde{Z}_R = R \; ; \; \tilde{Z}_C = -j/(\omega C) \; ; \; \tilde{Z}_L = j\omega L \tag{72}$$

Following the arguments of resistor networks, a network of impedances can be found to exhibit one total impedance as

$$\tilde{Z}_{series} = \sum \tilde{Z}_i \;\; ; \;\; \frac{1}{\tilde{Z}_{parallel}} = \sum \frac{1}{\tilde{Z}_i} \tag{73}$$

where in general these have both a real and an imaginary part. The imaginary part is called the *reactance*.

The average power dissipated per cycle is

$$W = \frac{V_0^2}{2R} = V_{RMS} I_{RMS} = I_{RMS}^2 R \tag{74}$$

where $V_{RMS} = V_0/\sqrt{2}$ and $I_{RMS} = I_0/\sqrt{2}$ are the root mean square values.

The introduction of complex impendance means that the complex differential equations we would have to deal with for circuits with R,C, and L's without complex numbers, are reduced to simple algebraic equations in complex numbers.

Maxwell's Equations

Ref. [1], chapter 32

We can summarise all the knowledge we have acquired about electric and magnetic fields and their interrelations in four simple equations (assuming no dielectric or magnetic materials)

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = Q_{enc}/\epsilon_0 \qquad \text{Gauss' Law} \tag{75}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \qquad \text{(no name)} \tag{76}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \text{Faraday's Law} \tag{77}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I_{enc} \qquad \text{Ampère-Maxwell Law} \tag{78}$$

where $d\mathbf{a}$ is a surface element and $d\mathbf{l}$ is a path element. We have not discussed the first part of the right hand side of equation (78), which is Maxwell's law of induction, that a changing electric flux induces a magnetic field (in the same way as the opposite is true). This originally came about from symmetry considerations with respect to Faraday's law, but is a crucial addition as you will see in your next course on electricity and magnetism. The part $\epsilon_0 d\Phi_E/dt$ is called the *displacement current*. This addition makes electromagnetic waves possible.

In differential form these equations simplify to

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0 \ ; \ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \ ; \ \boldsymbol{\nabla} \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \ ; \ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt} + \mu_0 \mathbf{J} \tag{79}$$

where we've used the divergence theorem and Stoke's theorem to eliminate the integrals [2].

References

[1] Halliday, Resnick and Walker, *Fundamentals of Physics*, 7th edition (extended), Wiley (2005).

[2] D. J. Griffiths, *Electrodynamics*, 3rd Edition, Addison-Wesley (2003).